LFC BUMPS ON SEPARABLE BANACH SPACES

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Abstract. In this note we construct a C^∞-smooth, LFC (Locally depending on Finitely many Coordinates) bump function, in every separable Banach space admitting a continuous, LFC bump function.

1. Introduction

The notion of a LFC function (a function that locally depends on finitely many coordinates) was introduced by Pechanec, Whitfield and Zizler in [15], where they showed that every Banach space which admits a LFC bump is saturated with copies of c_0. Nonetheless, the first use of LFC in the literature is Kuiper's construction (which appeared in [1]) of a $\mathcal{C}^\infty$-smooth, LFC equivalent norm on c_0. One of the most important application of LFC is the use of $\mathcal{C}^\infty$-smooth, LFC bumps on $c_0(\Gamma)$ in the construction of $\mathcal{C}^k$-smooth partitions of unity in reflexive Banach spaces admitting a C^k-smooth bump, due to Toruńczyk [16]. The existence of a LFC bump on the space implies additional properties: It was proved in [3] and [11] that it is Asplund. However, not every Asplund, c_0-saturated space admits a LFC bump function [11].

The LFC notion is closely related to the class of polyhedral Banach spaces (introduced by Klee [13]; see [[12], Chapter 15] for results and references). Fonf [4] proved that every polyhedral Banach space is saturated with copies of c_0. Fonf [5] characterized separable polyhedral Banach spaces as those Banach spaces admitting an equivalent LFC norm. Later, Hájek [6] characterized them as those admitting an equivalent $\mathcal{C}^\infty$-smooth and LFC norm. Since it is easier to work with functions on $\mathbb{R}^n$ than with functions defined on an infinite dimensional Banach space, the notion of LFC has been successfully used (implicitly and explicitly) in a large number of papers.

It remains an open problem whether every separable Banach space with a $\mathcal{C}^\infty$-smooth LFC bump is a polyhedral Banach space [9]. Hájek and Johanis conjectured that the answer is negative. They constructed an Orlicz space admitting a $\mathcal{C}^\infty$-smooth LFC bump and not satisfying Leung's sufficient condition on polyhedrality [14]. Hájek and Johanis proved in [8] that every separable Banach space with a Schauder basis and a continuous LFC bump, admits a $\mathcal{C}^\infty$-smooth and LFC bump function. This note extends the result of [8] and establish a characterization of the class of separable Banach spaces admitting a continuous, LFC bump as those

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separable Banach spaces with a C^∞-smooth LFC bump. This result answers a problem posed in [8], [7] and [3].

We use a standard Banach space notation. If X is a separable Banach space with norm $||\cdot||$, we denote by $B(x,r)$ the open ball centered at x with radius r. A function $b:X\longrightarrow\mathbb{R}$ is a bump function if it has a bounded and non-empty support. The notion of a function that locally depends on finitely many coordinates was first defined on Banach spaces with Schauder basis using the coordinate functionals [15]. Later, a generalization of this notion was considered by some authors using arbitrary continuous linear functionals.

Definition 1.1. *Let X be a Banach space, $A\subset X$ an open subset, E be an arbitrary set, $M\subset X^*$ and a mapping $b:A\longrightarrow E$.*

(a) *We say that b* depends only on M *on a subset $U\subset A$ if $b(x)=b(y)$ whenever $x,y\in U$ are such that $f(x)=f(y)$ for all $f\in M$. If $M=\{f_1,...,f_n\}$, this is equivalent to the existence of a mapping $g:\mathbb{R}^n\longrightarrow E$ such that $b(x)=g(f_1(x),...,f_n(x))$ for all $x\in U$.*

(b) *We say that b* locally depends on finitely many coordinates from M *(LFC-M for short) if for each $x\in A$ there are a neighbourhood $U_x\subset A$ of x and a finite subset $F_x\subset M$ such that b depends only on F_x on U_x. We say that b* depends locally on finitely many coordinates *(LFC for short) if it is LFC-X^*.*

(c) *A norm is said to be LFC, if it is LFC away from the origin.*

A simple example is the sup norm on c_0, which is LFC-$\{e_i^*\}$ away from the origin (where $\{e_i^*\}$ are the coordinate functionals in c_0). Indeed, for every $x\in c_0$, $x\neq 0$, there exists $n\in\mathbb{N}$ such that $|x(i)|<||x||_\infty/2$ for every $i\geq n$. Then the norm $||\cdot||_\infty$ depends only on $\{e_1^*,...,e_n^*\}$ on $B(x,\|x\|_\infty/4)$.

We shall use the fact that for every LFC mapping $b:A\longrightarrow E$ and every mapping $h:E\longrightarrow F$ (F and arbitrary set) the composition $h\circ b$ is also LFC. It can be readily verified that a continuous function $b:A\longrightarrow\mathbb{R}$ (where $A\subset X$ is an open subset of the Banach space X) is LFC-M for some $M\subset X^*$ if and only if for every $x\in A$, there are a neighborhood $V_x\subset A$ of x, a finite subset $\{f_1,\dots,f_{n_x}\}\subset M$ and a continuous function $g^x:\mathbb{R}^{n_x}\longrightarrow\mathbb{R}$ such that $g^x(f_1(y),\dots,f_{n_x}(y))=b(y)$ for every $y\in V_x$.

2. Continuous LFC bumps

We first show that it is possible to "join together" any finite number of neighborhoods, where we have local factorizations of a given LFC function, to obtain a new factorization of the LFC function in the union of these neighborhoods by a suitable composition through the space c_0.

Lemma 2.1. *Let X be a Banach space such that X^* is separable, and $b:X\longrightarrow\mathbb{R}$ be a continuous, LFC function on X. Let us consider $p\in\mathbb{N}$, $B_j=B(x_j,r_j)$ open balls, integers $n_j\in\mathbb{N}$, continuous functions $g^j:\mathbb{R}^{n_j}\longrightarrow\mathbb{R}$ and functionals $\{f_i^j\}_{i=1}^{n_j}\subset X^*$, for $j=1\dots,p$. Let us assume that for every $x\in B(x_j,2r_j)$,*

$$b(x)=g^j(f_1^j(x),...,f_{n_j}^j(x)).$$

Then, there exists a continuous linear map $T : X \longrightarrow c_0(\mathbb{N})$ and a continuous function $g : c_0(\mathbb{N}) \longrightarrow \mathbb{R}$ such that $b(x) = g(T(x))$ for every $x \in \bigcup_{j=1}^p B_j$.

Proof. Since X^* is a separable Banach space, there exists a one-to-one continuous linear mapping $i : X \longrightarrow c_0(\mathbb{N})$. Indeed, it is enough to take a sequence $\{g_k\}_{k=1}^\infty$ dense on S_{X^*} and define $i(x) = (g_k(x)/2^k)_{k=1}^\infty$. In addition, the linear mapping i satisfies that $x_n \xrightarrow{\omega} 0$ (weakly) whenever $\{x_n\}_{n=1}^\infty$ is a bounded sequence with $i(x_n) \to 0$ (in norm).

Let us consider the continuous, LFC function $b : X \longrightarrow \mathbb{R}$. We define $n = \sum_{j=1}^p n_j$, consider $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p}$ and the canonical proyection $p_j : \mathbb{R}^n \longrightarrow \mathbb{R}^{n_j}$ given by $p_j(v) = v_j$, for $v = (v_1, \ldots, v_p) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p}$. We can relabel the set of functionals $\{f_1^1, \ldots, f_{n_1}^1, \ldots, f_1^p, \ldots, f_{n_p}^p\}$ as $\{f_1, \ldots, f_n\}$ in such a way that $p_j(f_1(x), \ldots, f_n(x)) = (f_1^j(x), \ldots, f_{n_j}^j(x))$ for every $x \in X$ and $j = 1, \ldots, p$. Let us define $G^j : \mathbb{R}^n \longrightarrow \mathbb{R}$ as $G^j(x) = g^j(p_j(x))$. To simplify notation, we will use g^j to denote G^j in the rest of the proof (thus, we have $g^j(f_1(x), ..., f_n(x)) = b(x)$ for all $x \in B(x_j, 2r_j)$). We define

$$T : X \longrightarrow \mathbb{R}^n \times c_0(\mathbb{N}), \qquad T(x) = (f_1(x), ..., f_n(x), i(x)). \tag{2.1}$$

The function T is one-to-one, linear and continuous.

Let us first show the assertion of the lemma for $p = 2$. Since T is one-to-one, $T(B_1) \cap T(B_2) = T(B_1 \cap B_2)$. If $x \in T(B_1) \cap T(B_2) = T(B_1 \cap B_2)$, there exists $y \in B_1 \cap B_2$ such that $T(y) = x$. Thus $b(y) = g^1(f_1(y), ..., f_n(y)) = g^1(\pi(x)) = g^2(f_1(y), ..., f_n(y)) = g^2(\pi(x))$, where π is the projection of $\mathbb{R}^n \times c_0(\mathbb{N})$ onto $\mathbb{R}^n$ given by the n first coordinates.

Let us define $g : \overline{T(B_1) \cup T(B_2)} \longrightarrow \mathbb{R}$ as

$$g(x) = \begin{cases} g^1(\pi(x)) & \text{if } x \in \overline{T(B_1)} \\ g^2(\pi(x)) & \text{if } x \in \overline{T(B_2)}. \end{cases}$$

If $x \in T(B_1) \cap T(B_2)$, we have already showed that $g^1(\pi(x)) = g^2(\pi(x))$. To show that g is well defined and continuous on $\overline{T(B_1) \cup T(B_2)}$, it suffices to prove that $g^1(\pi(x)) = g^2(\pi(x))$ whenever $x \in \overline{T(B_1)} \cap \overline{T(B_2)}$. Assume, on the contrary, that there is $z \in \overline{T(B_1)} \cap \overline{T(B_2)}$ with $g^1(\pi(z)) \neq g^2(\pi(z))$. Then, there exist two sequences $\{x_m\} \subset B_1$ and $\{y_m\} \subset B_2$ such that $T(x_m) \to z$ and $T(y_m) \to z$. Since $\lim_m ||\pi(T(x_m)) - \pi(z)||_\infty = \lim_m ||\pi(T(y_m)) - \pi(z)||_\infty = 0$ and g^1 and g^2 are continuous, we have

$$g^1(\pi(z)) = \lim_{m\to\infty} g^1(\pi(T(x_m))) = \lim_{m\to\infty} g^1(f_1(x_m), ..., f_n(x_m)),$$
$$g^2(\pi(z)) = \lim_{m\to\infty} g^2(\pi(T(y_m))) = \lim_{m\to\infty} g^2(f_1(y_m), ..., f_n(y_m)).$$

Let $\delta > 0$ such that $|g^1(z_1, ..., z_n) - g^2(z_1, ..., z_n)| \geq \delta > 0$, where z_i is the i-coordinate of z. Since the functions g^1 and g^2 are continuous on the point $(z_1, ..., z_n)$, there exists $\eta > 0$ such that $|g^1(t_1, ..., t_n) - g^1(z_1, ..., z_n)| < \delta/4$ and $|g^2(t_1, ..., t_n) - g^2(z_1, ..., z_n)| < \delta/4$ whenever $t = (t_1, ..., t_n) \in \mathbb{R}^n$ and $\|(t_1, ..., t_n) - (z_1, ..., z_n)\|_\infty < \eta$. Let us take $0 < \varepsilon < \min\{\eta, r_2/2\}$. There exists $n_0 \in \mathbb{N}$ such that $||\pi(T(x_m)) - \pi(z)||_\infty < \varepsilon$ and $||\pi(T(y_m)) - \pi(z)||_\infty < \varepsilon$ whenever $m \geq n_0$. To simplify, we denote $\{x_m\}$ and $\{y_m\}$ as the subsequences $\{x_m\}_{m\geq n_0}$ and $\{y_m\}_{m\geq n_0}$.

Since $T(x_m - y_m) \to 0$, by the remark at the beginning of the proof, we obtain that $x_m - y_m \xrightarrow{\omega} 0$. From the fact that $\overline{\text{co}}^{\omega}(\{x_m - y_m : m \in \mathbb{N}\}) = \overline{\text{co}}(\{x_m - y_m : m \in \mathbb{N}\})$, we obtain convex combinations of $\{x_m - y_m\}$ converging (in norm) to 0, i.e. there are non-negative numbers $\{\lambda_i^\varepsilon\}_{i=1}^{m_\varepsilon}$ such that $\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon = 1$ and $\| \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i - \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon y_i\| < \varepsilon$. Since $\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i \in B_1$ and $\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon y_i \in B_2$, we have

$$\text{dist}(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i, B_2) \leq \| \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i - \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon y_i\| < \varepsilon.$$

Notice that $\varepsilon < r_2/2$ and then $\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i \in B(x_2, 2r_2) \cap B_1$. Therefore

$$b(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i) = g^2(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_1(x_i), ..., \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_n(x_i)) = g^2(\pi \circ T(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i)) = \tag{2.2}$$

$$= g^1(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_1(x_i), ..., \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_n(x_i)) = g^1(\pi \circ T(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i)). \tag{2.3}$$

We know that $||\pi(T(x_m)) - \pi(z)||_\infty < \varepsilon$ for every $m \in \mathbb{N}$. Thus, by convexity, we have that $||\pi \circ T(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i) - \pi(z)||_\infty < \varepsilon$. Since $\varepsilon < \eta$, we deduce

$$|g^1(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_1(x_i), ..., \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_n(x_i)) - g^1(z_1, ..., z_n)| < \delta/4, \tag{2.4}$$

$$|g^2(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_1(x_i), ..., \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_n(x_i)) - g^2(z_1, ..., z_n)| < \delta/4. \tag{2.5}$$

From equations (2.2), (2.3), (2.4) and (2.5) we deduce that $|g^1(z_1, ..., z_n) - g^2(z_1, ..., z_n)| < \delta/2$ which is a contradiction. This proves that the function g is well defined and continuous on the closed set $\overline{T(B_1) \cup T(B_2)}$. Now, by the Tietze theorem we can construct a continuous extension, which we shall denote also by g, on the space $\mathbb{R}^n \times c_0(\mathbb{N})$. Notice that the above arguments imply that $b|_{B_1 \cup B_2}$ is weakly (sequentially) uniformly continuous.

Finally, let us define $B(x) = g(T(x))$ for every $x \in X$. Then B is a continuous function and $B(x) = b(x)$ for every $x \in B_1 \cup B_2$.

Let us consider the general case of p balls. Since the function T defined in (2.1) is one-to-one, $\bigcap_{i \in I} T(B_i) = T(\bigcap_{i \in I} B_i)$ where $I \subset \{1, ..., p\}$. If $x \in \bigcap_{i \in I} T(B_i) = T(\bigcap_{i \in I} B_i)$ there exists $y \in \bigcap_{i \in I} B_i$ such that $T(y) = x$. Thus $b(y) = g^i(f_1(y), ..., f_n(y)) = g^i(\pi(x)) = g^j(f_1(y), ..., f_n(y)) = g^j(\pi(x))$ for every $i, j \in I$, where π is the projection of $\mathbb{R}^n \times c_0(\mathbb{N})$ onto $\mathbb{R}^n$ given by the n first coordinates. Let us define $g : \overline{\bigcup_{i=1}^p T(B_i)} \longrightarrow \mathbb{R}$ such that

$$g(x) = g^i(\pi(x)), \quad \text{if } x \in \overline{T(B_i)}.$$

Let us check that g is well defined and continuous in $\overline{\bigcup_{i=1}^p T(B_i)}$. Consider $z \in \bigcap_{i \in I} \overline{T(B_i)}$ where $I \subset \{1, ..., p\}$ and I has at least two elements. If i, $j \in I$ and $i \neq j$, it is enough to check that $g^i(\pi(x)) = g^j(\pi(x))$, whenever $x \in \overline{T(B_i) \cap T(B_j)}$. This equality is already proved for the case $p = 2$. Notice that the integer n considered in the case $p = 2$ for the two balls B_i and B_j is less or equal than the integer n considered in the general case of the p balls $B_1, \dots, B_p$, and thus the projections, both denoted as π, do not necessarily coincide. Nevertheless, this fact does not interfere in the proof, since we consider $g^k(\pi(x))$ as $g^k(p_k(x))$ in both case. Now, we can apply the Tietze theorem and find a continuous extension, which we shall denote also by g, defined on $\mathbb{R}^n \times c_0(\mathbb{N})$. Notice that the above arguments imply that $b|_{\cup_{i=1}^p B_i}$ is weakly (sequentially) uniformly continuous.

Finally, let us define $B(x) = g(T(x))$ for every $x \in X$. Then, B is a continuous function and $B(x) = b(x)$, for every $x \in \bigcup_{i=1}^p B_i$. □

Let us establish now the following characterization.

Theorem 2.2. *Let X be a separable Banach space. The following statements are equivalent:*

(1) *X admits a continuous, LFC bump.*
(2) *X admits a $\mathcal{C}^\infty$-smooth, LFC bump.*

Proof. We only need to prove (1) $\Rightarrow$ (2). Let $b : X \longrightarrow \mathbb{R}$ be a continuous, LFC bump. We can obtain, using a composition of b with a suitable real function, a continuous, LFC bump $b : X \longrightarrow [1, 2]$ such that $b(0) = 1$ and $b(x) = 2$ whenever $\|x\| \geq 1$. For every $x \in X$, there exist $r_x > 0$, $n_x \in \mathbb{N}$, functionals $\{f_1^x, ..., f_{n_x}^x\} \subset X^*$ and a continuous function $g^x : \mathbb{R}^{n_x} \longrightarrow \mathbb{R}$ such that

$$b(y) = g^x(f_1^x(y), ..., f_{n_x}^x(y)), \qquad \text{for every } y \in B(x, 2r_x).$$

Since X is separable, there exists a sequence of points $\{x_m\}_{m=1}^\infty \subset X$ such that $X = \bigcup_{m\in\mathbb{N}} B_m$ (where $r_m = r_{x_m}$ and $B_m = B(x_m, r_m)$). We can assume that $0 \in B_1$ and define the increasing sequence of open sets $V_j := B_1 \cup ... \cup B_j$. We know by a result of Fabian and Zizler [3] that, under our assumptions, X^* is separable. From Lemma 2.1, we obtain for every $j \in \mathbb{N}$, a continuous linear map $T_j : X \longrightarrow c_0(\mathbb{N})$ and a continuous function $g_j : c_0(\mathbb{N}) \longrightarrow \mathbb{R}$ such that $b(x) = g_j(T_j(x))$ for every $x \in V_j$.

Following the construction given by Hájek and Johanis in [8], let us choose two sequences of real numbers ε_j and η_j decreasing to 0 and 1 respectively, $0 < \varepsilon_j < \frac{1}{4}(\eta_j - \eta_{j+1})$ with $\eta_1 < 1 + \frac{1}{4}$ and $\varepsilon_1 < \frac{1}{8}$. We can uniformly approximate the continuous function $\eta_j\, g_j$ in $c_0(\mathbb{N})$ by a $\mathcal{C}^\infty$-smooth and LFC-$\{e_i^*\}$ function [16], which we shall denote by h_j, satisfying

$$|h_j(x) - \eta_j\, g_j(x)| < \varepsilon_j, \qquad \text{for every } x \in c_0(\mathbb{N}).$$

Let us define $H_j : X \longrightarrow \mathbb{R}$, $H_j(x) = h_j(T_j(x))$, for every $x \in X$ and $j \in \mathbb{N}$. Since T_j is linear and continuous and h_j is $\mathcal{C}^\infty$-smooth and LFC, we can easily deduce that H_j is $\mathcal{C}^\infty$-smooth and LFC. Indeed, for every $x \in X$, let us consider $V \subset c_0(\mathbb{N})$ a neighborhood of $T_j(x)$, a natural number s, a function $p : \mathbb{R}^s \longrightarrow \mathbb{R}$ and $\{e_1^*, ..., e_s^*\}$ (coordinate functionals on $c_0(\mathbb{N})$) such that $h_j(y) = p(e_1^*(y), ..., e_s^*(y))$ for every $y \in V$. Since T_j is continuous, the set $W = T_j^{-1}(V)$ is a neighborhood of x on X. Then $H_j(z) = p(e_1^* \circ T_j(z), ..., e_s^* \circ T_j(z))$ for all $z \in W$. Because $e_i^* \circ T_j \in X^*$, we conclude that H_j is LFC. In addition, we have

$$|H_j(x) - \eta_j\, b(x)| < \varepsilon_j, \qquad \text{for every } x \in V_j.$$

Let us define

$$\Phi : X \longrightarrow \ell_\infty(\mathbb{N}), \;\; \Phi(x) = (H_j(x))_j.$$

Φ is well defined since $\lim_j H_j(x) = b(x)$ for every $x \in X$. Let us check that Φ is continuous. Consider $x \in X$ and $\varepsilon > 0$. Since b is continuous, there is $\delta > 0$ such that $|b(x) - b(y)| < \frac{\varepsilon}{4}$ whenever $||x - y|| < \delta$. In addition, there exists $j_0 \in \mathbb{N}$ such that if $j \geq j_0$, then $x \in V_j$ and $\varepsilon_j < \frac{\varepsilon}{4}$. Thus, for every $y \in V_{j_0}$ with $||x - y|| < \delta$, we have

$$|H_j(x) - H_j(y)| \leq |H_j(x) - \eta_j b(x)| + \eta_j |b(x) - b(y)| + |\eta_j b(y) - H_j(y)| \leq 2\varepsilon_j + \eta_j \frac{\varepsilon}{4} < \varepsilon,$$

whenever $j \geq j_0$. From the above inequality and the fact that $H_1, \ldots, H_{j_0}$ are continuous at x, we can easily deduce the continuity of Φ at x.

Let us consider the open subset U of $\ell_\infty(\mathbb{N})$,

$$U = \{x \in \ell_\infty(\mathbb{N}) : |x_{j_0}| - \varepsilon_{j_0} > \sup_{j > j_0} |x_j| + \varepsilon_{j_0} \text{ for some } j_0 \in \mathbb{N}\}.$$

Let us prove that $\varphi(X) \subset U$. If $x \in V_{j_0}$ for some j_0 and $j > j_0$, we have

$$H_{j_0}(x) - \varepsilon_{j_0} > \eta_{j_0} b(x) - 2\varepsilon_{j_0} > \eta_{j_0+1} b(x) + 2\varepsilon_{j_0} > (\eta_j b(x) + \varepsilon_j) + \varepsilon_{j_0} > H_j(x) + \varepsilon_{j_0}$$

and thus $\Phi(X) \subset U$. By [8, Lemma 13], there exists a C^∞-smooth and LFC-$\{e_i^*\}$ function $F : U \to (0, \infty)$ (where $\{e_i^*\}$ are the coordinate functionals on $\ell_\infty(\mathbb{N})$) satisfying $||x||_\infty \leq F(x) \leq ||x||_\infty + \varepsilon_1$. Then the composition function defined as

$$B : X \longrightarrow \mathbb{R},\ B(x) = F(\Phi(x))$$

is $\mathcal{C}^\infty$-smooth and LFC. In addition,

(a) Since $0 \in V_j$ for every $j \in \mathbb{N}$, $H_j(0) < \eta_j \cdot b(0) + \varepsilon_j \leq \eta_1 + \varepsilon_1$. Thus, $B(0) \leq ||\Phi(x)||_\infty + \varepsilon_1 \leq \eta_1 + 2\varepsilon_1 \leq \frac{3}{2}$.

(b) If $\|x\| \geq 1$ and $j_0 \in \mathbb{N}$ verifies $x \in V_{j_0}$, then $H_{j_0}(x) > \eta_{j_0} b(x) - \varepsilon_{j_0} \geq 2\eta_{j_0} - \varepsilon_{j_0} > 2 - \varepsilon_1$ and $B(x) \geq ||\Phi(x)||_\infty > 2 - \varepsilon_1 \geq \frac{15}{8}$.

Therefore B is a separating function on X and by composing it with a suitable C^∞-smooth, real function we obtain a $\mathcal{C}^\infty$-smooth, LFC bump on X. □

Lemma 2.1 and Theorem 2.2 can be generalized using the concept of locally factorized functions.

Definition 2.3. *Let X, E and Y be Banach spaces, $A \subset X$ an open subset, $\mathcal{F}$ a family of Banach spaces and $b : A \to Y$ a continuous mapping.*

(a) *We say that b is factorized by E on a subset $U \subset A$ if there exists a continuous, linear map $T : X \longrightarrow E$ and a continuous function $G : E \longrightarrow Y$ such that $b(x) = G(T(x))$ for all $x \in U$.*

(b) *We say that b is locally factorized by E (b is LF-E, for short) if for each $x \in A$ there exists a neighbourhod $U_x \subset A$ of x such that b is factorized by E on U_x.*

(c) *We say that b is locally factorized by $\mathcal{F}$ (b is LF-$\mathcal{F}$, for short) if for each $x \in A$ there are a neighbourhood $U_x \subset A$ of x and a Banach space $E_x \in \mathcal{F}$ such that b is factorized by E_x on U_x.*

Every continuous, LFC function is LF-c_0. However, there exist LF-$\mathcal{F}$ functions that are not LFC. For a Banach space E with norm $||\cdot||$, let us consider $X = \sum_{c_0} E = \{(x_n)_{n=1}^\infty : x_n \in E \text{ and } \lim_n ||x_n|| = 0\}$ with the norm $||x|| = \sup\{||x_n|| : n \in \mathbb{N}\}$, for every $x \in X$. It can be readily verified that the norm in X is LF-E. Moreover, if $E = \ell_p$ with $1 \leq p \leq \infty$, then the norm in X is LF-ℓ_p. However, note that in this case, X does not admit a continuous, LFC bump, because $\sum_{c_0} E$ is not c_0-saturated. With the same arguments employed in Lemma 2.1 and Theorem 2.2, we can show the following more general statement.

Proposition 2.4. *Let X, E be separable Banach spaces and $\mathcal{F}$ a family of separable Banach spaces such that X admits a continuous, LF-E (LF-$\mathcal{F}$) bump. Assume that X^* is separable and E (every $E \in \mathcal{F}$, respectively) admits a bump function b satisfying one of the following properties:*

(1) *b is $\mathcal{C}^k$-smooth, where $k \in \mathbb{N} \cup \{\infty\}$,*
(2) *b is continuous and LFC,*
(3) *b is LFC and $\mathcal{C}^k$-smooth, where $k \in \mathbb{N} \cup \{\infty\}$.*

Then, X admits a bump function satisfying the same property.

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