A CLASS OF HAMILTON-JACOBI EQUATIONS ON BANACH-FINSLER MANIFOLDS

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During the last part of the editorial processing of this article, our dear friend and colleague Luis Sánchez-González passed away unexpectedly. We dedicate this paper to his memory.

ABSTRACT. The concept of subdifferentiability is studied in the context of C^1 Finsler manifolds (modeled on a Banach space with a Lipschitz C^1 bump function). A class of Hamilton-Jacobi equations defined on C^1 Finsler manifolds is studied and several results related to the existence and uniqueness of viscosity solutions are obtained.

1. INTRODUCTION

This work is mainly devoted to the study of a certain class of Hamilton-Jacobi equations defined on Banach-Finsler manifolds. Along the way, we also develop some techniques of subdifferential calculus which are needed in this context. This paper is a continuation of [20], where basic properties and a smooth variational principle were studied in the context of Banach-Finsler manifolds. In particular, we apply some of the results obtained in [20], as well as some techniques studied in the cases of Hamilton-Jacobi equations on $\mathbb{R}^n$, on Banach spaces and on Riemannian manifolds [18, 19, 14, 10, 13, 2, 3], in order to obtain our results about existence and uniqueness of viscosity solutions of a class of Hamilton-Jacobi equations on Banach-Finsler manifolds.

The concepts of subdifferentiability and viscosity solutions of Hamilton-Jacobi equations have been extensively studied by many authors. The notion of viscosity solution was introduced by M.G. Crandall and P.L. Lions (see for instance [7, 8]). It was H. Ishii who first introduced the method of Perron to derive the existence of viscosity solutions of Hamilton-Jacobi equations [18]. The literature about this subject is huge. For an introduction we can mention the books by G. Barles [4] and by P.L. Lions [22]. For a detailed account and further information, we refer the reader to the recent survey of H. Ishii [19] and references therein.

The study of the above mentioned concepts in (finite and infinite dimensional) Riemannian manifolds was introduced by D. Azagra, J. Ferrera and F. López-Mesas in [2, 3]. Let us also mention the related work of Y.S. Ledyaev and Q.J. Zhu [21] who studied subdifferentiability and generalized solutions of first-order partial differential equations on (finite dimensional) Riemannian manifolds.

In this work we attempt to continue the study of subdifferentiability and viscosity solutions of Hamilton-Jacobi equations in a non-Riemannian setting. In this way we consider the more general context of (finite and infinite dimensional) Finsler manifolds. Our manifolds will be modeled on a Banach space X which admits a C^1 Lipschitz bump function, which provides, as we will see, a quite natural setting for the class of Hamilton-Jacobi equations under our consideration.

The contents of the paper are arranged as follows. In the second section, we recall the definitions of C^1 Finsler manifold M modeled over a Banach space, Finsler metric over the manifold M (in the sense of Palais) and Fréchet subdifferentiability of a function $f : M \to (-\infty, \infty]$. Basic properties of the subdifferential are established, such as: a local fuzzy rule for the subdifferential of the sum,

via localizing charts and the corresponding fuzzy rule in Banach spaces ([12, 2]); the density of the points of subdifferentiability (in the domain) of a lower semicontinuous function; and also a Mean Value inequality for lower semicontinous functions defined on Finsler manifolds, in the same vein as the ones obtained by R. Deville [11] for Banach spaces and D. Azagra, J. Ferrera and F. López-Mesas [2] for Riemannian manifolds.

In the third section, we study the existence of a unique viscosity solution of the eikonal equation defined on a bounded open subset of a C^1 Finsler manifold modeled on a Banach space X with a C^1 Lipschitz bump function. The eikonal equation has been largely studied by many authors. In the works of L.A. Caffarelli and M.G. Crandall [6] and A. Siconolfi [25] the authors consider the construction of a Finsler metric associated to the eikonal equation defined on bounded open subsets of $\mathbb{R}^n$. Let us also mention the recent work of P. Angulo and L. Guijarro [1] related to the eikonal equation on bounded open subsets of (finite dimensional) Riemannian manifolds.

In the fourth section, we obtain a comparison and stability result for bounded and locally Lipschitz viscosity solutions of (stationary) Hamilton-Jacobi equations of the form $u(x) + H(x, ||du(x)||_x) = 0$ for $x \in M$, where the Hamiltonian $H : M \times \mathbb{R} \to \mathbb{R}$ satisfies a condition weaker than uniform continuity. Moreover, we determine a result on the existence of bounded viscosity solutions under additional conditions such as the coercivity of the Hamiltonian. Also, let us recall here the related results of J. Borwein, Q.J. Zhu, R. Deville, G. Godefroy, V. Zizler and E.M. El Haddad [5, 14, 10, 13, 15, 16, 17] obtained for Banach spaces.

In the fifth section, we study a comparison and monotony result for viscosity solutions of (evolution) Hamilton-Jacobi equations of the form $u_t(t, x) + H(t, x, ||u_x(t, x)||_x) = 0$ for $(t, x) \in [0, \infty) \times M$ and initial condition $u(0, x) = h(x)$ for $x \in M$, where the Hamiltonian $H : M \times \mathbb{R} \to \mathbb{R}$ satisfies a condition weaker than uniform continuity and the initial condition h is bounded and continuous. In order to establish the comparison result, additional conditions on u are required: u is bounded in $[0, T) \times M$ for every $T > 0$ and u is locally Lipschitz. Also, a result about existence of viscosity solutions (bounded in $[0, T) \times M$ for every $T > 0$) is determined within some specific conditions.

The notation we use is standard. The norm in a Banach space X is denoted by $|| \cdot ||$ and the dual norm in the dual Banach space X^* is denoted as $|| \cdot ||^*$. We will say that the norms $|| \cdot ||_1$ and $|| \cdot ||_2$ defined on X are K-equivalent ($K \geq 1$) whether $\frac{1}{K}||v||_1 \leq ||v||_2 \leq K||v||_1$, for every $v \in X$. A C^k bump function $b : X \to \mathbb{R}$ (where $k \in \mathbb{N} \cup \{\infty\}$) is a C^k smooth function on X with bounded, non-empty support, where $\text{supp}(b) = \overline{\{x \in X : b(x) \neq 0\}}$. A function $f : (A, d) \to \mathbb{R}$, where (A, d) is a metric space, is L-Lipschitz (with $L \geq 0$) if $|f(y) - f(z)| \leq Ld(y, z)$, for all $y, z \in A$. If M is a Banach-Finsler manifold, we denote by $T_x M$ the tangent space of M at x and by $T_x M^*$ the dual space of $T_x M$. Recall that the tangent bundle of M is $TM = \{(x, v) : x \in M \text{ and } v \in T_x M\}$ and the cotangent bundle of M is $TM^* = \{(x, \tau) : x \in M \text{ and } \tau \in T_x M^*\}$. We refer to [9] and [20] for additional definitions. For a set A, we call a function $f : A \to (-\infty, \infty]$ proper when the set $dom\, f := \{x \in M : f(x) < +\infty\}$ is nonempty.

2. Subdifferentials on Banach-Finsler manifolds

Let us begin with the definition of Finsler manifold in the sense of Palais and some basic properties.

Definition 2.1. *For $\ell \in \mathbb{N} \cup \{\infty\}$, let M be a (paracompact) C^ℓ Banach manifold modeled on a Banach space $(X, || \cdot ||)$. Consider TM the tangent bundle of M and a continuous map $|| \cdot ||_M : TM \to [0, \infty)$. We say that $(M, || \cdot ||_M)$ is a C^ℓ **Finsler manifold in the sense of Palais** (see [23, 9, 24]) if $|| \cdot ||_M$ satisfies the following conditions:*

(P1) For every $x \in M$, the map $||\cdot||_x := || \cdot ||_M|_{T_x M} : T_x M \to [0, \infty)$ is a norm on the tangent space $T_x M$ such that for every chart $\varphi : U \to X$ with $x \in U$, the norm $v \in X \mapsto ||d\varphi^{-1}(\varphi(x))(v)||_x$ is equivalent to $|| \cdot ||$ on X.

(P2) For every $x_0 \in M$, $\varepsilon > 0$ and every chart $\varphi : U \to X$ with $x_0 \in U$, there is an open neighborhood W of x_0 such that if $x \in W$ and $v \in X$, then

$$(1) \qquad \frac{1}{1+\varepsilon}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0} \leq ||d\varphi^{-1}(\varphi(x))(v)||_x \leq (1+\varepsilon)||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0}.$$

In terms of equivalence of norms, the above inequalities yield to the fact that the norms $||d\varphi^{-1}(\varphi(x))(\cdot)||_x$ and $||d\varphi^{-1}(\varphi(x_0))(\cdot)||_{x_0}$ are $(1+\varepsilon)$-equivalent.

Let us remark that every Riemannian manifold is a C^∞ Finsler manifold in the sense of Palais (see [23]). Throughout this work, we will assume that M is a (paracompact) connected C^1 Finsler manifold in the sense of Palais modeled on a Banach space X. For simplicity we will refer to them as C^1 Finsler manifolds.

Recall that the **length** of a piecewise C^1 smooth path $c : [a,b] \to M$ is defined as $\ell(c) := \int_a^b ||c'(t)||_{c(t)} \, dt$. Besides, if M is connected, then it is connected by piecewise C^1 smooth paths, and the associated **Finsler metric** d on M is defined as

$$d(p,q) = \inf\{\ell(c) : c \text{ is a piecewise } C^1 \text{ smooth path connecting } p \text{ to } q\}.$$

We will say that a Finsler manifold M is **complete** if it is complete for the metric d. The following result yields the local bi-Lipschitz behaviour of the charts of a Finsler manifold.

Lemma 2.2. [20, Lemma 2.4.] **(Bi-Lipschitz charts).** *Let us consider a C^1 Finsler manifold M modeled on a Banach space X with a C^1 bump function and $x_0 \in M$. Then, for every chart (U, φ) with $x_0 \in U$ satisfying inequality (1), there exists an open neighborhood $V \subset U$ of x_0 satisfying*

$$(2) \qquad (1+\varepsilon)^{-1}d(p,q) \leq |||\varphi(p) - \varphi(q)||| \leq (1+\varepsilon)d(p,q), \quad \text{ for every } p, q \in V,$$

where $||| \cdot |||$ is the (equivalent) norm $||d\varphi^{-1}(\varphi(x_0))(\cdot)||_{x_0}$ defined on X.

The concepts of subdifferential and superdifferential have been extensively studied for functions defined on $\mathbb{R}^n$, infinite dimensional Banach spaces and Riemannian manifolds. The straightforward definition in the case of Finsler manifolds is the following.

Definition 2.3. *Let M be a C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function and let $f : M \to (-\infty, +\infty]$ be a proper function. We define the set of **subdifferentials** of f at a point $x \in dom(f) = \{y \in M : f(y) < \infty\}$ as*

$$D^- f(x) = \{\Delta \equiv dg(x) : g : M \to \mathbb{R} \text{ is } C^1 \text{smooth and } f - g \text{ attains a local minimum at } x\} \subset T_xM^*,$$

*and the set of **superdifferentials** of f at x as*

$$D^+ f(x) = \{\Delta \equiv dg(x) : g : M \to \mathbb{R} \text{ is } C^1 \text{smooth and } f - g \text{ attains a local maximum at } x\} \subset T_xM^*.$$

If $D^- f(x) \neq \emptyset$ $(D^+ f(x) \neq \emptyset)$, we say that f is subdifferentiable (superdifferentiable) at x.

Notice that if a function f attains a local minimum at x, then $0 \in D^- f(x)$. Also notice that $D^- f(x) = -D^+(-f)(x)$. In addition, we can endow every subdifferential or superdifferential $\Delta \in T_xM^*$ of f at x with the dual norm

$$\|\Delta\|_x^* = \sup_{\xi \in S_{T_xM}} |\Delta(\xi)|, \qquad \text{where } S_{T_xM} = \{\xi \in T_xM : \|\xi\|_x = 1\}.$$

For simplicity we will write $\|\Delta\|_x$ for the dual norm $\|\Delta\|_x^*$. Basic properties related to subdifferentiability on Finsler manifolds can be deduced in the same way as D. Azagra, J. Ferrera and F. López-Mesas did in [2, Section 4] for Riemannian manifolds. Since these properties can be deduced without much difficulty by using the same techniques, we will omit some of the proofs.

Theorem 2.4. (Characterizations of subdifferentiability). *Let M be a C^1 Finsler manifold modeled on a Banach space X with a C^1 Lipschitz bump function. Consider a proper function $f : M \to (-\infty, +\infty]$, a point $x \in M$ and a functional $\Delta \in T_xM^*$. The following conditions are equivalent:*

(1) $\Delta \in D^- f(x)$.

(2) *There exists a function $g\colon M \to \mathbb{R}$ such that g is Fréchet differentiable at x, $f - g$ attains a local minimum at x and $\Delta = dg(x)$.*

(3) *For every chart $\varphi\colon U \subset M \to X$ with $x \in U$, if we set $\tau := \Delta \circ d\varphi^{-1}(\varphi(x))$, then*

$$\liminf_{h \to 0} \frac{(f \circ \varphi^{-1})(\varphi(x) + h) - f(x) - \tau(h)}{\|h\|} \geq 0,$$

i.e. $\tau \in D^-(f \circ \varphi^{-1})(\varphi(x))$.

(4) *There exists a chart $\varphi\colon U \subset M \to X$, with $x \in U$, such that if $\tau = \Delta \circ d\varphi^{-1}(\varphi(x))$, then*

$$\liminf_{h \to 0} \frac{(f \circ \varphi^{-1})(\varphi(x) + h) - f(x) - \tau(h)}{\|h\|} \geq 0,$$

i.e. $\tau \in D^-(f \circ \varphi^{-1})(\varphi(x))$.

Moreover, if f is locally bounded below and M admits C^1 smooth partitions of unity, we have the equivalent condition:

(5) *There exists a C^1 smooth function $g\colon M \to \mathbb{R}$, $f - g$ attains a global minimum at x and $\Delta = dg(x)$.*

Note that we can obtain an analogous result for the superdifferentiability of f. The proofs of (2) $\Longrightarrow$ (3) and (5) $\Longrightarrow$ (1) follow the lines of the Riemannian case [2]. The proof of (4) $\Longrightarrow$ (1) follows (via charts) from the case of Banach spaces with a C^1 smooth bump [15, Chapter 8].

Under the assumptions of Theorem 2.4, we get the following corollaries related to the subdifferentiability and differentiability of f at a point $x \in dom(f)$.

Corollary 2.5. *Let M be a C^1 Finsler manifold modeled on a Banach space X with a C^1 Lipschitz bump function. Consider a proper function $f\colon M \to (-\infty, +\infty]$, a chart $\varphi\colon U \subset M \to X$ and a point $x \in dom(f) \cap U$. Then,*

$$D^-f(x) = \left\{ \tau \circ d\varphi(x) \,:\, \tau \in X^*, \liminf_{h \to 0} \frac{(f \circ \varphi^{-1})(\varphi(x) + h) - f(x) - \tau(h)}{\|h\|} \geq 0 \right\}$$
$$= \{ \tau \circ d\varphi(x) \,:\, \tau \in D^-(f \circ \varphi^{-1})(\varphi(x)) \}.$$

Moreover, f is (Fréchet) differentiable at x if and only if there exist an open subset V in M with $x \in V$ and C^1 smooth functions $g, h\colon V \to \mathbb{R}$ such that

(1) *$g(z) \leq f(z) \leq h(z)$ for all $z \in V$, and*

(2) *$g(x) = f(x) = h(x)$ and $dg(x) = dh(x)$.*

The differentiability of f is therefore characterized as follows.

Corollary 2.6. (Criterion for differentiability). *Let M be a C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function. Consider a proper function $f\colon M \to (-\infty, +\infty]$ and a point $x \in dom(f)$. Then, f is (Fréchet) differentiable at x if and only if f is subdifferentiable and superdifferentiable at x. Moreover, if f is (Fréchet) differentiable at x, then $df(x)$ is the only subdifferential and superdifferential of f at x.*

As in the case of Banach spaces and Riemannian manifolds, the following relationship between the subdifferentiability and continuity holds.

Corollary 2.7. (Continuity properties). *Let M be a C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function. Consider a proper function $f\colon M \to (-\infty, +\infty]$ and a point $x \in dom(f)$. If f is subdifferentiable (superdifferentiable) at x, then f is lower semicontinuous (upper semicontinuous) at x.*

The next results are related to the subdifferentiability of the composition, sum and product of functions defined on Finsler manifolds.

Proposition 2.8. (Chain rule). *Let M, N be C^1 Finsler manifolds modeled on a Banach space with a C^1 Lipschitz bump function. Let $g\colon M \to N$ and $f\colon N \to (-\infty, +\infty]$ be two functions such*

that f is subdifferentiable at $g(x)$ and g is Fréchet differentiable at x. Then $f \circ g$ is subdifferentiable at x and

$$\{\Delta \circ dg(x) \, : \, \Delta \in D^-f(g(x))\} \subset D^-(f \circ g)(x).$$

Corollary 2.9. *Let M, N be C^1 Finsler manifolds modeled on a Banach space with a C^1 Lipschitz bump function and assume that $\varphi \colon M \to N$ is a C^1 diffeomorphism. Then $f \colon M \to (-\infty, +\infty]$ is subdifferentiable at x if and only if $f \circ \varphi^{-1}$ is subdifferentiable at $\varphi(x)$, and*

$$D^-f(x) = \{\Delta \circ d\varphi(x) \colon \Delta \in D^-(f \circ \varphi^{-1})(\varphi(x))\}.$$

Proposition 2.10. *Let M be a C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function and consider the functions $f, g \colon M \to (-\infty, +\infty]$. Then the following statements hold:*

(1) **(Sum rule).** *$D^-f(x) + D^-g(x) \subset D^-(f + g)(x)$.*
(2) **(Product rule).** *If $f, g \colon M \to [0, \infty)$, then $f(x)D^-g(x) + g(x)D^-f(x) \subset D^-(fg)(x)$.*

Note that there are analogous statements of Propositions 2.8 and 2.10 and Corollary 2.9 for superdifferentials.

Proposition 2.11. (Geometrical and topological properties of the subdifferencial). *Let M be a C^1 Finsler manifold modeled on a Banach space X with a C^1 Lipschitz bump function. For every function $f : M \to (-\infty, \infty]$ and $x \in dom(f)$, the sets $D^-f(x)$ and $D^+f(x)$ are closed and convex subsets of T_xM^*. Moreover, if f is locally Lipschitz, then these sets are bounded.*

The following results are fundamental for the study of viscosity solutions of the Hamilton-Jacobi equations on Finsler manifolds given in the next sections.

Proposition 2.12. (Fuzzy rule for the subdifferential of the sum). *Let M be a C^1 Finsler manifold modeled on a Banach space X with a C^1 Lipschitz bump function. Let $f, g \colon M \to \mathbb{R}$ be two functions such that f is lower semicontinuous and g is locally uniformly continuous. Then, for every $x \in M$, every chart (U, φ) with $x \in U$, every $\Delta \in D^-(f + g)(x)$ and $\varepsilon > 0$, there exist $x_1, x_2 \in U$, $\Delta_1 \in D^-f(x_1)$, $\Delta_2 \in D^-g(x_2)$ such that*

(1) $d(x_1, x) < \varepsilon$ and $d(x_2, x) < \varepsilon$,
(2) $|f(x_1) - f(x)| < \varepsilon$ and $|g(x_2) - g(x)| < \varepsilon$,
(3) $\|\Delta_1 \circ d\varphi(x_1)^{-1} + \Delta_2 \circ d\varphi(x_2)^{-1} - \Delta \circ d\varphi(x)^{-1}\| < \varepsilon$.
(4) $d(x_1, x_2) \cdot \max \left\{ \|\Delta_1 \circ d\varphi(x_1)^{-1}\|, \|\Delta_2 \circ d\varphi(x_2)^{-1}\| \right\} < \varepsilon$.

The proof of the above fuzzy rule follows from the analogous results for Banach spaces [5, Theorem 2.12] and [13, Theorem 4.2 in Section 4.2] applied to the functions $f \circ \varphi^{-1}$ and $g \circ \varphi^{-1}$ defined in a neighborhood of $\varphi(x)$, Lemma 2.2 and Corollary 2.5. Recall that φ is locally bi-Lipschitz (Lemma 2.2) and then $g \circ \varphi^{-1}$ is locally uniformly continuous. It is worth noticing that the hypothesis given in the fuzzy rule for the subdifferential of the sum can be weakened by a more technical assumption (see [5, Section 2] and [13, Section 4.2]). Let us remark that up to our knowledge it is not known whether the fuzzy rule holds for every pair of lower semicontinuous functions with finite values $u, v : X \to \mathbb{R}$, where X is a Banach spaces with a C^1 Lipschitz bump.

Recall that the smooth variational principle of Deville-Godefroy-Zizler for a Banach space X with a C^1 Lipschitz bump function [14, 15] provides the subdifferentiability of a lower semicontinuous function $f : X \to (-\infty, \infty]$ on a dense subset of $dom(f) = \{y \in X : f(y) < \infty\}$. There is a similar statement for Finsler manifolds.

Proposition 2.13. (Density of the set of points of subdifferentiability). *Let M be a C^1 Finsler manifold modeled on a Banach space X with a C^1 Lipschitz bump function. If $f \colon M \to (-\infty, +\infty]$ is proper and lower semicontinuous, then the subset of points of $dom(f)$ where f is subdifferentiable is dense in $dom(f)$.*

Let us give an outline of the proof: Given a point $x \in dom(f)$ and a chart (U, φ) with $x \in U$, we consider the lower semicontinuous function $L : X \to (-\infty, \infty]$ defined as $L = f \circ \varphi^{-1}$ in a closed neighborhood C of $\varphi(x)$ (C small enough such that $C \subset \varphi(U)$) and $L = \infty$ in $X \setminus C$. The analogous

result on Banach spaces establishes that there is a sequence of subdifferentiable points of L in X with limit $\varphi(x)$. Thus, by Corollary 2.5, there is a sequence of subdifferentiable points of f in U with limit x.

Let us recall the well-known concepts of lower and upper semicontinuous envelopes of a function.

Definition 2.14. *Let M be a C^1 Finsler manifold modeled on a Banach space X. For a function $u : \Omega \to \mathbb{R}$ defined on an open subset $\Omega \subset M$, the* **upper semicontinuous envelope** *u^* of u is defined by*

$$u^*(x) = \inf\{v(x) : v : \Omega \to \mathbb{R} \text{ is continuous and } u \leq v \text{ on } \Omega\} \qquad \text{for any } x \in \Omega.$$

The **lower semicontinuous envelope** u_* is defined in a similar way. Recall that

$$u^*(x) = \lim_{r \to 0^+} \Big(\sup_{y \in B(x,r)} u(y) \Big) \quad \text{and} \quad u_*(x) = \lim_{r \to 0^+} \Big(\inf_{y \in B(x,r)} u(y) \Big) \quad \text{for } x \in \Omega,$$

where $B(x, r)$ denotes the open ball of center x and radius $r > 0$ in the Finsler manifold M. The following result of stability of superdifferentials is fundamental in the theory of viscosity solutions.

Proposition 2.15. (Stability of the superdifferentials). *Let M be a C^1 Finsler manifold modeled on a Banach space X with a C^1 Lipschitz bump function. Let Ω be an open subset of M. Let $\mathcal{F}$ be a locally uniformly bounded family of upper semicontinuous functions from Ω into $\mathbb{R}$ and $u = \sup\{v : v \in \mathcal{F}\}$ on Ω. Then, for every $x \in \Omega$ and every $\Delta \in D^+u^*(x)$, there exist sequences $\{v_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}$ and $\{(x_n, \Delta_n)\}_{n \in \mathbb{N}}$ in TM^* with $x_n \in \Omega$ and $\Delta_n \in D^+v_n(x_n)$ for every $n \in \mathbb{N}$, such that*

(i) $\lim_{n \to \infty} v_n(x_n) = u^*(x)$, and
(ii) $\lim_{n \to \infty}(x_n, \Delta_n) = (x, \Delta)$ in the cotangent bundle TM^*, i.e. $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} \|\Delta_n \circ d\varphi(x_n)^{-1} - \Delta \circ d\varphi(x)^{-1}\| = 0$ for every chart (U, φ) on M with $x \in U$. (Notice that, in general, we assume that $\Delta_n \circ d\varphi(x_n)^{-1}$ are defined only for $n \geq n_0$, where n_0 depends on the chart (U, φ)).

Let us point out that the proof of Proposition 2.15 follows the lines of the Riemannian case: for a fixed chart (A, ψ) of M with $x \in A \subset \Omega$, we consider the functions $u \circ \psi^{-1} = \sup\{v \circ \psi^{-1} : v \in \mathcal{F}\}$ and $(u \circ \psi^{-1})^* = u^* \circ \psi^{-1}$, which are defined in the open neighborhood $\psi(A)$ of $\psi(x) \in X$. Next, we apply the analogous result for Banach spaces to the function $u^* \circ \psi^{-1}$ [15, Chapter VIII. Proposition 1.6] and Corollary 2.5 to obtain the assertions (i) and (ii) for the chart (A, ψ). Next, it can be easily checked that, in fact, condition (ii) holds for every chart (U, φ) with $x \in U$.

Now, let us give a local version of Deville's mean value inequality, which will be essential in order to prove the uniqueness of the eikonal equation on Finsler manifolds. Recall that, for a Finsler manifold M, the open (closed) ball of center x and radius $r > 0$ is denoted by $B(x, r)$ ($\overline{B}(x, r)$).

Theorem 2.16. (Local Deville's mean value inequality for Finsler manifolds). *Let M be a C^1 Finsler manifold modeled on a Banach space X with a C^1 Lipschitz bump function and consider $p \in M$ and $\delta > 0$. Let $f : B(p, 4\delta) \subset M \to \mathbb{R}$ be a lower semicontinuous function satisfying $\|\xi\|_x \leq K$ for every $\xi \in D^-f(x)$ and $x \in B(p, 4\delta)$. Then f is K-Lipschitz on $B(p, \delta)$.*

Proof. Let us fix $\varepsilon > 0$ and consider for every pair of points $x, y \in B(p, \delta)$ a continuous piecewise C^1 smooth path $\gamma : [0, T] \to M$ such that $\gamma(0) = x$, $\gamma(T) = y$ and $\ell(\gamma) < d(x, y) + \min\{\delta, \varepsilon\}$. Notice that in this case, $\ell(\gamma) < d(x, y) + \delta \leq d(x, p) + d(p, y) + \delta < 3\delta$. Thus, for every $z \in \gamma([0, T])$, $d(p, z) \leq d(p, x) + d(x, z) < \delta + \ell(\gamma) < \delta + 3\delta = 4\delta$ and this yields $\gamma([0, T]) \subset B(p, 4\delta)$.

Now, for all $z \in \gamma([0, T])$ we consider a chart $\varphi_z : U_z \to V_z$ such that

(a) $z \in U_z \subset B(p, 4\delta)$, $\varphi_z(U_z) = V_z$ and V_z is an open convex subset of X,
(b) φ_z satisfies the Palais condition (1) for $1 + \varepsilon$, and
(c) φ_z is $(1 + \varepsilon)$-bi-Lipschitz in U_z for the norm in X denoted as

$$\|v\|_z := \|d\varphi_z^{-1}(\varphi_z(z))(v)\|_z \text{ for all } v \in X,$$

(see Definition 2.1 and Lemma 2.2 for more details).

For every $t \in [0, T]$, we select real numbers $0 \leq r_t < s_t \leq T$ satisfying: (1) $r_0 = 0$ and $\gamma(0) = x \in \gamma([0, s_0]) \subset U_x$, (2) $r_t < t < s_t$ and $\gamma(t) \in \gamma([r_t, s_t]) \subset U_{\gamma(t)}$ whenever $t \in (0, T)$, and (3) $s_T = T$ and $\gamma(T) = y \in \gamma([r_T, T]) \subset U_y$.

By compactness of $[0, T]$, there exists a finite set of points $\{t_1, \ldots, t_n\} \subset [0, T]$ with $t_1 = 0$ and $t_n = T$ satisfying

$$(3) \qquad \gamma([0, T]) = \bigcup_{k=1}^{n} \gamma([r_{t_k}, s_{t_k}]).$$

Let us denote $z_k = \gamma(t_k)$, $\varphi_k := \varphi_{z_k}$, $U_k := U_{z_k}$, $V_k := V_{z_k}$, $r_k := r_{t_k}$ and $s_k := s_{t_k}$ for $k = 1, \ldots, n$. By reordering and splitting the intervals if needed, we may assume that $r_1 = 0$, $r_{k+1} = s_k$ for $k = 1, \ldots, n-1$ and $s_n = T$.

Consider the function $\Phi_k : V_k \subset X \to \mathbb{R}$ defined by $\Phi_k = f \circ \varphi_k^{-1}$. By applying Corollary 2.9 we know that, for all $a \in V_k$,

$$D^- \Phi_k(a) = \{\Delta \circ d\varphi_k^{-1}(a) \, : \, \Delta \in D^- f\left(\varphi_k^{-1}(a)\right)\}.$$

Now, for any $k = 1, \ldots, n$ we consider in X the norm

$$\||v\||_k := \||v\||_{z_k} = \|d\varphi_k^{-1}(\varphi_k(z_k))(v)\|_{z_k} \quad \text{for all } v \in X.$$

If $z \in U_k$, for a continuous linear operator $T : (T_z M, \|\cdot\|_z) \to (X, \||\cdot\||_k)$ we set the norm

$$\||T\||_{z,k} := \sup\{\||T(v)\||_k \, : \, \|v\|_z \leq 1\}.$$

Moreover, if T is an isomorphism we denote

$$\||T^{-1}\||_{k,z} = \sup\{\|T^{-1}(v)\|_z \, : \, \||v\||_k \leq 1\}.$$

From the Palais condition (1), we obtain for all $z \in U_k$ and $v \in T_z M$,

$$\||d\varphi_k(z)(v)\||_k = \|d\varphi_k^{-1}(\varphi_k(z_k))(d\varphi_k(z)(v))\|_{z_k} \leq (1+\varepsilon)\|d\varphi_k^{-1}(\varphi_k(z))(d\varphi_k(z)(v))\|_z = (1+\varepsilon)\|v\|_z$$

and

$$\||d\varphi_k(z)(v)\||_k = \|d\varphi_k^{-1}(\varphi_k(z_k))(d\varphi_k(z)(v))\|_{z_k} \geq (1+\varepsilon)^{-1}\|d\varphi_k^{-1}(\varphi_k(z))(d\varphi_k(z)(v))\|_z = (1+\varepsilon)^{-1}\|v\|_z.$$

Therefore, for all $z \in U_k$,

$$(1+\varepsilon)^{-1} \leq \||d\varphi_k(z)\||_{z,k} \leq (1+\varepsilon) \quad \text{and thus} \quad (1+\varepsilon)^{-1} \leq \||d\varphi_k^{-1}(\varphi_k(z))\||_{k,z} \leq (1+\varepsilon).$$

Now, for all $a \in V_k$, with $z = \varphi_k^{-1}(a) \in U_k$ and $\Delta \in D^- f(\varphi_k^{-1}(a))$, we have

$$\||\Delta \circ d\varphi_k^{-1}(a)\||_k^* \leq \|\Delta\|_z^* \||d\varphi_k^{-1}(a)\||_{k,z} \leq K(1+\varepsilon).$$

Therefore, $\||\Lambda\||_k^* \leq K(1+\varepsilon)$ for all $\Lambda \in D^- \Phi_k(a)$ and $a \in V_k$.

Let us define $x_k = \gamma(r_k)$ for all $k = 1, \ldots, n$ and $x_{n+1} = \gamma(s_n)$. The function $\Phi_k : V_k \to \mathbb{R}$ is lower semicontinuous at every point of the open convex set $V_k \subset X$. Let us apply Deville's mean value inequality for Banach spaces ([11]; see also [10, 13]) to the function Φ_k defined in the open convex subset V_k of the Banach space $(X, \||\cdot\||_k)$ to obtain that Φ_k is $K(1+\varepsilon)$-Lipschitz with respect to the norm $\||\cdot\||_k$ and then,

$$|f(x_{k+1}) - f(x_k)| = |\Phi_k(\varphi_k(x_{k+1})) - \Phi_k(\varphi_k(x_k))| \leq K(1+\varepsilon)\||\varphi_k(x_{k+1}) - \varphi_k(x_k)\||_k$$

for all $k = 1, \ldots, n$. In addition, since φ_k is $(1+\varepsilon)$-bi-Lipschitz in U_k with the norm $\||\cdot\||_k$ (Lemma 2.2), we obtain

$$|f(x_{k+1}) - f(x_k)| \leq K(1+\varepsilon)^2 d(x_{k+1}, x_k)$$

for all $k = 1, \ldots, n$. Consequently,

$$|f(x) - f(y)| \leq \sum_{k=1}^{n} |f(x_{k+1}) - f(x_k)| \leq K(1+\varepsilon)^2 \sum_{k=1}^{n} d(x_{k+1}, x_k) \leq K(1+\varepsilon)^2 \ell(\gamma) \leq K(1+\varepsilon)^2(d(x,y)+\varepsilon).$$

By letting $\varepsilon \to 0$, we get the inequality

$$|f(x) - f(y)| \leq K d(x, y).$$

Finally, since $x, y \in B(p, \delta)$ are arbitrary, f is K-Lipschitz in $B(p, \delta)$. $\qquad \square$

By applying the same techniques of the above theorem, we can prove a global mean value inequality for Finsler manifolds.

Theorem 2.17. (Deville's mean value inequality for Finsler manifolds). *Let M be a C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function. Let $f \colon M \to \mathbb{R}$ be a lower semicontinuous function such that $\|\xi\|_x \leq K$ for every $\xi \in D^- f(x)$ and $x \in M$. Then, f is K-Lipschitz.*

3. The eikonal equation on Banach-Finsler manifolds

Let M be a **complete** C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function and assume that $\Omega \subset M$ is a non-empty bounded open subset with $\partial\Omega \neq \emptyset$. Let us consider the **eikonal equation**,

(EEq)
$$\begin{cases} \|du(x)\|_x = 1, & \text{for all } x \in \Omega, \\ u(x) = 0, & \text{for all } x \in \partial\Omega \end{cases}$$

which is a well-known Hamilton-Jacobi equation. Our purpose throughout this section is to prove that this equation has a unique viscosity solution. Let us first see that (EEq) does not have a classical solution.

Proposition 3.1. (EEq) *does not have a classical solution.*

Proof. Assume that there exists a classical solution $u : \overline{\Omega} \to \mathbb{R}$ of (EEq), i.e. u is continuous in $\overline{\Omega}$, (Fréchet) differentiable in Ω and $\|du(x)\|_x = 1$ for all $x \in \Omega$. We extend u to M as $u(z) = 0$ for $z \in \Omega^c$. By applying Theorem 2.16, let us check that u is 1-Lipschitz:

(i) For $x, y \in \Omega$, consider a piecewise C^1 smooth path $\gamma : [a, b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = y$.

   (a) If $\gamma([a, b]) \subset \Omega$, we apply the compactness of $\gamma([a, b])$ to find a finite number of balls $\{B(x_i, 4\delta_i) : i = 1 \ldots, m\}$ such that

$$\bigcup_{i=1}^m B(x_i, 4\delta_i) \subset \Omega \ \text{ and } \ \gamma([a, b]) \subset \bigcup_{i=1}^m B(x_i, \delta_i).$$

   We may assume that there are auxiliary points $a = t_1 < t_2 < \cdots t_{n+1} = b$ such that for every $k \in \{1, \ldots, n\}$ there is $i \in \{1, \ldots, m\}$ satisfying $\gamma([t_k, t_{k+1}]) \subset B(x_i, \delta_i)$. By applying Theorem 2.16, we deduce that

$$|u(x) - u(y)| \leq \sum_{k=1}^n |u(\gamma(t_{k+1})) - u(\gamma(t_k))| \leq \sum_{k=1}^n d(\gamma(t_{k+1}), \gamma(t_k)) \leq \ell(\gamma).$$

   (b) If there is $b' \in (a, b]$ such that $\gamma([a, b')) \subset \Omega$ and $\gamma(b') \in \partial\Omega$, by taking the restrictions $\gamma|_{[a,t]}$ with $t < b'$ and the limit $t \to b'$, we obtain

$$|u(\gamma(b')) - u(\gamma(a))| \leq \ell(\gamma|_{[a,b']}).$$

   Thus, if $\gamma([a, b]) \not\subset \Omega$, consider the points $a < a' \leq b' < b$ such that $\gamma([a, a')) \subset \Omega$, $\gamma(a') \in \partial\Omega$, $\gamma((b', b]) \subset \Omega$ and $\gamma(b') \in \partial\Omega$. Then, by the preceding observation

$$|u(\gamma(a))| = |u(\gamma(a)) - u(\gamma(a'))| \leq d(\gamma(a), \gamma(a')) \leq \ell(\gamma|_{[a,a']})$$

   and

$$|u(\gamma(b))| = |u(\gamma(b')) - u(\gamma(b))| \leq d(\gamma(b'), \gamma(b)) \leq \ell(\gamma|_{[b',b]}).$$

   Therefore, $|u(\gamma(a)) - u(\gamma(b))| \leq \ell(\gamma|_{[a,a']}) + \ell(\gamma|_{[b',b]}) \leq \ell(\gamma)$.

Thus, by taking the infimum of the lengths of all piecewise C^1 smooth paths γ connecting x and y, we obtain $|u(x) - u(y)| \leq d(x, y)$.

(ii) For $x \in \Omega$, $y \in \Omega^c$ and any piecewise C^1 smooth path $\gamma : [a, b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = y$, consider the point a' in the segment $[a, b]$ such that $\gamma([a, a')) \subset \Omega$ and $\gamma(a') \in \partial\Omega$. By the preceding cases,

$$|u(x) - u(y)| = |u(\gamma(a))| = |u(\gamma(a)) - u(\gamma(a'))| \leq d(\gamma(a), \gamma(a')) \leq \ell(\gamma|_{[a,a']}) \leq \ell(\gamma).$$

Again, by taking the infimum of the lengths of all piecewise C^1 smooth paths γ connecting x and y, we obtain $|u(x) - u(y)| \leq d(x, y)$.

(iii) For $x, y \in \Omega^c$ the inequality is clear.

Since Ω is a bounded subset and u is Lipschitz, u is bounded on $\overline{\Omega}$. Notice that $-u$ is also a classical solution of (EEq), and thus we may assume that $s = \sup\{u(x) : x \in \Omega\} > 0$. Let us fix $0 < \varepsilon < \min\{1, s\}$ and apply the Ekeland variational principle to $u : M \to \mathbb{R}$ (recall that we are assuming the completeness of M) to find a point $\overline{x} \in M$ such that $s \leq u(\overline{x}) + \varepsilon$ and $u(x) \leq u(\overline{x}) + \varepsilon d(x, \overline{x})$ for all $x \in M$. Necessarily, $\overline{x} \in \Omega$ (otherwise, $s \leq \varepsilon$, which is a contradiction).

Now, for each $v \in T_{\overline{x}} M$ with $\|v\|_{\overline{x}} = 1$, let us consider a piecewise C^1 smooth path $\gamma_v : [0, T] \to M$, parametrized by the arc length, such that $\gamma_v(0) = \overline{x}$ and $\gamma_v'(0) = v$. Since $d(\gamma_v(0), \gamma_v(t)) \leq \ell\left(\gamma_v|_{[0,t]}\right) = t$ for all $t \in [0, T]$, we have that $u(\gamma_v(t)) - u(\gamma_v(0)) \leq \varepsilon d(\gamma_v(t), \gamma_v(0)) \leq \varepsilon t$ for all $t \in [0, T]$. Therefore,

$$du(\overline{x})(v) = \lim_{t \to 0^+} \frac{u(\gamma_v(t)) - u(\gamma_v(0))}{t} \leq \varepsilon$$

and consequently, $\|du(\overline{x})\|_{\overline{x}} \leq \varepsilon < 1$. This contradicts that u is a classical solution of (EEq). $\qquad\square$

Let us consider the more general Hamilton-Jacobi equation

(EEq2)
$$\begin{cases} \|du(x)\|_x = 1, & \text{for all } x \in \Omega, \\ u(x) = h(x), & \text{for all } x \in \partial\Omega \end{cases}$$

where $\Omega \subset M$ is a non-empty bounded open subset with $\partial\Omega \neq \emptyset$ and $h : \partial\Omega \to \mathbb{R}$ is 1-Lipschitz. The definition of viscosity solution of the Hamilton-Jacobi equation (EEq2) on a Finsler manifold is the following.

Definition 3.2. *Let us consider a function $u \colon \overline{\Omega} \to \mathbb{R}$.*

- *(1) u is a **viscosity subsolution of** (EEq2) whenever u is upper semicontinuous, $\|\Lambda\|_x \leq 1$ for all $\Lambda \in D^+ u(x)$ with $x \in \Omega$ and $u \leq h$ on $\partial\Omega$.*
- *(2) u is a **viscosity supersolution of** (EEq2) whenever u is lower semicontinuous, $\|\Delta\|_x \geq 1$, for all $\Delta \in D^- u(x)$ with $x \in \Omega$ and $u \geq h$ on $\partial\Omega$.*
- *(3) u is a **viscosity solution of** (EEq2) if u is simultaneously a viscosity subsolution and a viscosity supersolution of (EEq2), i. e. u is a continuous function and verifies*
    - *(i) $\|\Delta\|_x \geq 1$ for all $\Delta \in D^- u(x)$ with $x \in \Omega$,*
    - *(ii) $\|\Lambda\|_x \leq 1$ for all $\Lambda \in D^+ u(x)$ with $x \in \Omega$, and*
    - *(iii) $u(x) = h(x)$ for all $x \in \partial\Omega$.*

The next theorem shows that the equation (EEq2) has a unique viscosity solution.

Theorem 3.3. *The function $u \colon \overline{\Omega} \to \mathbb{R}$, defined by $u(x) = \inf\{h(y) + d(y, x) : y \in \partial\Omega\}$ is the unique viscosity solution of (EEq2).*

Proof. Since h is 1-Lipschitz, $h(x) - h(y) \leq d(x, y)$ for every $x, y \in \partial\Omega$, and then $h(x) \leq h(y) + d(x, y)$. By taking the infimum over all $y \in \partial\Omega$ we have $h(x) \leq \inf\{h(y) + d(x, y) : y \in \partial\Omega\} \leq h(x) + d(x, x) = h(x)$. Thus $u(x) = h(x)$ for $x \in \partial\Omega$.

Now, let us check the conditions (i) and (ii) given in the definition of viscosity solution. We can consider u defined in M with the same expression $u(x) = \inf\{h(y) + d(y, x) : y \in \partial\Omega\}$ for $x \in \Omega^c$. Let us first check (i). Consider $\Delta \in D^- u(x)$ with $x \in \Omega$ and fix $\varepsilon > 0$. Then, for every $\delta > 0$, there exists $x_\delta \in \partial\Omega$ such that

$$h(x_\delta) + d(x_\delta, x) \leq u(x) + \frac{\delta\varepsilon}{2}.$$

Let us point out that, in the Finsler distance, it is possible to approximate $d(z, w)$ for $z, w \in M$ by the length of a C^1 smooth path connecting z and w and parametrized by the arc length. Let us give an outline of this fact: For a piecewise C^1 smooth path $\rho : [a, b] \to M$ connecting z and w whose length approximates $d(z, w)$ and for any $r > 0$, we can find a finite collection of points $a = t_1 < \cdots < t_{n+1} = b$ and a finite family of $(1+r)$-bi-Lipschitz charts $\{(A_i, \psi_i)\}_{i=1}^n$ given by Lemma 2.2 such that $\rho([t_i, t_{i+1}]) \subset A_i$ and $\psi_i(A_i)$ is open and convex in X. Now, we proceed in X to construct a C^1 smooth path $\sigma_i : [t_i, t_{i+1}] \to X$ connecting $\psi_i(\rho(t_i))$ and $\psi_i(\rho(t_{i+1}))$ such that the length of

σ_i for the norm $|||u|||_{\rho(t_i)} := ||d\psi_i^{-1}(\psi_i(\rho(t_i)))(u)||_{\rho(t_i)}$ approximates $|||\psi_i(\rho(t_i)) - \psi_i(\rho(t_{i+1}))|||_{\rho(t_i)}$, $\sigma_i([t_i, t_{i+1}]) \subset \psi_i(A_i)$, $\sigma_i'(t) \neq 0$ for every $t \in [t_i, t_{i+1}]$ and $i = 1, \ldots, n$ and $(\psi_i^{-1} \circ \sigma_i)'(t_{i+1}) = (\psi_{i+1}^{-1} \circ \sigma_{i+1})'(t_{i+1})$ for every $i = 1, \ldots, n-1$. In this way, the length of $\psi_i^{-1} \circ \sigma_i : [t_i, t_{i+1}] \to M$ approximates $d(\rho(t_i), \rho(t_{i+1}))$. Now the path given by the union $\sigma := \cup_{i=1}^n (\psi_i^{-1} \circ \sigma_i) : [a, b] \to M$ is a C^1 smooth path connecting z and w, $\sigma'(t) \neq 0$ for every $t \in [a, b]$ and $\ell(\sigma)$ approximates the distance $d(z, w)$ for $r > 0$ small enough. Now, we can reparametrize σ by the arc length to obtain the required C^1 smooth path.

Thus, we may assume that there are C^1 smooth paths $\gamma_\delta : [0, T_\delta] \to M$ parametrized by the arc length with $||\gamma_\delta'(t)||_{\gamma_\delta(t)} = 1$ for all $t \in [0, T_\delta]$ connecting x and x_δ (i.e., $\gamma_\delta(0) = x$ and $\gamma_\delta(T_\delta) = x_\delta$) and verifying

$$\ell(\gamma_\delta) = T_\delta \leq d(x, x_\delta) + \frac{\delta\varepsilon}{2}.$$

Notice that $\delta < T_\delta$ whenever $\delta < d(x, \partial\Omega)$. So, let us define $z_\delta := \gamma_\delta(\delta) \in M$ for $\delta < d(x, \partial\Omega)$. Then $d(x, z_\delta) \leq \ell(\gamma_\delta|_{[0,\delta]}) = \delta$ and thus $\lim_{\delta \to 0} d(x, z_\delta) = 0$. Since $\Delta \in D^- u(x)$, there exists a C^1 smooth function $g : M \to \mathbb{R}$ such that $u - g$ attains a local minimum at x and $\Delta = dg(x)$. Therefore $u(x) - g(x) \leq u(y) - g(y)$, for all y in a neighbourhood of x. Thus, $u(x) - g(x) \leq u(z_\delta) - g(z_\delta)$ for $\delta > 0$ small enough. This yields

$$g(z_\delta) - g(x) \leq u(z_\delta) - u(x) = \inf\{h(y) + d(y, z_\delta) : y \in \partial\Omega\} - \inf\{h(y) + d(y, x) : y \in \partial\Omega\}$$

$$\leq \inf\{h(y) + d(y, z_\delta) : y \in \partial\Omega\} - h(x_\delta) - d(x_\delta, x) + \frac{\delta\varepsilon}{2}$$

$$\leq h(x_\delta) + d(x_\delta, z_\delta) - h(x_\delta) - d(x_\delta, x) + \frac{\delta\varepsilon}{2}$$

$$\leq \ell\left(\gamma_\delta|_{[\delta, T_\delta]}\right) - \ell(\gamma_\delta) + \delta\varepsilon = -\ell\left(\gamma_\delta|_{[0,\delta]}\right) + \delta\varepsilon = -\delta + \delta\varepsilon = \delta(\varepsilon - 1).$$

This implies

$$\frac{g(z_\delta) - g(x)}{\delta} = \frac{g \circ \gamma_\delta(\delta) - g \circ \gamma_\delta(0)}{\delta} \leq \varepsilon - 1.$$

Since $g \circ \gamma_\delta$ is C^1 smooth, by the mean value theorem there is $\tau_\delta \in [0, \delta]$ such that

$$\frac{|g \circ \gamma_\delta(\delta) - g \circ \gamma_\delta(0)|}{\delta} = |(g \circ \gamma_\delta)'(\tau_\delta)| \leq ||dg(\gamma_\delta(\tau_\delta))||_{\gamma_\delta(\tau_\delta)} ||\gamma_\delta'(\tau_\delta)||_{\gamma_\delta(\tau_\delta)} = ||dg(\gamma_\delta(\tau_\delta))||_{\gamma_\delta(\tau_\delta)}.$$

Clearly, $d(x, \gamma_\delta(\tau_\delta)) \leq \ell(\gamma_\delta|_{[0,\tau_\delta]}) = \tau_\delta \leq \delta$ and thus $\lim_{\delta \to 0} d(x, \gamma_\delta(\tau_\delta)) = 0$. Since the function $z \to ||dg(z)||_z$ is continuous, $\lim_{\delta \to 0} ||dg(\gamma_\delta(\tau_\delta))||_{\gamma_\delta(\tau_\delta)} = ||dg(x)||_x$. Thus $||dg(x)||_x \geq 1 - \varepsilon$. This inequality holds for every $\varepsilon > 0$, and consequently $||\Delta||_x \geq 1$.

Now, let us show (ii). Take $\Lambda \in D^+ u(x)$, $x \in \Omega$. There exists a C^1 smooth function $g : M \to \mathbb{R}$ such that $u - g$ attains a local maximum at x and $\Lambda = dg(x)$. Therefore $u(y) - g(y) \leq u(x) - g(x)$, for all y in a neighborhood of x. For each $v \in T_x M$ with $||v||_x = 1$, choose a (piecewise) C^1 smooth path parametrized by the arc length $\gamma_v : [0, T] \to M$ such that $\gamma_v(0) = x$ and $\gamma_v'(0) = v$. Then

$$d(\gamma_v(0), \gamma_v(t)) \leq \ell\left(\gamma_v|_{[0,t]}\right) = t, \text{ for all } t \in [0, T].$$

It can be easily checked that $u(x) = \inf\{h(y) + d(y, x) : y \in \partial\Omega\}$ is 1-Lipschitz in M, and thus for $t > 0$ small enough

$$g(\gamma_v(t)) - g(\gamma_v(0)) \geq u(\gamma_v(t)) - u(\gamma_v(0)) \geq -d(\gamma_v(t), \gamma_v(0)) \geq -t,$$

and

$$dg(x)(v) = \lim_{t \to 0^+} \frac{g(\gamma_v(t)) - g(\gamma_v(0))}{t} \geq -1.$$

Therefore $dg(x)(-v) \leq 1$ and we can conclude that $||\Lambda||_x = ||dg(x)||_x \leq 1$.

Remark 3.4. *Following the above argument, we can prove the next statement: Let M be a C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function. Assume that $\Omega \subset M$ is a non-empty open subset and consider a function $f : \Omega \to \mathbb{R}$. If f is pointwise K-Lipschitz at*

$x \in \Omega$, that is $|f(x) - f(y)| \leq Kd(x,y)$ for all y in a neighborhood of x, then $\|\Delta\|_x \leq K$ for every $\Delta \in D^+ f(x) \cup D^- f(x)$.

Finally, we will check the uniqueness of the viscosity solution. Suppose that there exist two viscosity solutions $u, v \colon \overline{\Omega} \to \mathbb{R}$. In particular, their superdifferentials at every point $x \in \Omega$ are $\| \cdot \|_x$-bounded above by 1. Thus, $-u$ and $-v$ have subdifferentials $\| \cdot \|_x$-bounded above by 1 in Ω. By applying Theorem 2.16 we can deduce that $-u$ and $-v$ are locally 1-Lipschitz in Ω. We consider u and v defined in $M \setminus \Omega$ as $u(x) = v(x) = \inf\{h(y) + d(y,x) : y \in \partial\Omega\}$. Thus, $u, v : M \to \mathbb{R}$ are continuous, locally 1-Lipschitz in Ω, and 1-Lipschitz in $M \setminus \Omega$. Following an analogous proof to the one given in Proposition 3.1, it can be deduced that u and v are 1-Lipschitz in M.

Since Ω is bounded and u and v are 1-Lipschitz, we know that u and v are bounded in $\overline{\Omega}$. In fact, we may assume that the boundary data h is non-negative in $\partial\Omega$. Otherwise, we consider $S > 0$ large enough so that $\widetilde{h} = h + S$ is non-negative in $\partial\Omega$ and the Hamilton-Jacobi equation

$$(4) \qquad \begin{cases} \|d\widetilde{u}(x)\|_x = 1, & \text{for all } x \in \Omega, \\ \widetilde{u}(x) = \widetilde{h}(x), & \text{for all } x \in \partial\Omega. \end{cases}$$

Notice that a function $\widetilde{u}$ is a viscosity solution of (4) if and only if $u = \widetilde{u} - S$ is a viscosity solution of (EEq2).

Now, if we prove that $\theta u(x) \leq v(x)$ for all $x \in \overline{\Omega}$ and all $\theta \in (0,1)$, then we will have $u \leq v$. Analogously, it can be proved $v \leq u$, and thus $u = v$.

Assume, by contradiction, that $\sup_{\overline{\Omega}}(\theta u - v) > 0$ for some $\theta \in (0,1)$. We know that $\theta u - v$ is continuous and bounded. Hence, by applying the Ekeland variational principle to the function $\theta u - v : \overline{\Omega} \to \mathbb{R}$ for $0 < \varepsilon < \sup_{\overline{\Omega}}(\theta u - v)$, we can find $\overline{x} \in \overline{\Omega}$ such that

$$\sup_{\overline{\Omega}}(\theta u - v) < (\theta u - v)(\overline{x}) + \varepsilon$$

and

$$(\theta u - v)(x) \leq (\theta u - v)(\overline{x}) + \varepsilon d(x, \overline{x}), \text{ for all } x \in \overline{\Omega}.$$

Necessarily, $\overline{x} \in \Omega$, otherwise, $\sup_{\overline{\Omega}}(\theta u - v) < (\theta u - v)(\overline{x}) + \varepsilon = (\theta - 1)h(\overline{x}) + \varepsilon \leq \varepsilon$, which is a contradiction. Since $(\theta u - v)(\cdot) - \varepsilon d(\cdot, \overline{x})$ attains a local maximum at $\overline{x}$, we have $0 \in D^+ (\theta u(\cdot) - v(\cdot) - \varepsilon d(\cdot, \overline{x}))(\overline{x})$, which yields $0 \in D^-(\varepsilon d(\cdot, \overline{x}) + v(\cdot) - \theta u(\cdot))(\overline{x})$.

Let (U, φ) be a chart with $\overline{x} \in U \subset \Omega$ satisfying the Palais condition for $1 + \varepsilon$. Let us consider in X the norm $\|v\|_{\overline{x}} = \|d\varphi^{-1}(\varphi(\overline{x}))(v)\|_{\overline{x}}$ for all $v \in X$. For a continuous linear operator $T : (T_x M, \| \cdot \|_x) \to (X, \|\cdot\|_{\overline{x}})$, where $x \in U$, we consider the norm

$$\|\|T\|\|_{x,\overline{x}} = \sup\{\|T(v)\|_{\overline{x}} : \|v\|_x \leq 1\}.$$

Moreover, if T is an isomorphism we consider the norm

$$\|\|T^{-1}\|\|_{\overline{x},x} = \sup\{\|T^{-1}(v)\|_x : \|v\|_{\overline{x}} \leq 1\}.$$

From the Palais condition, we obtain for all $x \in U$ and $v \in T_x M$,

$$\|d\varphi(x)(v)\|_{\overline{x}} = \|d\varphi^{-1}(\varphi(\overline{x}))(d\varphi(x)(v))\|_{\overline{x}} \leq (1 + \varepsilon)\|d\varphi^{-1}(\varphi(x))(d\varphi(x)(v))\|_x = (1 + \varepsilon)\|v\|_x$$

and

$$\|d\varphi(x)(v)\|_{\overline{x}} = \|d\varphi^{-1}(\varphi(\overline{x}))(d\varphi(x)(v))\|_{\overline{x}} \geq (1 + \varepsilon)^{-1}\|d\varphi^{-1}(\varphi(x))(d\varphi(x)(v))\|_x = (1 + \varepsilon)^{-1}\|v\|_x.$$

Therefore, for all $x \in U$,

$$(1 + \varepsilon)^{-1} \leq \|\|d\varphi(x)\|\|_{x,\overline{x}} \leq (1 + \varepsilon) \text{ and thus } (1 + \varepsilon)^{-1} \leq \|\|d\varphi^{-1}(\varphi(x))\|\|_{\overline{x},x} \leq (1 + \varepsilon).$$

For a continuous linear functional $L : (X, \|\cdot\|_{\overline{x}}) \to \mathbb{R}$, we will consider the norm

$$\|\|L\|\|_{\overline{x}} = \sup\{|L(v)| : \|v\|_{\overline{x}} \leq 1\}.$$

By applying Proposition 2.12 (the fuzzy rule for the subdifferential of the sum) to the function $\varepsilon d(\cdot, \overline{x}) + v(\cdot) - \theta u(\cdot)$, we find points $x_1, x_2, x_3 \in U \subset \Omega$, functionals $\Delta_1 \in D^-(-\theta u)(x_1)$, $\Delta_2 \in D^- v(x_2)$, $\Delta_3 \in D^- (\varepsilon d(\cdot, \overline{x}))(x_3)$ such that

$$(5) \qquad \left\| \left\| \Delta_1 \circ d\varphi(x_1)^{-1} + \Delta_2 \circ d\varphi(x_2)^{-1} + \Delta_3 \circ d\varphi(x_3)^{-1} - 0 \circ d\varphi(\overline{x})^{-1} \right\| \right\|_{\overline{x}} \leq \varepsilon.$$

For convenience, we define

- $\Lambda_1 := -\dfrac{1}{\theta}\Delta_1 \in -\dfrac{1}{\theta}D^-(-\theta u)(x_1) = D^+ u(x_1)$,
- $\Lambda_2 := \Delta_2 \in D^- v(x_2)$,
- $\Lambda_3 := \dfrac{1}{\varepsilon}\Delta_3 \in \dfrac{1}{\varepsilon}D^- (\varepsilon d(\cdot, \overline{x}))(x_3) = D^- (d(\cdot, \overline{x}))(x_3)$.

Thus, we can rewrite (5) as $\left\| \left\| \theta\Lambda_1 \circ d\varphi(x_1)^{-1} - \Lambda_2 \circ d\varphi(x_2)^{-1} - \varepsilon\Lambda_3 \circ d\varphi(x_3)^{-1} \right\| \right\|_{\overline{x}} \leq \varepsilon$, and then $\left\| \left\| \theta\Lambda_1 \circ d\varphi(x_1)^{-1} - \Lambda_2 \circ d\varphi(x_2)^{-1} \right\| \right\|_{\overline{x}} - \varepsilon \left\| \left\| \Lambda_3 \circ d\varphi(x_3)^{-1} \right\| \right\|_{\overline{x}} \leq \varepsilon$. Since $d(\cdot, \overline{x})$ is 1-Lipschitz, we have $\|\Lambda_3\|_{x_3} \leq 1$. Hence,

$$(6) \qquad \left\| \left\| \theta\Lambda_1 \circ d\varphi(x_1)^{-1} - \Lambda_2 \circ d\varphi(x_2)^{-1} \right\| \right\|_{\overline{x}} \leq \varepsilon + \varepsilon\|\Lambda_3\|_{x_3} \left\| \left\| d\varphi(x_3)^{-1} \right\| \right\|_{\overline{x}, x_3} \leq \varepsilon(\varepsilon + 2).$$

In addition, we have

$$\begin{aligned}
\|\theta\Lambda_1 - \Lambda_2 \circ d\varphi(x_2)^{-1} \circ d\varphi(x_1)\|_{x_1} &= \|[\theta\Lambda_1 - \Lambda_2 \circ d\varphi(x_2)^{-1} \circ d\varphi(x_1)] \circ d\varphi(x_1)^{-1} \circ d\varphi(x_1)\|_{x_1} \\
&\leq \left\| \left\| [\theta\Lambda_1 - \Lambda_2 \circ d\varphi(x_2)^{-1} \circ d\varphi(x_1)] \circ d\varphi(x_1)^{-1} \right\| \right\|_{\overline{x}} \||d\varphi(x_1)|\|_{x_1, \overline{x}} \\
&\leq \left\| \left\| \theta\Lambda_1 \circ d\varphi(x_1)^{-1} - \Lambda_2 \circ d\varphi(x_2)^{-1} \right\| \right\|_{\overline{x}} (1 + \varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
\|\Lambda_2\|_{x_2} &= \|\Lambda_2 \circ [d\varphi(x_2)^{-1} \circ d\varphi(x_1)] \circ [d\varphi(x_1)^{-1} \circ d\varphi(x_2)]\|_{x_2} \\
&\leq \|\Lambda_2 \circ d\varphi(x_2)^{-1} \circ d\varphi(x_1)\|_{x_1} \left\| \left\| d\varphi(x_1)^{-1} \right\| \right\|_{\overline{x}, x_1} \||d\varphi(x_2)|\|_{x_2, \overline{x}} \\
&\leq \|\Lambda_2 \circ d\varphi(x_2)^{-1} \circ d\varphi(x_1)\|_{x_1} (1 + \varepsilon)^2.
\end{aligned}$$

Let us check that these inequalities give us a contradiction. Since u and v are viscosity solutions, we have $\|\Lambda_1\|_{x_1} \leq 1$ and $\|\Lambda_2\|_{x_2} \geq 1$. Therefore, we can write

$$\begin{aligned}
\left\| \left\| \theta\Lambda_1 \circ d\varphi(x_1)^{-1} - \Lambda_2 \circ d\varphi(x_2)^{-1} \right\| \right\|_{\overline{x}} &\geq \|\theta\Lambda_1 - \Lambda_2 \circ d\varphi(x_2)^{-1} \circ d\varphi(x_1)\|_{x_1}(1 + \varepsilon)^{-1} \\
&\geq \left(\|\Lambda_2 \circ d\varphi(x_2)^{-1} \circ d\varphi(x_1)\|_{x_1} - \|\Lambda_1\|_{x_1}\theta \right)(1 + \varepsilon)^{-1} \\
&\geq \left(\|\Lambda_2\|_{x_2}(1 + \varepsilon)^{-2} - \|\Lambda_1\|_{x_1}\theta \right)(1 + \varepsilon)^{-1} \\
&\geq \left((1 + \varepsilon)^{-2} - \theta \right)(1 + \varepsilon)^{-1}.
\end{aligned}$$

Finally,

$$(7) \qquad \varepsilon(\varepsilon + 2) \geq \left\| \left\| \theta\Lambda_1 \circ d\varphi(x_1)^{-1} - \Lambda_2 \circ d\varphi(x_2)^{-1} \right\| \right\|_{\overline{x}} \geq \left((1 + \varepsilon)^{-2} - \theta \right)(1 + \varepsilon)^{-1}.$$

By letting $\varepsilon \to 0$, we have a contradiction. $\qquad \square$

4. A CLASS OF HAMILTON-JACOBI EQUATIONS ON BANACH-FINSLER MANIFOLDS

Let M be a **complete** and C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function and $H: M \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Recall that we refer to the completeness of M for the Finsler metric d. Let us consider the Hamilton-Jacobi equation

$$(E1) \qquad\qquad u(x) + H(x, \|du(x)\|_x) = 0.$$

The aim of this section is to study the existence and uniqueness of the viscosity solutions $u: M \to \mathbb{R}$ of (E1), under certain assumptions.

Definition 4.1. *Let us consider a function $u: M \to \mathbb{R}$.*

*(1) The function u is a **viscosity subsolution of** (E1) if u is upper semicontinuous and $u(x) + H(x, \|\Delta\|_x) \leq 0$ for every $x \in M$ and $\Delta \in D^+ u(x)$.*

*(2) u is a **viscosity supersolution of** (E1) if u is lower semicontinuous and $u(x) + H(x, \|\Delta\|_x) \geq 0$ for every $x \in M$ and $\Delta \in D^- u(x)$.*

(3) u is a ***viscosity solution of*** (E1) *if u is simultaneously a viscosity subsolution and a viscosity supersolution of* (E1).

Let us consider the analogous definition for Finsler manifolds of the condition (A) given in [5, Theorem 3.2] for Banach spaces.

Definition 4.2. *The Hamiltonian H in* (E1) *satisfies condition (A) whenever there are a constant $C \geq 0$ and a continuous function $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $\omega(0,0) = 0$ such that for any $x_1, x_2 \in M$ and any $t_1, t_2 \in \mathbb{R}$,*

$$|H(x_1, t_1) - H(x_2, t_2)| \leq \omega(d(x_1, x_2), t_1 - t_2) + C \max\{|t_1|, |t_2|\} d(x_1, x_2).$$

Remark 4.3. *Let us recall that every uniformly continuous Hamiltonian H in* (E1) *satisfies condition (A). In addition, condition (A) implies that H is uniformly continuous in $M \times [-K, K]$ for every $K > 0$.*

Let us give now a generalization for Finsler manifolds of the results given in [2, Theorem 6.13], [5, Theorem 3.2], [12, Proposition 3.3] and [13, Theorem 6.1].

Theorem 4.4. *Let M be a complete C^1 Finsler manifold modeled on a Banach space X with a C^1 Lipschitz bump function and let $H : M \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian of* (E1). *Assume that H satisfies condition (A). If u is a viscosity subsolution and v is a viscosity supersolution of* (E1), *both functions are bounded and for every $x \in M$ either u or v is Lipschitz in a neighborhood of x, then*

$$\inf_M (v - u) \geq 0.$$

Proof. Let us fix $\varepsilon > 0$. By applying the Ekeland variational principle to $v - u$, we can find a point $\overline{x} \in M$ such that

$$(8) \qquad \inf_M (v - u) > (v - u)(\overline{x}) - \varepsilon$$

and

$$(v - u)(y) \geq (v - u)(\overline{x}) - \varepsilon d(y, \overline{x}), \quad \text{for all } y \in M.$$

Since $(v - u)(y) + \varepsilon d(y, \overline{x})$ attains a minimum at $\overline{x}$, $0 \in D^-(v - u + \varepsilon d(\cdot, \overline{x}))(\overline{x})$.

Let us assume that u is Lipschitz in a neighborhood of $\overline{x}$ (the other case is analogous). Thus, there is an open subset $A \subset M$ with $\overline{x} \in A$ and a constant $K_{\overline{x}} > 0$ such that u is $K_{\overline{x}}$–Lipschitz in A. Let us consider, as we did in Theorem 3.3, the norm $|||w|||_{\overline{x}} = ||d\varphi^{-1}(\varphi(\overline{x}))(w)||_{\overline{x}}$ for $w \in X$.

Let (U, φ) be a chart with $\overline{x} \in U \subset A$ satisfying the Palais condition for $1 + \overline{\varepsilon}$, where $\overline{\varepsilon} = \min\{\varepsilon, \varepsilon K_{\overline{x}}^{-1}\}$ such that $\varphi : (U, d) \to (\varphi(U), ||| \cdot |||_{\overline{x}})$ is $(1 + \varepsilon)$-bi-Lipschitz (Lemma 2.2).

In addition, we consider for $x \in U$ and $d\varphi(x) : (T_x M, || \cdot ||_x) \to (X, |||\cdot|||_{\overline{x}})$, the norms

$$|||d\varphi(x)|||_{x, \overline{x}} = \sup\{|||d\varphi(x)(v)|||_{\overline{x}} : ||v||_x \leq 1\},$$

$$|||d\varphi^{-1}(\varphi(x))|||_{\overline{x}, x} = \sup\{||d\varphi^{-1}(\varphi(x))(v)||_x : |||v|||_{\overline{x}} \leq 1\}.$$

We obtained in the proof of Theorem 3.3 that, for $x \in U$,

$$(1 + \overline{\varepsilon})^{-1} \leq |||d\varphi(x)|||_{x, \overline{x}} \leq (1 + \overline{\varepsilon}) \quad \text{and} \quad (1 + \overline{\varepsilon})^{-1} \leq |||d\varphi^{-1}(\varphi(x))|||_{\overline{x}, x} \leq (1 + \overline{\varepsilon}).$$

Finally, for a linear functional $L : (X, ||| \cdot |||_{\overline{x}}) \to \mathbb{R}$, let us consider the norm

$$|||L|||_{\overline{x}} = \sup\{|L(v)| : |||v|||_{\overline{x}} \leq 1\}.$$

Notice that we can consider a Lipschitz extension of $u|_A$ to M, denoted by $\tilde{u} : M \to \mathbb{R}$, in order to apply the local fuzzy rule to $v - \tilde{u} + \varepsilon d(\cdot, \overline{x})$. Thus, by applying Proposition 2.12, we get points $x_1, x_2, x_3 \in U$ and functionals $\Delta_1 \in D^-(-u)(x_1)$, $\Delta_2 \in D^- v(x_2)$ and $\Delta_3 \in D^-(\varepsilon d(\cdot, \overline{x}))(x_3)$ such that

 (i) $d(x_i, \overline{x}) < \varepsilon$, for $i = 1, 2, 3$,
 (ii) $|v(x_2) - v(\overline{x})| < \varepsilon$ and $|u(x_1) - u(\overline{x})| < \varepsilon$ and
 (iii) $|||\Delta_1 \circ d\varphi(x_1)^{-1} + \Delta_2 \circ d\varphi(x_2)^{-1} + \Delta_3 \circ d\varphi(x_3)^{-1} - 0 \circ d\varphi(\overline{x})^{-1}|||_{\overline{x}} < \varepsilon$.
 (iv) $\max\{|||\Delta_1 \circ d\varphi(x_1)^{-1}|||_{\overline{x}}, |||\Delta_2 \circ d\varphi(x_2)^{-1}|||_{\overline{x}}\} \cdot d(x_1, x_2) < \varepsilon$.

Let us denote $\Lambda_1 = -\Delta_1 \in D^+u(x_1)$, $\Lambda_2 = \Delta_2 \in D^-v(x_2)$ and $\Lambda_3 = \varepsilon^{-1}\Delta_3 \in D^-(d(\cdot, x_3))(x_3)$. Then,

$$(9) \qquad ||| - \Lambda_1 \circ d\varphi(x_1)^{-1} + \Lambda_2 \circ d\varphi(x_2)^{-1} + \varepsilon\Lambda_3 \circ d\varphi(x_3)^{-1} |||_{\overline{x}} < \varepsilon.$$

From (8) and condition (ii) we get

$$(10) \qquad \inf_M (v - u) > (v - u)(\overline{x}) - \varepsilon > v(x_2) - u(x_1) - 3\varepsilon.$$

Since u is a viscosity subsolution of (E1) and v is a viscosity supersolution of (E1), we get

$$(11) \qquad -u(x_1) \geq H(x_1, \|\Lambda_1\|_{x_1}),$$
$$v(x_2) \geq -H(x_2, \|\Lambda_2\|_{x_2}).$$

Consequently, by inequalities (10) and (11),

$$\inf_M (v - u) > H(x_1, \|\Lambda_1\|_{x_1}) - H(x_2, \|\Lambda_2\|_{x_2}) - 3\varepsilon \geq$$

$$(12) \qquad \geq -\big[\omega(d(x_1, x_2), \|\Lambda_1\|_{x_1} - \|\Lambda_2\|_{x_2}) + C \max\{\|\Lambda_1\|_{x_1}, \|\Lambda_2\|_{x_2}\} d(x_1, x_2)\big] - 3\varepsilon,$$

where ω is the function and $C \geq 0$ is the constant given in condition (A) for H. Now, inequality (9) above yields

$$\Big| |||\Lambda_1 \circ d\varphi(x_1)^{-1}|||_{\overline{x}} - |||\Lambda_2 \circ d\varphi(x_2)^{-1}|||_{\overline{x}} \Big| \leq |||\varepsilon\Lambda_3 \circ d\varphi(x_3)^{-1}|||_{\overline{x}} + \varepsilon.$$

Recall that the function $d(\cdot, \overline{x})$ is 1-Lipschitz and thus $\|\Lambda_3\|_{x_3} \leq 1$. Therefore,

$$|||\varepsilon\Lambda_3 \circ d\varphi(x_3)^{-1}|||_{\overline{x}} \leq \varepsilon\|\Lambda_3\|_{x_3}|||d\varphi(x_3)^{-1}|||_{\overline{x}, x_3} \leq \|\Lambda_3\|_{x_3}\varepsilon(1 + \varepsilon) \leq \varepsilon(1 + \varepsilon).$$

Now,

$$|||\Lambda_2 \circ d\varphi(x_2)^{-1}|||_{\overline{x}} \geq \|\Lambda_2\|_{x_2}|||d\varphi(x_2)|||_{x_2, \overline{x}}^{-1} \geq \|\Lambda_2\|_{x_2}(1 + \overline{\varepsilon})^{-1}$$

and

$$\big|\big|\big|\Lambda_1 \circ d\varphi(x_1)^{-1}\big|\big|\big|_{\overline{x}} \leq \|\Lambda_1\|_{x_1}\big|\big|\big|d\varphi(x_1)^{-1}\big|\big|\big|_{\overline{x}, x_1} \leq \|\Lambda_1\|_{x_1}(1 + \overline{\varepsilon}).$$

Therefore,

$$-\|\Lambda_1\|_{x_1}(1 + \overline{\varepsilon}) + \|\Lambda_2\|_{x_2}(1 + \overline{\varepsilon})^{-1} \leq \varepsilon(2 + \varepsilon).$$

Since u is $K_{\overline{x}}$–Lipschitz in U, we have $\|\Lambda_1\|_{x_1} \leq K_{\overline{x}}$ and, by computing, we obtain

$$\|\Lambda_2\|_{x_2} - \|\Lambda_1\|_{x_1} < \varepsilon(2 + \varepsilon) + \overline{\varepsilon}\|\Lambda_1\|_{x_1} + \frac{\overline{\varepsilon}}{1 + \overline{\varepsilon}}\|\Lambda_2\|_{x_2}$$
$$\leq \varepsilon(2 + \varepsilon) + \overline{\varepsilon}\|\Lambda_1\|_{x_1} + \overline{\varepsilon}\big(\varepsilon(2 + \varepsilon) + (1 + \overline{\varepsilon})\|\Lambda_1\|_{x_1}\big)$$
$$\leq \varepsilon(4 + 4\varepsilon + \varepsilon^2).$$

In an analogous way we obtain $\|\Lambda_1\|_{x_1} - \|\Lambda_2\|_{x_2} < \varepsilon(4 + 4\varepsilon + \varepsilon^2)$. Also, condition (iv) yields

$$\varepsilon > \max\big\{|||\Delta_1 \circ d\varphi(x_1)^{-1}|||_{\overline{x}}, |||\Delta_2 \circ d\varphi(x_2)^{-1}|||_{\overline{x}}\big\} \cdot d(x_1, x_2) \geq$$

$$(13) \qquad \geq (1 + \varepsilon)^{-1} \max\big\{\|\Delta_1\|_{x_1}, \|\Delta_2\|_{x_2}\big\} \cdot d(x_1, x_2).$$

In addition, $d(x_1, x_2) < 2\varepsilon$ and, by the continuity of ω and inequality (13), we obtain

$$\omega\big(d(x_1, x_2), \|\Lambda_1\|_{x_1} - \|\Lambda_2\|_{x_2}\big) + C \max\big\{\|\Lambda_1\|_{x_1}, \|\Lambda_2\|_{x_2}\big\} \cdot d(x_1, x_2) \to 0 \text{ as } \varepsilon \to 0.$$

Finally, inequality (12) yields $\inf_M(v - u) \geq 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.5.   *(1) If we assume in the assumptions of Theorem 4.4 that either u or v is L-Lipschitz, then it is enough to assume that the Hamiltonian H is uniformly continuous in $M \times [0, R]$ for some $R > L$.*

(2) It is worth noticing that Theorem 4.4 holds (with few modifications in the proof) for the weaker condition on H denoted as (*) in [13]: a Hamiltonian H of (E1) verifies condition (*) if

$$|H(x_1, t) - H(x_2, t)| \to 0 \quad as \quad d(x_1, x_2)(1 + |t|) \to 0 \quad uniformly \ on \ t \in \mathbb{R}, \ x_1, x_2 \in M \ and$$
$$|H(x, t_1) - H(x, t_2)| \to 0 \quad as \quad |t_1 - t_2| \to 0 \quad uniformly \ on \ x \in M, t_1, t_2 \in \mathbb{R}.$$

A few modifications of Theorem 4.4 yield the following results on the stability of the viscosity solutions.

Proposition 4.6. *Let M be a complete C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function and let H_1, $H_2 : M \times \mathbb{R} \to \mathbb{R}$ be two Hamiltonians of* (E1). *Assume that H_1 and H_2 verify condition (A). If u is a viscosity subsolution of* (E1) *for the Hamiltonian H_1 and v is a viscosity supersolution of* (E1) *for the Hamiltonian H_2, the functions u and v are bounded and for every $x \in M$ either u or v is Lipschitz in a neighborhood of x, then*

$$\sup_M (u - v) \leq \sup_{M \times \mathbb{R}} (H_2 - H_1).$$

An immediate consequence of Proposition 4.6 is the next result. First, let us recall the definition of a equi-continuous family of functions.

Definition 4.7. *Let Γ be a topological space and let S be an arbitrary set. A family of functions $\{f_\gamma : S \to \mathbb{R}\}_{\gamma \in \Gamma}$ is equi-continuous at $\gamma_0 \in \Gamma$ if for every $\varepsilon > 0$ there exists an open neighborhood U of γ_0 such that $|f_\gamma(s) - f_{\gamma_0}(s)| < \varepsilon$ for all $\gamma \in U$ and $s \in S$. A family $\{f_\gamma : \gamma \in \Gamma\}$ is equi-continuous if it is equi-continuous at every $\gamma_0 \in \Gamma$.*

Corollary 4.8. *Let M be a complete C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function, and Γ a topological space. Let $H_\gamma : M \times \mathbb{R} \to \mathbb{R}$ be Hamiltonians of* (E1) *satisfying condition (A) for all $\gamma \in \Gamma$. Let us assume that:*
  (1) the family of functions $\{H_\gamma : \gamma \in \Gamma\}$ is equi-continuous and
  (2) for every $\gamma \in \Gamma$, the function $u_\gamma : X \to \mathbb{R}$ is a locally Lipschitz viscosity solution of (E1) *for the Hamiltonian H_γ.*

Then, for every $\gamma_0 \in \Gamma$ and every $\varepsilon > 0$, there exists an open neighborhood U of γ_0 such that $||u_\gamma - u_{\gamma_0}||_\infty = \sup\{|u_\gamma(x) - u_{\gamma_0}(x)| : x \in M\} < \varepsilon$ for all $\gamma \in U$.

In the following results, we adapt Perron's method to Finsler manifolds and, in particular to prove the existence and uniqueness of the bounded viscosity solutions on a class of Hamilton-Jacobi equations of the form (E1). Let us consider the more general class of Hamilton-Jacobi equations of the form

(E2) $$F(x, du(x), u(x)) = 0,$$

where $F : TM^* \times \mathbb{R} \to \mathbb{R}$ is a continuous Hamiltonian. Let us recall that the topology of TM^* satisfies the first axiom of countability: for each point $(x, \Lambda) \in TM^*$ and a fixed chart (U, ψ) such that $x \in U$, the family

$$U_n^n(x) := \{(y, \Delta) \in TU^* : d(y, x) < \frac{1}{n}, \text{ and } ||\Lambda \circ d\psi(x)^{-1} - \Delta \circ d\psi(y)^{-1}|| < \frac{1}{n}\}$$

is a countable neighborhood basis of (x, Λ). Also, a sequence $\{(x_n, \Delta_n)\}_{n \in \mathbb{N}} \subset TM^*$ converges to (x, Δ) in TM^* iff
  (1) $\lim_{n \to \infty} d(x_n, x) = 0$ and
  (2) $\lim_{n \to \infty} ||\Delta_n \circ d\varphi(x_n)^{-1} - \Delta \circ d\varphi(x)^{-1}|| = 0$ for every chart (U, φ) on M with $x \in U$. Equivalently, there is a chart (U, φ) on M with $x \in U$ such that $\lim_{n \to \infty} ||\Delta_n \circ d\varphi(x_n)^{-1} - \Delta \circ d\varphi(x)^{-1}|| = 0$. (Let us recall that, in general, we assume $\Delta_n \circ d\varphi(x_n)^{-1}$ defined only for $n \geq n_0$, where n_0 depends on the chart (U, φ).)

Notice that we can define the continuity of F (given in (E2)) in terms of sequences: the Hamiltonian F is continuous at $(x, \Delta, t) \in TM^* \times \mathbb{R}$ if $\lim_{n \to \infty} F(x_n, \Delta_n, t_n) = F(x, \Delta, t)$ for every sequence $\{(x_n, \Delta_n, t_n)\}_{n \in \mathbb{N}} \subset TM^* \times \mathbb{R}$ with limit (x, Δ, t).

It can be easily checked that condition (2) above implies $\lim_n ||\Delta_n||_{x_n} = ||\Delta||_x$ and thus, for a continuous function $H : M \times \mathbb{R} \to \mathbb{R}$, the Hamilton-Jacobi equation considered in (E1)

$$F(x, du(x), u(x)) := u(x) + H(x, ||du(x)||_x) = 0$$

is a particular case of (E2). Let us recall that a function $u : M \to \mathbb{R}$

(1) is a **viscosity subsolution of** (E2) if u is upper semicontinuous and $F(x, \Delta, u(x)) \leq 0$ for every $x \in M$ and $\Delta \in D^+ u(x)$,

(2) is a **viscosity supersolution of** (E2) if u is lower semicontinuous and $F(x, \Delta, u(x)) \geq 0$ for every $x \in M$ and $\Delta \in D^- u(x)$,

(3) is a **viscosity solution of** (E2) if u is simultaneously a viscosity subsolution and a viscosity supersolution of (E2).

Lemma 4.9. *Let M be a C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function. Let Ω be an open subset of M. Let $\mathcal{F}$ be a locally uniformly bounded family of functions from Ω into $\mathbb{R}$ and $u = \sup\{v : v \in \mathcal{F}\}$ on Ω. If every $v \in \mathcal{F}$ is a viscosity subsolution of (E2) on Ω, where the Hamiltonian $F : T\Omega^* \times \mathbb{R} \to \mathbb{R}$ is continuous, then u^* is also a viscosity subsolution of (E2) on Ω.*

Proof. Let us consider $x \in \Omega$ and $\Delta \in D^+ u^*(x)$. By Proposition 2.15 (stability of the superdifferentials) there exist sequences $\{v_n\}$ in $\mathcal{F}$ and $\{(x_n, \Delta_n)\}_{n \in \mathbb{N}}$ in TM^* with $x_n \in \Omega$ and $\Delta_n \in D^+ v_n(x_n)$ for every $n \in \mathbb{N}$, such that

(i) $\lim_{n \to \infty} v_n(x_n) = u^*(x)$, and

(ii) $\lim_{n \to \infty} (x_n, \Delta_n) = (x, \Delta)$ in TM^* (i.e. $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} ||\Delta_n \circ d\varphi(x_n)^{-1} - \Delta \circ d\varphi(x)^{-1}|| = 0$ for every chart (U, φ) with $x \in U$).

Since v_n is a viscosity subsolution of (E2) on Ω for every $n \in \mathbb{N}$, we have $F(x_n, \Delta_n, v_n(x_n)) \leq 0$ for every $n \in \mathbb{N}$. Hence, $F(x, \Delta, u^*(x)) \leq 0$ and u^* is a viscosity subsolution of (E2) on Ω. $\square$

Remark 4.10. *In particular, in the above context, the supremum of two viscosity subsolutions of (E2) on Ω is a viscosity subsolution of (E2) on Ω.*

Proposition 4.11. *Let M be a C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function. Let Ω be an open subset of M and let $F : T\Omega^* \times \mathbb{R} \to \mathbb{R}$ be a continuous Hamiltonian on Ω. Assume that there are two continuous functions $s_0, s_1 : \Omega \to \mathbb{R}$, which are respectively a viscosity subsolution and a viscosity supersolution of (E2) on Ω and $s_0 \leq s_1$ on Ω. Let us define the family*

$$\mathcal{F} = \{w : \Omega \to \mathbb{R} : s_0 \leq w \leq s_1 \text{ on } \Omega \text{ and } w \text{ is a viscosity subsolution of (E2) on } \Omega\},$$

and the function $u = \sup \mathcal{F}$. Then, u^* is a viscosity subsolution of (E2) on Ω, $(u^*)_*$ is a viscosity supersolution of (E2) on Ω and $s_0 \leq (u^*)_* \leq u^* \leq s_1$.

Proof. The proof is similar to the one given in [13, Theorem 6.4]. We shall give it here for completeness. Notice that, since s_0 and s_1 are continuous in Ω, the family $\mathcal{F}$ is locally bounded on Ω. Thus, by Lemma 4.9, u^* is a viscosity subsolution of (E2) on Ω.

Let us suppose that $v = (u^*)_*$ is not a viscosity supersolution of (E2). Then, there exist $x_0 \in M$ and $\Delta_0 \in D^- v(x_0)$ such that $F(x_0, \Delta_0, v(x_0)) < 0$. According to the definition of the subdifferential, there is a C^1 smooth function $g : M \to \mathbb{R}$ such that $v - g$ attains a local minimum at x_0 and $\Delta_0 = dg(x_0)$. Then, there exists an open neighborhood U of x_0, where $v(x) - g(x) \geq v(x_0) - g(x_0)$ for all $x \in U$. Notice that $\widetilde{g}(x) = g(x) + v(x_0) - g(x_0)$ is also a C^1 smooth function with $\Delta_0 = d\widetilde{g}(x_0)$ and $v - \widetilde{g}$ attains a local minimum at x_0, and thus we may assume

(14) $F(x_0, dg(x_0), v(x_0)) < 0, \quad v(x_0) = g(x_0) \quad \text{and} \quad g(x) \leq v(x) \quad \text{for all } x \in U.$

It is clear that $g \leq v \leq s_1$ on U. Let us check that, in fact, $g(x_0) < s_1(x_0)$. Indeed, otherwise $s_1 - g$ would attain a local minimum at x_0 and thus $dg(x_0) \in D^- s_1(x_0)$. Since s_1 is a viscosity supersolution, $0 \leq F(x_0, dg(x_0), s_1(x_0)) = F(x_0, dg(x_0), v(x_0)) < 0$, which is a contradiction.

Since M is modeled on a Banach space X with a C^1 Lipschitz bump function, we can choose $\delta > 0$ and a C^1 Lipschitz bump function $b : M \to [0, 1]$ with

(1) $B(x_0, 2\delta) \subset U$,

(2) $b(x_0) > 0$,

(3) $b(x) = 0$ whenever $d(x, x_0) \geq \delta$, and

(4) $\sup\{|b(x)| : x \in M\}$ and $\sup\{||db(x)||_x : x \in M\}$ small enough so that

$$F(x, dg(x) + db(x), g(x) + b(x)) < 0 \quad \text{whenever } d(x, x_0) < 2\delta, \text{ and}$$
$$g(x) + b(x) \le s_1(x) \quad \text{for every } x \in U.$$

Clearly, $g + b$ is a viscosity subsolution of (E2) on $B(x_0, 2\delta)$. Now, define

$$w(x) = \begin{cases} \max\{g(x) + b(x), u^*(x)\} & \text{for all } x \in B(x_0, 2\delta), \\ u^*(x) & \text{for all } x \in \Omega \setminus B(x_0, 2\delta). \end{cases}$$

On the one hand, $u^*(x) \ge v(x) \ge g(x) = g(x) + b(x)$ for all $x \in U \setminus \overline{B}(x_0, \delta)$. Therefore, $w(x) = u^*(x)$ for all $x \in \Omega_1 := \Omega \setminus \overline{B}(x_0, \delta)$ and then, w is a viscosity subsolution of (E2) on Ω_1. On the other hand, w is the supremum of two viscosity subsolutions on $\Omega_2 := B(x_0, 2\delta)$. Thus w is a viscosity subsolution of (E2) on Ω_2, and consequently it is a viscosity subsolution on $\Omega = \Omega_1 \cup \Omega_2$.

Since $s_0 \le w \le s_1$, we have $w \in \mathcal{F}$ and then, $w \le u \le u^*$ on Ω, and $u^*(x) \ge w(x) \ge g(x) + b(x)$ on $B(x_0, \delta)$. Therefore, $v(x) = (u^*)_*(x) \ge g(x) + b(x)$ on $B(x_0, \delta)$. In particular, $v(x_0) = (u^*)_*(x_0) \ge g(x_0) + b(x_0) > g(x_0)$ which contradicts (14). $\square$

Corollary 4.12. *Let M be a complete C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function. Let $H : M \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian of (E1). Assume that there are constants $K_0, K_1 > 0$ such that $K_0 \le H(x, 0) \le K_1$ for all $x \in M$ and at least one of the following conditions holds:*

(i) *H is uniformly continuous and $\liminf_{t \to \infty} H(x, t) > K_1$ for each $x \in M$.*
(ii) *H satisfies condition (A), there is a constant K_1' such that $\liminf_{t \to \infty} H(x, t) \ge K_1' > K_1$ for each $x \in M$ and the limit is locally uniform on M.*

Then, there exists a unique bounded viscosity solution u of the equation (E1). Moreover, if we define the family

$$\mathcal{F} := \{w : M \to \mathbb{R} : -K_1 \le w \le -K_0 \text{ on } M, \text{ and } w \text{ is a viscosity subsolution of (E1)}\},$$

then, the viscosity solution is $u = \sup\{w : w \in \mathcal{F}\}$ and u is locally Lipschitz.

Proof. It can be easily checked that the functions $s_0(x) = -K_1$ and $s_1(x) = -K_0$ are respectively a viscosity subsolution and a viscosity supersolution of (E1). Let us take u^* the upper semicontinuous envelope of $u := \sup\{w : w \in \mathcal{F}\}$, and $(u^*)_*$ the lower semicontinuous envelope of u^*. By Proposition 4.11, u^* and $(u^*)_*$ are respectively a viscosity subsolution and a viscosity supersolution of (E1) and $-K_1 \le (u^*)_* \le u^* \le -K_0$.

Notice that if w is a viscosity subsolution of (E1) with $-K_1 \le w \le -K_0$ in M, then $H(x, ||\Delta||_x) \le -w(x) \le K_1$ for each $x \in M$ and $\Delta \in D^+w(x)$. Let us fix $x \in M$. Since H satisfies either condition (i) or (ii) above, there are constants $r_x, R_x > 0$ (depending only on H and x) such that $H(z, t) > K_1$ whenever $z \in B(x, r_x)$ and $t > R_x$. Therefore $||\Delta||_z \le R_x$ for all $z \in B(x, r)$ and $\Delta \in D^+w(z)$. By applying Theorem 2.16, we conclude that $-w$ is R_x–Lipschitz in $B(x, \frac{r_x}{4})$, and so is w.

This implies that the function $u = \sup\{w : w \in \mathcal{F}\}$ satisfies the same Lipschitz condition: u is R_x–Lipschitz in $B(x, \frac{r_x}{4})$. Thus, by the definition of upper and lower semicontinuous envelopes, we have $u = u^* = (u^*)_*$. This yields $u = \sup\{w : w \in \mathcal{F}\}$ is a bounded and locally Lipschitz viscosity solution of (E1).

Finally, if $g : M \to \mathbb{R}$ is a bounded viscosity solution of (E1), according to Theorem 4.4, necessarily $g = u$. This provides the uniqueness of the bounded viscosity solution of (E1) and finishes the proof. $\square$

Remark 4.13. *Notice that a uniformly continuous Hamiltonian $H : M \times \mathbb{R} \to \mathbb{R}$ of (E1) satisfies condition (i) given in Corollary 4.12 whenever $H(x, \cdot)$ is coercive for each $x \in M$, i.e. $\lim_{t \to \infty} H(x, t) = +\infty$ for each $x \in M$. Also a Hamiltonian H of (E1) with property (A) satisfies condition (ii) given in Corollary 4.12 whenever H is uniformly coercive in M, i.e. $\lim_{t \to \infty} H(x, t) = +\infty$ uniformly on M.*

Examples 4.14. *Let us consider some examples regarding Corollary 4.12. Recall that M is a complete C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function.*

(1) Let us consider the Hamilton-Jacobi equation

$$u(x) + \min\{\|du(x)\|_x, a\} - \cos d(x_0, x) = 0,$$

where $a > 2$ is a fixed real number and x_0 is a fixed point in the Finsler manifold M. The Hamiltonian $H : M \times \mathbb{R} \to \mathbb{R}$, $H(x,t) = \min\{t,a\} - \cos d(x_0, x)$ is uniformly continuous. Moreover, $-1 \le H(x,0) = -\cos d(x_0, x) \le 1$ for $x \in M$, and $\lim_{t\to\infty} H(x,t) = a - \cos d(x_0, x) \ge a - 1 > 1$, uniformly in $x \in M$. By Corollary 4.12, there is a unique bounded viscosity solution u such that $-1 \le u \le 1$. Moreover, if $t \ge a$ then $H(x,t) > 1$. Thus, every superdifferential of u is bounded above by a and u is a-Lipschitz.

(2) Let us consider the Hamilton-Jacobi equation

$$u(x) + \|du(x)\|_x - \cos d(x_0, x) = 0.$$

The Hamiltonian $H(x,t) = t - \cos d(x_0, x)$ is uniformly continuous, $-1 \le H(x,0) \le 1$ for all $x \in M$ and $\lim_{t\to\infty} H(x,t) = \infty$ uniformly in M. By Corollary 4.12, there is a unique bounded viscosity solution u, which is locally Lipschitz and $-1 \le u \le 1$. Moreover, if $t > 2$ then $H(x,t) > 1$. Thus, the superdifferentials of u are bounded by 2 and u is 2-Lipschitz.

(3) For $0 < a < b$, let us consider the Hamilton-Jacobi equation

$$u(x) + \min\{\|du(x)\|_x, 1\} - \frac{a + d(x_0, x)}{b + d(x_0, x)} = 0.$$

The Hamiltonian $H(x,t) = \min\{t,1\} - \frac{a+d(x_0,x)}{b+d(x_0,x)}$ is uniformly continuous, $-1 \le H(x,0) = -\frac{a+d(x_0,x)}{b+d(x_0,x)} \le -\frac{a}{b}$ for all $x \in M$ and $\lim_{t\to\infty} H(x,t) = 1 - \frac{a+d(x_0,x)}{b+d(x_0,x)} > 0$ for every $x \in M$. By Corollary 4.12, there is a unique bounded viscosity solution u, which is locally Lipschitz and $\frac{a}{b} \le u \le 1$. Notice that, if $\min\{t,1\} > 1 - \frac{a}{b}$, then $H(x,t) > -\frac{a}{b}$. Therefore, the norm of the superdifferentials of u are bounded above by $1 - \frac{a}{b}$ and thus u is $(1 - \frac{a}{b})$-Lipschitz.

(4) Let us consider the Hamilton-Jacobi equation

$$u(x) + \frac{1 + 2\|du(x)\|_x}{1 + \|du(x)\|_x + d(x_0, x)} = 0.$$

The Hamiltonian $H(x,t) = \frac{1+2|t|}{1+|t|+d(x_0,x)}$ is uniformly continuous. In addition, $0 \le H(x,0) = \frac{1}{1+d(x_0,x)} \le 1$ for all $x \in M$ and $\lim_{t\to\infty} H(x,t) = \frac{1+2|t|}{1+|t|+d(x_0,x)} = 2$ for every $x \in M$. Moreover, it can be easily checked that for every $x \in M$, if $t > d(x_0,x)$ then $H(x,t) > 1$. Therefore, by Corollary 4.12, there is a unique bounded viscosity solution u, which is locally Lipschitz and $-1 \le u \le 0$. Moreover, for every $R > 0$, u is R-Lipschitz in $B(x_0, \frac{R}{4})$.

(5) A generalization of the example (2) is the Hamilton-Jacobi equation

$$u(x) + \|du(x)\|_x - f(x) = 0,$$

where $f : M \to \mathbb{R}$ is uniformly continuous and bounded. The Hamiltonian $H(x,t) = t - f(x)$ is uniformly continuous, $K_0 := \inf_M f \le H(x,0) = f(x) \le \sup_M f := K_1$ for all $x \in M$ and $\lim_{t\to\infty} H(x,t) = \infty$ uniformly in M. By Corollary 4.12, there is a unique bounded viscosity solution u, which is locally Lipschitz and $-K_1 \le u \le K_0$. Moreover, if $t > K_1 - K_0$ then $H(x,t) > K_1$. Thus, the superdifferentials of u are bounded by $K_1 - K_0$ and u is $(K_1 - K_0)$-Lipschitz.

5. A class of evolution Hamilton-Jacobi equations on Banach-Finsler manifolds

Let M be a **complete** C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function. Let us consider a continuous function $H : [0, \infty) \times M \times \mathbb{R} \to \mathbb{R}$ and the Hamilton-Jacobi equation

(E3)
$$\begin{cases} u_t + H(t, x, \|u_x\|_x) = 0, & (t > 0) \\ u(0, x) = h(x), \end{cases}$$

where $u : [0, \infty) \times M \to \mathbb{R}$ and $h : M \to \mathbb{R}$ is the initial condition which we assume to be bounded and continuous.

Definition 5.1. *Let us consider a function $u : [0, \infty) \times M \to \mathbb{R}$.*
  (1) *u is a **viscosity subsolution of** (E3) if u is upper semicontinuous, $\alpha + H(t, x, \|\Delta\|_x) \leq 0$ for every $(\alpha, \Delta) \in D^+ u(t, x)$ and $(t, x) \in \mathbb{R}^+ \times M$ and $u(0, x) \leq h(x)$ for every $x \in M$.*
  (2) *u is a **viscosity supersolution of** (E3) if u is lower semicontinuous, $\alpha + H(t, x, \|\Delta\|_x) \geq 0$ for every $(\alpha, \Delta) \in D^- u(t, x)$ and $(t, x) \in \mathbb{R}^+ \times M$ and $u(0, x) \geq h(x)$ for every $x \in M$.*
  (3) *u is a **viscosity solution of** (E3) if u is simultaneously a viscosity subsolution and a viscosity supersolution of (E3).*

Let us consider the analogous condition (A) for Hamiltonians of (E3).

Definition 5.2. *The Hamiltonian H of (E3) satisfies condition (A) whenever there are a constant $C \geq 0$ and a continuous function $\omega : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $\omega(0, 0, 0) = 0$ such that for any $t_1, t_2 \in [0, \infty)$, $x_1, x_2 \in M$ and $r_1, r_2 \in \mathbb{R}$,*

$$|H(t_1, x_1, r_1) - H(t_2, x_2, r_2)| \leq \omega(|t_1 - t_2|, d(x_1, x_2), r_1 - r_2) + C \max\{|r_1|, |r_2|\}(|t_1 - t_2| + d(x_1, x_2)).$$

Remark 5.3. *Let us recall that every uniformly continuous Hamiltonian H of (E3) satisfies condition (A). In addition, condition (A) implies that H is uniformly continuous in $[0, \infty) \times M \times [-K, K]$ for every $K > 0$.*

In the next result we follow the ideas of [5], [13, Theorem 6.2], [3], [12], [16] and [19] to obtain a generalization for Finsler manifolds.

Theorem 5.4. *Let M be a complete and C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function and let $H : [0, \infty) \times M \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian of (E3). Assume that H verifies condition (A). If u is a viscosity subsolution and v is a viscosity supersolution of (E3), for every $T > 0$ both functions are bounded in $[0, T) \times M$ and for every $(t, x) \in (0, \infty) \times M$ either u or v is Lipschitz in a neighborhood of (t, x), then*

$$\inf_{[0, \infty) \times M} (v - u) \geq 0.$$

Proof. Assume, by contradiction, that there is $(a, z) \in (0, \infty) \times M$ such that $v(a, z) - u(a, z) < 0$. Let us fix $T > 0$ large enough so that $\inf_{[0,T) \times M} (v - u) < 0$. For $\delta > 0$, let us set

$$u_\delta(t, x) = u(t, x) - \frac{\delta}{T - t}, \qquad (t, x) \in [0, T) \times M.$$

It is easy to check that u_δ is a viscosity subsolution and v is a viscosity supersolution of

$$u_t(t, x) + H(t, x, \|u_x(t, x)\|_x) = 0, \qquad (t, x) \in [0, T) \times M$$

with initial condition $u_\delta(0, x) + \frac{\delta}{T} \leq h(x) \leq v(0, x)$ for $x \in M$. Let us fix $\delta > 0$ small enough so that $\inf_{(0,T) \times M} (v - u_\delta) < 0 < \frac{\delta}{T} \leq \inf_{\{0\} \times M} (v - u_\delta)$. Moreover, the boundedness of $v - u$ in $[0, T]$ yields the existence of $0 < T' < T$ such that

$$\inf_{(0,T') \times M} (v - u_\delta) < 0 < \inf_{\{0,T'\} \times M} (v - u_\delta).$$

Thus, we may assume $u \equiv u_\delta$ and v are a viscosity subsolution and a viscosity supersolution respectively of

$$u_t(t, x) + H(t, x, \|u_x(t, x)\|_x) = 0, \qquad (t, x) \in [0, T') \times M$$

with initial condition

$$u(0, x) + \frac{\delta}{T} \leq h(x) \leq v(0, x), \quad x \in M,$$

where u and v are bounded in $[0, T'] \times M$, for every $(t, x) \in (0, T') \times M$ either u or v is Lipschitz in a neighborhood of (t, x) and

(15) $$\inf_{(0,T') \times M} (v - u) < 0 < \inf_{\{0,T'\} \times M} (v - u)$$

Let us fix $\eta > 0$ small enough so that $\varphi : \mathbb{R} \times M \to \mathbb{R}$ defined as

$$\varphi(t,x) = \begin{cases} v(t,x) - u(t,x) + \eta t, & \text{if } (t,x) \in [0,T'] \times M, \\ \infty, & \text{otherwise.} \end{cases}$$

verifies

(16) $$\inf_{(0,T')\times M} \varphi < 0 \quad \text{and} \quad \inf_{\{0,T'\}\times M} \varphi > 0.$$

Since v and $-u$ are lower semicontinuous in $[0,T) \times M$ and bounded in $[0,T'] \times M$, the function φ is lower semicontinuous and bounded below. Therefore, we can apply the Ekeland variational principle to φ and any $\varepsilon > 0$ (in the complete metric space $\mathbb{R} \times M$ with associated distance $D((r,y),(s,z)) = |r - s| + d(y,z)$) in order to find $(\bar{t}, \overline{x}) \in [0,T'] \times M$ such that

$$\varphi(\bar{t}, \overline{x}) < 0$$

and

$$\varphi(t,x) \geq \varphi(\bar{t}, \overline{x}) - \varepsilon(|t - \bar{t}| + d(x, \overline{x})), \quad \text{for all } (t,x) \in \mathbb{R} \times M.$$

Thus $\varphi(t,x) + \varepsilon(|t - \bar{t}| + d(x, \overline{x}))$ attains the minimum at $(\bar{t}, \overline{x})$ and then $0 \in D^-\big(\varphi + \varepsilon(|\cdot - \bar{t}| + d(\cdot, \overline{x}))\big)(\bar{t}, \overline{x})$. The boundedness conditions given in (16) yield $\bar{t} \in (0, T')$.

By assumption, let us assume that there is an open subset $(a,b) \times A \subset (0,T') \times M$ with $(\bar{t}, \overline{x}) \in (a,b) \times A$ and a constant $K_{(\bar{t}, \overline{x})} > 0$ such that v is $K_{(\bar{t}, \overline{x})}$–Lipschitz in $(a,b) \times A$ (the other case is analogous). Let (U, φ) be a chart with $\overline{x} \in U \subset A$ satisfying the Palais condition for $1 + \overline{\varepsilon}$, where $\overline{\varepsilon} = \min\{\varepsilon, \varepsilon K_{(\bar{t}, \overline{x})}^{-1}\}$.

The set $(0,T') \times M$ is a Finsler manifold with the same smoothness properties as M, i.e. $(0,T') \times M$ is a C^1 Finsler manifold modeled over a Banach space with a C^1 Lipschitz bump function. Moreover, if (U, φ) is the above chart in M with $\overline{x} \in U$ satisfying the Palais condition for $1 + \overline{\varepsilon}$, then (V, ϕ) with $V = (a,b) \times U$ and $\phi(t,x) = (t, \varphi(x))$ is a chart in $(0,T') \times M$ with $(\bar{t}, \overline{x}) \in V$. In addition, this chart satisfies the Palais condition for $1 + \overline{\varepsilon}$ for the norms in the tangent space $T_{(t,x)}((0,T') \times M)$ defined as $\|(r,v)\|_{(t,x)} = |r| + \|v\|_x$. Notice that, in this case, the dual norm in $T_{(t,x)}((0,T') \times M)^*$ is $\|(s,\Lambda)\|_{(t,x)}^* = \max\{|s|, \|\Lambda\|_x^*\}$.

Let us recall that there is a Lipschitz extension $\tilde{v} : \mathbb{R} \times M \to \mathbb{R}$ of the restriction $v|_V$, there is a lower semicontinuous extension $\tilde{u} : \mathbb{R} \times M \to \mathbb{R}$ of the function $-u : [0,T'] \times M \to \mathbb{R}$ and $g(t,x) = \eta t + \varepsilon(|t - \bar{t}| + d(x, \overline{x}))$ is Lipschitz in $\mathbb{R} \times M$. Thus, by applying Proposition 2.12 (the fuzzy rule for the subdifferential of the sum) to $\tilde{v} - \tilde{u} + g$, we find $t_1, t_2, t_3 \in (a,b)$, $x_1, x_2, x_3 \in U$, $(\alpha_1, \Delta_1) \in D^- v(t_1, x_1)$, $(\alpha_2, \Delta_2) \in D^-(-u)(t_2, x_2)$ and $(\alpha_3, \Delta_3) \in D^- g(t_3, x_3)$ such that

(i) $|t_i - \bar{t}| < \varepsilon$ and $d(x_i, \overline{x}) < \varepsilon$ for $i = 1,2,3$,
(ii) $|v(t_1, x_1) - v(\bar{t}, \overline{x})| < \varepsilon$, $|u(t_2, x_2) - u(\bar{t}, \overline{x})| < \varepsilon$ and $|g(t_3, x_3) - g(\bar{t}, \overline{x})| < \varepsilon$,
(iii) $|\alpha_1 + \alpha_2 + \alpha_3| < \varepsilon$ and $\||\Delta_1 \circ d\varphi(x_1)^{-1} + \Delta_2 \circ d\varphi(x_2)^{-1} + \Delta_3 \circ d\varphi(x_3)^{-1}\||_{\overline{x}} < \varepsilon$, where $\||\cdot\||_{\overline{x}}$ is defined as in the proof of Theorem 4.4, i.e. $\||w\||_{\overline{x}} = \|d\varphi^{-1}(\varphi(\overline{x}))(w)\|_{\overline{x}}$ for $w \in X$, and
(iv) $\max\big\{\||\Delta_1 \circ d\varphi(x_1)^{-1}\||_{\overline{x}}, \||\Delta_2 \circ d\varphi(x_2)^{-1}\||_{\overline{x}}\big\}\big(|t_1 - t_2| + d(x_1, x_2)\big) < \varepsilon$.

Let us write $\Lambda_1 := \Delta_1 \in D_x^- v(t_1, x_1) = \pi_2(D^- v(t_1, x_1))$ where $\pi_2 : \mathbb{R} \times TM^* \to TM^*$ is the canonical proyection over TM^*, $\Lambda_2 = -\Delta_2 \in D_x^+ u(t_2, x_2) = \pi_2(D_x^+ u(t_2, x_2))$ and $\Lambda_3 = \Delta_3 \in D_x^- g(t_3, x_3) = D^-(\varepsilon d(\cdot, \overline{x}))(x_3)$. Notice that $(-\alpha_2, \Lambda_2) \in D^+ u(t_2, x_2)$.
The second inequality in (iii) yields

$$\left| \||\Lambda_1 \circ d\varphi(x_1)^{-1}\||_{\overline{x}} - \||\Lambda_2 \circ d\varphi(x_2)^{-1}\||_{\overline{x}} \right| \leq \||\Lambda_3 \circ d\varphi(x_3)^{-1}\||_{\overline{x}} + \varepsilon.$$

The function $\varepsilon d(\cdot, \overline{x})$ is ε-Lipschitz and thus $\|\Lambda_3\|_{x_3} \leq \varepsilon$. Therefore,

$$\||\Lambda_3 \circ d\varphi(x_3)^{-1}\||_{\overline{x}} \leq \|\Lambda_3\|_{x_3} \||d\varphi(x_3)^{-1}\||_{\overline{x},x_3} = \|\Lambda_3\|_{x_3}(1 + \varepsilon) < \varepsilon(1 + \varepsilon),$$

where the norms $\||\cdot\||_{x, \overline{x}}$ and $\||\cdot\||_{\overline{x}, x}$ for $x \in U$ are defined as in Theorem 4.4. Also,

$$\||\Lambda_2 \circ d\varphi(x_2)^{-1}\||_{\overline{x}} \geq \|\Lambda_2\|_{x_2} \||d\varphi(x_2)\||_{x_2, \overline{x}}^{-1} \geq \|\Lambda_2\|_{x_2}(1 + \overline{\varepsilon})^{-1}$$

and

$$\||\Lambda_1 \circ d\varphi(x_1)^{-1}\||_{\overline{x}} \leq \|\Lambda_1\|_{x_1} \||d\varphi(x_1)^{-1}\||_{\overline{x}, x_1} \leq \|\Lambda_1\|_{x_1}(1 + \overline{\varepsilon}).$$

Therefore,

$$\|\Lambda_2\|_{x_2}(1+\overline{\varepsilon})^{-1} - \|\Lambda_1\|_{x_1}(1+\overline{\varepsilon}) < \varepsilon(2+\varepsilon).$$

Since v is $K_{(\overline{t},\overline{x})}$–Lipschitz in V, then $\|\Lambda_1\|_{x_1} \leq K_{(\overline{t},\overline{x})}$ and, by computing, we obtain

$$\|\Lambda_2\|_{x_2} - \|\Lambda_1\|_{x_1} < \varepsilon(2+\varepsilon) + \overline{\varepsilon}\|\Lambda_1\|_{x_1} + \frac{\overline{\varepsilon}}{1+\overline{\varepsilon}}\|\Lambda_2\|_{x_2}$$
$$\leq \varepsilon(2+\varepsilon) + \overline{\varepsilon}\|\Lambda_1\|_{x_1} + \overline{\varepsilon}\big(\varepsilon(2+\varepsilon) + (1+\overline{\varepsilon})\|\Lambda_1\|_{x_1}\big)$$
$$\leq \varepsilon(4 + 4\varepsilon + \varepsilon^2).$$

In an analogous way we obtain $\|\Lambda_1\|_{x_1} - \|\Lambda_2\|_{x_2} < \varepsilon(4 + 4\varepsilon + \varepsilon^2)$.

Now, since u is a viscosity subsolution and v is a viscosity supersolution of (E3) and the fact that $(\alpha_1, \Lambda_1) \in D^-v(t_1, x_1)$, $(-\alpha_2, \Lambda_2) \in D^+u(t_2, x_2)$, we have

$$\alpha_1 + H(t_1, x_1, \|\Lambda_1\|_{x_1}) \geq 0,$$
$$-\alpha_2 + H(t_2, x_2, \|\Lambda_2\|_{x_2}) \leq 0.$$

Thus

$$\alpha_1 + \alpha_2 + H(t_1, x_1, \|\Lambda_1\|_{x_1}) - H(t_2, x_2, \|\Lambda_2\|_{x_2}) \geq 0.$$

From condition (iii), we obtain

$$-\alpha_3 + \varepsilon + H(t_1, x_1, \|\Lambda_1\|_{x_1}) - H(t_2, x_2, \|\Lambda_2\|_{x_2}) \geq 0.$$

Since $\alpha_3 \in D^-(\eta t + \varepsilon|t - \overline{t}|)(t_3)$, we have that $\eta - \varepsilon \leq \alpha_3 \leq \eta + \varepsilon$ and thus

$$(17) \qquad -\eta + 2\varepsilon + H(t_1, x_1, \|\Lambda_1\|_{x_1}) - H(t_2, x_2, \|\Lambda_2\|_{x_2}) \geq 0.$$

Therefore,

$$-\eta \geq -2\varepsilon - H(t_1, x_1, \|\Lambda_1\|_{x_1}) + H(t_2, x_2, \|\Lambda_2\|_{x_2})$$
$$\geq -2\varepsilon - \omega\big(|t_1 - t_2|, d(x_1, x_2), \|\Lambda_1\|_{x_1} - \|\Lambda_2\|_{x_2}\big)$$
$$\quad - C\max\big\{\|\Lambda_1\|_{x_1}, \|\Lambda_2\|_{x_2}\big\}\big(|t_1 - t_2| + d(x_1, x_2)\big)$$
$$\geq -2\varepsilon - \omega\big(|t_1 - t_2|, d(x_1, x_2), \|\Lambda_1\|_{x_1} - \|\Lambda_2\|_{x_2}\big)$$
$$\quad - C(1+\varepsilon)\max\big\{|||\Lambda_1 \circ d\varphi(x_1)^{-1}|||_{\overline{x}}, |||\Lambda_2 \circ d\varphi(x_2)^{-1}|||_{\overline{x}}\big\}\big(|t_1 - t_2| + d(x_1, x_2)\big),$$

where ω is the function and $C \geq 0$ is the constant given in condition (A) for H. In addition, $|t_1 - t_2| < 2\varepsilon$ and $d(x_1, x_2) < 2\varepsilon$. Now, from the continuity of ω and condition (iv), we obtain $H(t_1, x_1, \|\Lambda_1\|_{x_1}) - H(t_2, x_2, \|\Lambda_2\|_{x_2}) \to 0$ as $\varepsilon \to 0$ and thus $-\eta \geq 0$, which is a contradiction. $\qquad \square$

Remark 5.5. *(1) We say that a function $f : (0,\infty) \times M \to \mathbb{R}$ is L-Lipschitz in the second variable if $|f(t,y) - f(t,z)| \leq Ld(y,z)$, for all $(t,y), (t,z) \in (0,\infty) \times M$. The assumptions on u and v in Theorem 5.4 can be weakened in the following way: u is a viscosity subsolution of (E3) and v is a viscosity supersolution of (E3), for every $T > 0$ both functions are bounded in $[0, T] \times M$, for every $(t, x) \in (0, \infty) \times M$ either u or v is uniformly continuous in a neighborhood of (t, x), and finally for every $(t, x) \in (0, \infty) \times M$ either u or v is Lipschitz in the second variable in a neighborhood of (t, x).*

(2) Let us assume in the hypothesis of Theorem 5.4 the additional condition: there is $L > 0$ such that either u or v is L-Lipschitz in the second variable in $(0, \infty) \times M$. Then, it is enough to assume that the Hamiltonian H is uniformly continuous in $[0, \infty) \times M \times [0, R]$ for some $R > L$.

Let us consider the example $H : [0, \infty) \times M \times \mathbb{R} \to \mathbb{R}$, $H(t, x, m) = r(t, x)m$, where $r : [0, \infty) \times M \to \mathbb{R}$ is a bounded and uniformly continuous function and the associated Hamilton-Jacobi equation

$$(E3^*) \qquad \begin{cases} u_t(t, x) + r(t, x)\|u_x(t, x)\|_x = 0, & (t, x) \in (0, \infty) \times M, \\ u(0, x) = h(x), & x \in M. \end{cases}$$

The Hamiltonian H is uniformly continuous in $[0, \infty) \times M \times [0, R]$ for every $R > 0$. Let us denote by $\mathcal{L}$ the family of locally uniformly continuous functions $u : [0, \infty) \times M \to \mathbb{R}$ which

are Lipschitz in the second variable in $(0, \infty) \times M$ and bounded in $[0, T] \times M$ for every $T > 0$. Then, there is at most one function within $\mathcal{L}$ which is a viscosity solution of equation (E3*).

(3) It is worth noticing that Theorem 5.4 holds (with few modifications in the proof) for the weaker condition (*) for H given in [13] (see Remark 4.5): a Hamiltonian H of (E3) verifies condition (*) if

$$|H(t_1, x_1, r) - H(t_2, x_2, r)| \to 0 \text{ as } (d(x_1, x_2) + |t_1 - t_2|)(1 + |r|) \to 0 \text{ uniformly on } t_1, t_2, r \in \mathbb{R}, x_1, x_2 \in M,$$

$$|H(t, x, r_1) - H(t, x, r_2)| \to 0 \quad as \quad |r_1 - r_2| \to 0 \quad uniformly \ on \ \ x \in M, t, r_1, r_2 \in \mathbb{R}.$$

A few modifications of Theorem 5.4 yield the following result on the monotonicity of the viscosity solutions.

Proposition 5.6. *Let M be a complete C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function and let $H_1, H_2 : M \times \mathbb{R} \to \mathbb{R}$ be two Hamiltonians of (E3) verifying condition (A) such that $H_1 \leq H_2$. Let us assume that v is a viscosity supersolution of (E3) with Hamiltonian H_1 and initial condition $v(0, x) = h_1(x)$ (for $x \in M$) and u is a viscosity subsolution of (E3) with Hamiltonian H_2 and initial condition $u(0, x) = h_2(x)$ (for $x \in M$), where h_1 and h_2 are bounded and continuous on M, and $h_2 \leq h_1$. In addition, let us assume that for every $T > 0$ the functions u and v are bounded in $[0, T) \times M$ and for every $(t, x) \in (0, \infty) \times M$ either u or v is Lipschitz in a neighborhood of (t, x). Then,*

$$\sup_{[0,\infty) \times M} (u - v) \leq \sup_{[0,\infty) \times M} (H_2 - H_1) + \sup_{M} (h_2 - h_1).$$

Let us give an outline of the proof of Proposition 5.6 for completeness. Let us assume, by contradiction, that

$$\inf_{[0,\infty) \times M} (v - u) < \inf_{[0,\infty) \times M} (H_1 - H_2) + \inf_{M} (h_1 - h_2).$$

Let us consider the function $v - u - i$, where $i := \inf_M (h_1 - h_2)$. Then $\inf_{[0,\infty) \times M}(v - u - i) < \inf_{[0,\infty) \times M}(H_1 - H_2) \leq 0$. Notice that $(v - u - i)(0, x) \geq 0$ for all $x \in M$. We can obtain analogous inequalities for this function to the one given in (15) for $v - u$. In particular, equation (17) becomes

$$
\begin{aligned}
0 \leq{} & -\eta + 2\varepsilon + H_1(t_1, x_1, \|\Lambda_1\|_{x_1}) - H_2(t_2, x_2, \|\Lambda_2\|_{x_2}) \\
={} & -\eta + 2\varepsilon + H_1(t_1, x_1, \|\Lambda_1\|_{x_1}) - H_1(t_2, x_2, \|\Lambda_2\|_{x_2}) \\
& + H_1(t_2, x_2, \|\Lambda_2\|_{x_2}) - H_2(t_2, x_2, \|\Lambda_2\|_{x_2}) \\
\leq{} & -\eta + 2\varepsilon + H_1(t_1, x_1, \|\Lambda_1\|_{x_1}) - H_1(t_2, x_2, \|\Lambda_2\|_{x_2}) \\
& + \sup_{[0,\infty) \times M} (H_1 - H_2),
\end{aligned}
$$

where $|t_1 - t_2| < 2\varepsilon$, $d(x_1, x_2) < 2\varepsilon$, $|\, \|\Lambda_1\|_{x_1} - \|\Lambda_2\|_{x_2}| < \varepsilon(4 + 4\varepsilon + \varepsilon^2)$ and $C \max\{\|\Lambda_1\|_{x_1}, \|\Lambda_2\|_{x_2}\}(|t_1 - t_2| + d(x_1, x_2)) < \varepsilon(1 + \varepsilon)$. By letting $\varepsilon \to 0$, property (A) for H_1 yields $0 \leq -\eta + \sup_{[0,\infty) \times M}(H_1 - H_2)$, which is a contradiction because $\eta > 0$ and $H_1 \leq H_2$.

Finally, let us give existence results of viscosity subsolutions, supersolutions and solutions for Hamilton-Jacobi equations of the form (E3). The proofs are analogous to those given in the preceding section. The first one is a straightforward consequence of Proposition 4.11.

Corollary 5.7. *Let M be a C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function and an open subset Ω of M. Let us consider the C^1 Finsler manifold $N = (0, \infty) \times M$ and the open subset $A = (0, \infty) \times \Omega$ of N, a continuous Hamiltonian $F : TA^* \times \mathbb{R} \to \mathbb{R}$ and a continuous function $h : \Omega \to \mathbb{R}$. Consider the Hamilton-Jacobi equation*

$$\text{(E4)} \qquad \begin{cases} F(t, x, u_t(t, x), u_x(t, x), u(t, x)) = 0, & (t, x) \in A, \\ u(0, x) = h(x), & x \in \Omega. \end{cases}$$

Assume that there are continuous functions $s_0, s_1 : [0, \infty) \times \Omega \to \mathbb{R}$ with $s_0 \leq s_1$ and $s_0(0, x) = s_1(0, x) = h(x)$ for all $x \in \Omega$ such that s_0 and s_1 are respectively a viscosity subsolution and a

viscosity supersolution of (E4). *Let us consider the family*

$$\mathcal{F} = \{w : [0, \infty) \times \Omega \to \mathbb{R} : s_0 \leq w \leq s_1 \text{ and } w \text{ is a viscosity subsolution of (E4)}\}.$$

Let us define $u = \sup \mathcal{F}$. *Then,* u^* *is a viscosity subsolution of* (E4) *and* $(u^*)_*$ *is a viscosity supersolution of* (E4).

Proof. First, let us recall that for a function $g : [0, \infty) \times \Omega \to \mathbb{R}$ and the restriction $r = g|_A$, we have $r^*(t, x) = g^*(t, x)$ and $r_*(t, x) = g_*(t, x)$ for $(t, x) \in A$. Also, recall that N is a C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function.

Thus, the inequality $F(t, x, (u^*)_t(t, x), (u^*)_x(t, x), u^*(t, x)) \leq 0$ for all the superdifferentials of u^* in A is a consequence of Proposition 4.11 for the open subset A of the Finsler manifold N. For the initial condition, notice that $s_0 \leq u \leq s_1$ and s_0, s_1 are continuous. Therefore, $s_0 \leq u^* \leq s_1$. In particular, $s_0(0, x) \leq u^*(0, x) \leq s_1(0, x)$ for all $x \in \Omega$ and thus $s_0(0, x) = u^*(0, x) = s_1(0, x) = h(x)$ for all $x \in \Omega$.

Analogously, $v = (u^*)_*$ is a supersolution: The inequality $F(t, x, v_t(t, x), v_x(t, x), v(t, x)) \geq 0$ for all the subdifferentials of v in A is a consequence of Proposition 4.11 for the open subset A of the Finsler manifold N. The initial condition is obtained from the fact that $s_0 \leq u^* \leq s_1$ and s_0, s_1 are continuous. Thus, $s_0 \leq (u^*)_* \leq s_1$ and then $s_0(0, x) = (u^*)_*(0, x) = s_1(0, x) = h(x)$ for all $x \in \Omega$. $\square$

Corollary 5.8. *Let M be a complete C^1 Finsler manifold modeled on a Banach space with a C^1 Lipschitz bump function. Let $H : [0, \infty) \times M \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian of* (E3). *Assume that H verifies condition (A), the initial condition $h : M \to \mathbb{R}$ is L-Lipschitz and bounded, and there are constants $K_0, K_1 \in \mathbb{R}$ such that*

$$K_0 = \inf\{H(t, x, m) : (t, x) \in (0, \infty) \times M, |m| \leq L\}$$

and

$$K_1 = \sup\{H(t, x, m) : (t, x) \in (0, \infty) \times M, |m| \leq L\}.$$

Let us define

$$\mathcal{F} = \{w : [0, \infty) \times M \to \mathbb{R} : w \text{ is a subsolution of (E3) and}$$
$$- K_1 t + h(x) \leq w(t, x) \leq -K_0 t + h(x) \text{ for } (t, x) \in (0, \infty) \times M\}$$

and $u = \sup \mathcal{F}$. Then, u^* is a viscosity subsolution and $(u^*)_*$ is a viscosity supersolution (E3). *Moreover,*

   (1) if u^* is continuous, then $u^* = (u^*)_*$ and u^* is a viscosity solution of (E3);
   (2) if u^* is locally Lipschitz, then u^* is the unique viscosity solution of (E3) *which is bounded in $[0, T) \times M$ for every $T > 0$.*

Proof. Notice that $s_0(t, x) = -K_1 t + h(x)$, for $(t, x) \in [0, \infty) \times M$ is a viscosity subsolution of (E3) and $s_1(t, x) = -K_0 t + h(x)$, for $(t, x) \in [0, \infty) \times M$ is a viscosity supersolution of (E3). Corollary 5.7 yields u^* and $(u^*)_*$ are respectively a viscosity subsolution and a viscosity supersolution of (E3).

If, in addition, we assume that u^* is continuous, then by the definition of lower semicontinuous envelope, $u^* = (u^*)_*$ and therefore it is a viscosity solution of (E3).

If, in addition, we assume that u^* is locally Lipschitz, the inequality $s_0 \leq u^* \leq s_1$ in $[0, \infty) \times M$ yields the boundedness of u^* in $[0, T) \times M$ for all $T > 0$. Therefore, we can apply the comparison result given in Theorem 5.4 to obtain that u^* is the unique viscosity solution of (E3) which is bounded on $[0, T) \times M$ for all $T > 0$. Thus, if there exists w a different viscosity solution of (E3), then there is $T_0 > 0$ such that w is not bounded in $[0, T_0) \times M$. $\square$

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