ON SMOOTH EXTENSIONS OF VECTOR-VALUED FUNCTIONS DEFINED ON CLOSED SUBSETS OF BANACH SPACES

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ABSTRACT. Let X and Z be Banach spaces, A a closed subset of X and a mapping $f : A \to Z$. We give necessary and sufficient conditions to obtain a C^1 smooth mapping $F : X \to Z$ such that $F_{|_A} = f$, when either (i) X and Z are Hilbert spaces and X is separable, or (ii) X^* is separable and Z is an absolute Lipschitz retract, or (iii) $X = L_2$ and $Z = L_p$ with $1 < p < 2$, or (iv) $X = L_p$ and $Z = L_2$ with $2 < p < \infty$, where L_p is any separable Banach space $L_p(S, \Sigma, \mu)$ with (S, Σ, μ) a σ-finite measure space.

1. Introduction

In this note, we study how the techniques given in [2] and [12] can be applied to obtain a C^1 smooth extension of a C^1 smooth and *vector-valued* function defined on a closed subset of a Banach space. More precisely, if X and Z are Banach spaces, A is a closed subset of X and $f : A \to Z$ is a mapping, under what conditions does there exist a C^1 smooth mapping $F : X \to Z$ such that the restriction of F to A is f? This note is the second part of our previous paper [12], where we studied the problem of the extension of a C^1 smooth and *real-valued* function defined on a closed subset of a non-separable Banach space X to a C^1 smooth function defined on X.

The notation we use is standard. In addition, we shall follow, whenever possible, the notations given in [2] and [12]. We shall denote a norm in a Banach space X by $|| \cdot ||_X$ (or $|| \cdot ||$ if the Banach space X is understood). We denote by $B_X(x, r)$ the open ball with center $x \in X$ and radius $r > 0$ (or $B(x, r)$ if the Banach space X is understood). We write the closed ball as $\overline{B}_X(x, r)$ (or $\overline{B}(x, r)$). We denote by $\mathcal{L}(X, Z)$ the space of all bounded and linear maps from the Banach space X to the Banach space Z. If A is a subset of X, the restriction of a mapping $f : X \to Z$ to A is denoted by $f_{|_A}$. We say that $G : X \to Z$ is an extension of $g : A \to Z$ if $G_{|_A} = g$. A mapping $f : A \to Z$ is L-Lipschitz whether $||f(x) - f(y)|| \leq L||x - y||$ for every $x, y \in A$ and the Lipschitz constant of f is $\text{Lip}(f) := \sup\{\frac{||f(x)-f(y)||}{||x-y||} : x, y \in A,\, x \neq y\}$.

The C^1 smooth extension problem for real-valued functions defined on a subset of an infinite-dimensional Banach space has been recently studied in [2], where it has been shown that, if X is a Banach space with separable dual, $Y \subset X$ is a closed subspace of X and $f : Y \to \mathbb{R}$ is a C^1 smooth function, then there exists a

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C^1 smooth extension of f to X. Also, a detailed review on the theory of smooth extensions is provided in [2]. A generalization of the results in [2] was given in [12] for non-separable Banach spaces, whenever the spaces satisfy a certain approximation property (property (*) for $(X,\mathbb{R})$; see definition below). When X satisfies this approximation property, A is a closed subset of X and $f : A \to \mathbb{R}$ is a function, the existence of a C^1 smooth extension of f is characterized by the following property (called "condition (E)" in [12]):

Definition 1.1. *Let X and Z be Banach spaces and $A \subset X$ a closed subset.*

(1) *We say that the mapping $f : A \to Z$ satisfies* ***the mean value condition*** *if there exists a continuous map $D : A \to \mathcal{L}(X,Z)$ such that for every $y \in A$ and every $\varepsilon > 0$, there is an open ball $B(y,r)$ in X such that*

$$||f(z) - f(w) - D(y)(z - w)|| \leq \varepsilon ||z - w||,$$

for every $z, w \in A \cap B(y,r)$. In this case, we say that f satisfies the mean value condition on A for the map D.

(2) *We say that the mapping $f : A \to Z$ satisfies* ***the mean value condition for a bounded map*** *if it satisfies the mean value condition for a bounded and continuous map $D : A \to \mathcal{L}(X,Z)$, i.e. $\sup\{||D(y)|| : y \in A\} < \infty$.*

It is a straightforward consequence of the mean value theorem that, whenever $f : A \to Z$ is the restriction of a C^1 smooth mapping $F : X \to Z$ (C^1 smooth and Lipschitz mapping), then $f : A \to Z$ satisfies the mean value condition for $F'_{|A}$ (the mean value condition for the bounded map $F'_{|A}$, respectively).

In this note we adapt the proofs given in the real-valued case [2, 12] to obtain, under certain conditions, C^1 smooth extensions of C^1 smooth mappings $f : A \to Z$. More precisely, let us consider the following properties.

Definition 1.2. (1) *The pair of Banach spaces (X,Z) has* **property (*)** *if there is a constant $C_0 \geq 1$, which depends only on X and Z, such that for every subset $A \subset X$, every Lipschitz mapping $f : A \to Z$ and every $\varepsilon > 0$, there is a C^1 smooth and Lipschitz mapping $g : X \to Z$ such that $||f(x) - g(x)|| < \varepsilon$ for all $x \in A$ and $\mathrm{Lip}(g) \leq C_0 \,\mathrm{Lip}(f)$.*

(2) *The pair of Banach spaces (X,Z) has* **property (A)** *if there is a constant $C \geq 1$, which depends only on X and Z, such that for every Lipschitz mapping $f : X \to Z$ and every $\varepsilon > 0$, there exists a C^1 smooth and Lipschitz mapping $g : X \to Z$ such that $||f(x) - g(x)|| < \varepsilon$ for all $x \in X$ and $\mathrm{Lip}(g) \leq C \,\mathrm{Lip}(f)$.*

(3) *The pair of Banach spaces (X,Z) has* **property (E)** *if there is a constant $K \geq 1$, which depends only on X and Z, such that for every subset A of X and every Lipschitz mapping $f : A \to Z$, there exists a Lipschitz extension $F : X \to Z$ such that $\mathrm{Lip}(F) \leq K \,\mathrm{Lip}(f)$.*

(4) *A Banach space X has* **property (*)**, **property (A)** *or* **property (E)** *whenever the pair $(X,\mathbb{R})$ does.*

Remark 1.3. (1) *Clearly, a pair of Banach spaces (X,Z) satisfies property (A) whenever it satisfies property (*). In Section 2 we shall prove that, in general, these properties are not equivalent.*

(2) *It is easy to prove that a pair of Banach spaces* (X, Z) *satisfies property (*) provided that* (X, Z) *satisfies properties (A) and (E). Moreover, if* Z *is a dual Banach space, then* (X, Z) *satisfies property (*) if and only if* (X, Z) *satisfies properties (A) and (E). Indeed, let us assume that* (X, Z) *satisfies property (*) and consider a Lipschitz mapping* $f : A \to Z$*, where* A *is a subset of* X*. Then, for every* $n \in \mathbb{N}$*, there is a* C^1 *smooth, Lipschitz mapping* $f_n : X \to Z$ *such that* $||f(x) - f_n(x)|| \leq \frac{1}{n}$ *for every* $x \in A$ *and* $\mathrm{Lip}(f_n) \leq C_0 \,\mathrm{Lip}(f)$*. Then, for every* $x \in X$*, the sequence* $\{f_n(x)\}_n$ *is bounded. Since the closed balls in* $(Z, || \cdot ||^*)$ *are weak*-compact, there exists for every free ultrafilter* $\mathcal{U}$ *in* $\mathbb{N}$*, the weak*-limit*

$$\widehat{f}(x) := w^* - \lim_{\mathcal{U}} f_n(x).$$

Clearly, $\widehat{f} : X \to Z$ *is an extension of* $f : A \to Z$ *and* $\mathrm{Lip}(\widehat{f}) \leq C_0 \,\mathrm{Lip}(f)$*.*

(3) *It is worth mentioning that property (E) can be obtained from the following approximation property: for every subset* $A \subset X$*, every Lipschitz function* $f : A \to Z$ *and every* $\varepsilon > 0$*, there exists a Lipschitz mapping* $g : X \to Z$ *such that* $||f(x) - g(x)|| < \varepsilon$ *for every* $x \in A$ *and* $\mathrm{Lip}(g) \leq (1 + \varepsilon) \,\mathrm{Lip}(f)$*. In particular, a pair of Banach spaces* (X, Z) *has property (E), whenever* (X, Z) *satisfies property (A) with constant* $C_0 = 1 + \varepsilon$ *for any* $\varepsilon > 0$*.*

(4) X *satisfies property (*) if and only if it satisfies property (A). Indeed, this is a consequence of the fact that* X *always has property (E): if* A *is a closed subset of* X *and* $f : A \to \mathbb{R}$ *is a Lipschitz function, then the function* F *defined on* X *as*

$$F(x) = \inf_{a \in A} \{f(a) + \mathrm{Lip}(f) ||x - a||\}$$

is a Lipschitz extension of f *to* X *and* $\mathrm{Lip}(F) = \mathrm{Lip}(f)$*.*

In Section 2 we shall give some examples of pairs of Banach spaces (X, Z) satisfying property (*). In particular, when either (i) X and Z are Hilbert spaces with X separable, or (ii) X^* is separable and Z is a Banach space which is an absolute Lipschitz retract, or (iii) $X = L_2$ and $Z = L_p$ with $1 < p < 2$, or (iv) $X = L_p$ and $Z = L_2$ with $2 < p < \infty$. Throughout this paper, the space L_p denotes any separable Banach space $L_p(S, \Sigma, \mu)$ with (S, Σ, μ) a σ-finite measure space. We also give an example of a pair of Banach spaces satisfying property (A) but not property (*).

In Section 3, it is stated that, if the pair of Banach spaces (X, Z) satisfies property (*), then every mapping $f : A \to Z$, where A is a closed subset of X, is the restriction of a C^1 smooth mapping (C^1 smooth and Lipschitz mapping) $F : X \to Z$ if and only if f satisfies the mean value condition (the mean value condition for a bounded map and f is Lipschitz, respectively). We also prove that property (*) is necessary in order to obtain the above C^1 smooth and Lipschitz extension result.

In Section 4, it is proved that every C^1 smooth mapping $f : Y \to Z$ defined on a closed subspace of X admits a C^1 smooth extension to X, whenever the pair of Banach spaces (X, Z) satisfies property (*) and every bounded and linear operator $T : Y \to Z$ can be extended to a bounded and linear operator on X. Moreover, we obtain some results on bounded and linear extension morphisms on the Banach space $C^1_L(X, Z)$ of all C^1 smooth and Lipschitz mappings $f : X \to Z$.

2. On the properties (*), (A) and (E)

In this section, we present examples of pairs of Banach spaces (X, Z) satisfying property (*). The first examples are pairs of Banach spaces satisfying properties (A) and (E) and thus property (*).

Example 2.1. *Let X and Z be Banach spaces such that X is finite dimensional. Then, the pair (X, Z) satisfies properties (A) and (E).* On the one hand, W.B. Johnson, J. Lindenstrauss and G. Schechtman have shown in [14] that every pair of Banach space (X, Z) with X n-dimensional satisfies property (E) with constant $K(n) \geq 1$ (which depends only on the dimension of X). On the other hand, the classical convolution techniques for smooth approximation in finite dimensional spaces provide property (A) for (X, Z).

Example 2.2. *Let X and Z be Hilbert spaces with X separable. Then (X, Z) satisfies the properties (A) and (E).* M.D. Kirszbraun has shown in [18] (see [4, Theorem 1.12]) that the pair (X, Z) satisfies property (E) with $K = 1$, whenever X and Z are Hilbert spaces. Also, R. Fry has proven in [9] (see also [11, Theorem H]) that (X, Z) satisfies property (A) when X is a separable Hilbert space.

Example 2.3. *The pairs (L_2, L_p) for $1 < p < 2$ and (L_p, L_2) for $2 < p < \infty$ satisfy properties (A) and (E).* K. Ball showed that for every $1 < p < 2$ the pair (L_2, L_p) satisfies property (E) with constant $K(p) \geq 1$ depending only on p [3]. I. G. Tsar'kov proved that for every $2 < p < \infty$ the pair (L_p, L_2) satisfies property (E) with constant $K(p) \geq 1$ depending only on p [28]. Also, the results in [9, Theorem 1] yield the fact that (X, Z) satisfies property (A).

Recall that a subset A of a metric space Z is called a *Lipschitz retract* of Z if there is a *Lipschitz retraction* from Z to A, i.e. there is a Lipschitz map $r : Z \to A$ such that $r_{|_A} = id_A$. A metric space Z is called an *absolute Lipschitz retract* if it is a Lipschitz retract of any metric space W containing Z. The spaces $(c_0(\mathbb{N}), ||\cdot||_\infty)$, $(\ell_\infty(\mathbb{N}), ||\cdot||_\infty)$ and $(C(K), ||\cdot||_\infty)$ for every compact metric space K are absolute Lipschitz retracts (see [4] and [19] for more information and examples of absolute Lipschitz retracts). An absolute Lipschitz retract space satisfies the following Lipschitz extension property.

Proposition 2.4. [4, Proposition 1.2] *Let Z be a metric space. The following are equivalent:*

- (i) *Z is an absolute Lipschitz retract.*
- (ii) *There is $K \geq 1$, which only depends on Z, such that for every metric space X, every subset $A \subset X$ and every Lipschitz mapping $f : A \to Z$, there is a Lipschitz extension $F : X \to Z$ of f such that $\operatorname{Lip}(F) \leq K \operatorname{Lip}(f)$.*

By combining the above characterization and the results in [11], we obtain the following proposition.

Proposition 2.5. *Let X be a Banach space such that there are a set $\Gamma \neq \emptyset$ and a bi-Lipschitz homeomorphism $\varphi : X \to c_0(\Gamma)$ with C^1 smooth coordinate functions $e^*_\gamma \circ \varphi : X \to \mathbb{R}$. Let Z be a Banach space which is an absolute Lipschitz retract. Then the pair (X, Z) satisfies properties (A) and (E).*

Proof. Let us take the mapping $f \circ \varphi^{-1} : \varphi(X) \to Z$ which is $\mathrm{Lip}(\varphi^{-1})\,\mathrm{Lip}(f)$-Lipschitz. By Proposition 2.4, there is a Lipschitz extension $\widetilde{f} : c_0(\Gamma) \to Z$ of $f \circ \varphi^{-1}$ with $\mathrm{Lip}(\widetilde{f}) \leq K\,\mathrm{Lip}(\varphi^{-1})\,\mathrm{Lip}(f)$ and K is the constant given in Proposition 2.4. Now, from the results in [10] we can find a C^∞ smooth and Lipschitz mapping $h : c_0(\Gamma) \to Z$ which locally depends on finitely many coordinate functionals $\{e^*_\gamma\}_{\gamma\in\Gamma}$, such that $||\widetilde{f}(x) - h(x)|| < \varepsilon$ and $\mathrm{Lip}(h) = \mathrm{Lip}(\widetilde{f})$. Let us define $g : X \to Z$ as $g(x) := h(\varphi(x))$ for every $x \in X$. The mapping g is C^1 smooth, $||f(x) - g(x)|| < \varepsilon$ for all $x \in X$ and $\mathrm{Lip}(g) \leq C\,\mathrm{Lip}(f)$, with $C := K\,\mathrm{Lip}(\varphi)\,\mathrm{Lip}(\varphi^{-1})$. □

This provides the following example.

Example 2.6. *Let X and Z be Banach spaces such that X^* is separable and Z is an absolute Lipschitz retract. Then, the pair (X, Z) satisfies properties (A) and (E).* Notice that P. Hájek and M. Johanis [11] proved the existence of a bi-Lipschitz homeomorphism with C^k smooth coordinate functions in every separable Banach space with a C^k smooth and Lipschitz bump function.

We also obtain an example of a pair of Banach spaces satisfying property (A) but not property (*).

Example 2.7. *Although the pairs (L_p, L_2) and (L_2, L_q) with $1 < p < 2$ and $2 < q < \infty$ satisfy property (A) (see [9]), they do not satisfy property (E) whenever L_p, L_q and L_2 are infinite dimensional [24].* Thus, Remark 1.3 (2) implies that the pairs (L_p, L_2) and (L_2, L_q) do not satisfy property (*) whenever $1 < p < 2 < q < \infty$ and L_p, L_q and L_2 are infinite dimensional. Therefore, properties (A) and (*) are not equivalent in general.

Next, let us prove that property (*) is necessary to obtain certain C^1 smooth and Lipschitz extensions.

Proposition 2.8. *Let (X, Z) be a pair of Banach spaces such that there is a constant $C \geq 1$, which only depends on X and Z, such that for every closed subset $A \subset X$ and every Lipschitz mapping $f : A \to Z$ satisfying the mean value condition for a bounded map D with $M = \sup\{||D(y)|| : y \in A\} < \infty$, there exists a C^1 smooth and Lipschitz extension G of f to X with $\mathrm{Lip}(G) \leq C(M + \mathrm{Lip}(f))$. Then, the pair (X, Z) satisfies property (*).*

Proof. Let A be a subset of X, $f : A \to Z$ an L-Lipschitz mapping and $\varepsilon > 0$. Let us take a $\frac{\varepsilon}{(C+1)L}$-net in A which we shall denote by N, i.e. a subset N of A such that (i) $||z - y|| \geq \frac{\varepsilon}{(C+1)L}$ for every $z, y \in N$, (ii) for every $x \in A$ there is a point $y \in N$ such that $||x - y|| \leq \frac{\varepsilon}{(C+1)L}$. Clearly, N is a closed subset of X and $f_{|_N} : N \to Z$ is an L-Lipschitz mapping on N satisfying the mean value condition for the bounded map given by $D(x) = 0 \in \mathcal{L}(X, Z)$ for every $x \in N$. Then, by assumption, there exists a C^1 smooth and CL-Lipschitz mapping $G : X \to Z$ such that $G_{|_N} = f_{|_N}$. For any $x \in A$, let us choose $y \in N$ such that $||x - y|| \leq \frac{\varepsilon}{(C+1)L}$. Then, $G(y) = f(y)$ and

$$||f(x) - G(x)|| \leq ||f(x) - f(y)|| + ||G(x) - G(y)|| \leq (L + CL)||x - y|| \leq \varepsilon.$$

□

3. On smooth extension of mappings

The main results of this note are the following theorems.

Theorem 3.1. *Let (X,Z) be a pair of Banach spaces with property (*), $A \subset X$ a closed subset of X and a mapping $f : A \to Z$. Then, f satisfies the mean value condition if and only if there is a C^1 smooth extension G of f to X.*

Theorem 3.2. *Let (X,Z) be a pair of Banach spaces with property (*), $A \subset X$ a closed subset of X and a mapping $f : A \to Z$. Then, f is Lipschitz and satisfies the mean value condition for a bounded map if and only if there is a C^1 smooth and Lipschitz extension G of f to X.*

Moreover, if f is Lipschitz and satisfies the mean value condition for a bounded map $D : A \to \mathcal{L}(X,Z)$ with $M := \sup\{||D(y)|| : y \in A\} < \infty$, then we can obtain a C^1 smooth and Lipschitz extension G with $\mathrm{Lip}(G) \leq (1 + C_0)(M + \mathrm{Lip}(f))$, where C_0 is the constant given by property (*) (which depends only on X and Z).

Remark 3.3. *By Theorem 3.2 and Proposition 2.8 we obtain that property (*) is equivalent to the following property: for every closed subset $A \subset X$ and every Lipschitz mapping $f : A \to Z$ satisfying the mean value condition for a bounded map D with $M = \sup\{||D(y)|| : y \in A\} < \infty$, there exists a C^1 smooth and Lipschitz extension G of f to X with $\mathrm{Lip}(G) \leq C(M + \mathrm{Lip}(f))$, where $C \geq 1$ depends only on X and Z.*

Moreover, Example 2.7 and Proposition 2.8 reveal that there exist Lipschitz mappings $h : A \to L_2$ and $h' : B \to L_q$ defined on closed subsets A of L_p and B of L_2 with $1 < p < 2$ and $2 < q < \infty$ satisfying the mean value condition on A and B for a bounded map which cannot be extended to C^1 smooth and Lipschitz mappings on L_p and L_2, respectively, i.e. the conclusion of Theorem 3.2 does not hold for the pairs (L_p, L_2) and (L_2, L_q) with $1 < p < 2$ and $2 < q < \infty$. In particular, property (A) is a necessary condition but it is not a sufficient condition to obtain the conclusion of Theorem 3.2.

Before giving a proof of Theorems 3.1 and 3.2, we shall give some consequences. From the examples of Section 2, Theorem 3.1 and Theorem 3.2, we obtain the following corollary. Recall that L_p denotes any separable Banach space $L_p(S, \Sigma, \mu)$ with (S, Σ, μ) a σ-finite measure space.

Corollary 3.4. *Let X and Z be Banach spaces and assume that at least one of the following conditions holds:*

- (i) *X is finite dimensional,*
- (ii) *X and Z are Hilbert spaces and X is separable,*
- (iii) *$X = L_2$ and $Z = L_p$ with $1 < p < 2$,*
- (iv) *$X = L_p$ and $Z = L_2$ with $2 < p < \infty$,*
- (v) *there are a set $\Gamma \neq \emptyset$ and a bi-Lipschitz homeomorphism $\varphi : X \to c_0(\Gamma)$ with C^1 smooth coordinate functions (for example, when X^* is separable), and Z is an absolute Lipschitz retract.*

Let A be a closed subset of X and $f : A \to Z$ a mapping. Then, f satisfies the mean value condition (mean value condition for a bounded map and f is Lipschitz) on A if and only if there is a C^1 smooth (C^1 smooth and Lipschitz, respectively) extension G of f to X.

Moreover, if f is Lipschitz and satisfies the mean value condition for a bounded map $D : A \to \mathcal{L}(X, Z)$ with $M := \sup\{||D(y)|| : y \in A\} < \infty$, then we can obtain a C^1 smooth and Lipschitz extension G with $\operatorname{Lip}(G) \leq (1 + C_0)(M + \operatorname{Lip}(f))$, where $C_0 \geq 1$ is the constant given by property (*) (which depends only on X and Z).

Let us now turn to the proofs of the Theorems 3.1 and 3.2, which follow the ideas of the real-valued case. First of all, let us notice that if the pair (X, Z) satisfies property (*), X does too, i.e. there is a constant $C_0 \geq 1$ (which depends only on X) such that for every subset $A \subset X$, every Lipschitz function $f : A \to \mathbb{R}$ and every $\varepsilon > 0$, there is a C^1 smooth and Lipschitz function $g : X \to \mathbb{R}$ such that $|g(x) - f(x)| < \varepsilon$ for all $x \in A$ and $\operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f)$. Indeed, let us take $e \in Z$ with $||e|| = 1$ and $\varphi \in Z^*$ with $||\varphi|| = 1$ and $\varphi(e) = 1$. Let $f : A \to \mathbb{R}$ be an L-Lipschitz function and $\varepsilon > 0$. The mapping $h : A \to Z$ defined as $h(x) = f(x)e$ for all $x \in A$, is L-Lipschitz. Since the pair (X, Z) satisfies property (*), there exists a C^1 smooth and Lipschitz mapping $\widetilde{g} : X \to Z$ with $||h(x) - \widetilde{g}(x)|| < \varepsilon$ for all $x \in A$ and $\operatorname{Lip}(\widetilde{g}) \leq C_0 L$. The required function $g : X \to \mathbb{R}$ can be defined as $g(x) := \varphi(\widetilde{g}(x))$. Next, we shall need the following lemmas.

Lemma 3.5. *Let (X, Z) be a pair of Banach spaces with property (*). Then, for every subset $A \subset X$, every Lipschitz mapping $f : A \to B_Z(0, R)$ (with $R \in (0, \infty)$) and every $\varepsilon > 0$, there is a C^1 smooth and Lipschitz mapping $h : X \to Z$ such that*

(i) $||f(x) - h(x)|| < \varepsilon$ *for every* $x \in A$,
(ii) $||h(x)|| < C_0 \operatorname{Lip}(f)^{1/2} + R + \varepsilon$ *for every* $x \in X$, *and*
(iii) $\operatorname{Lip}(h) \leq C_0((1 + 2C_0) \operatorname{Lip}(f) + 2(R + \varepsilon) \operatorname{Lip}(f)^{1/2})$.

Proof. Without loss of generality we may assume that $\operatorname{Lip}(f) > 0$. By property (*) there is a C^1 smooth and Lipschitz mapping $g : X \to Z$ such that

$$||f(x) - g(x)|| < \varepsilon \qquad \text{for all } x \in A, \text{ and } \quad \operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f).$$

Let us define $W := \{x \in X : \operatorname{dist}(x, \overline{A}) \geq \frac{1}{\operatorname{Lip}(f)^{1/2}}\}$. Since X satisfies property (*), there is a C^1 smooth function $h_A : X \to [0, 1]$ such that $h_A(x) = 1$ whenever $x \in \overline{A}$, $h_A(x) = 0$ whenever $x \in W$ and $\operatorname{Lip}(h_A) \leq 2C_0 \operatorname{Lip}(f)^{1/2}$. Let us define $h : X \to Z$ as $h(x) := g(x)h_A(x)$, which is C^1 smooth and $||f(x) - h(x)|| < \varepsilon$ for all $x \in A$ (recall that $h_A(x) = 1$ for all $x \in A$).

Since $h_A(x) = 0$ for all $x \in W$, we have that $h(x) = 0$ for all $x \in W$. Also, $||h(x)|| \leq ||g(x)|| \leq R + \varepsilon$ for all $x \in \overline{A}$. Now, for each $x \notin W$ there is $x_0 \in \overline{A}$ such that $||x - x_0|| < \frac{1}{\operatorname{Lip}(f)^{1/2}}$ and thus,

$$||g(x)|| \leq ||g(x) - g(x_0)|| + ||g(x_0)|| \leq C_0 \operatorname{Lip}(f)||x - x_0|| + R + \varepsilon < C_0 \operatorname{Lip}(f)^{1/2} + R + \varepsilon.$$

Therefore, $||h(x)|| < C_0 \operatorname{Lip}(f)^{1/2} + R + \varepsilon$ for every $x \in X$. Now, if $x \in \operatorname{int}(W)$, then $h'(x) = 0$. Also, if $x \notin \operatorname{int}(W)$, then

$$\begin{aligned} ||h'(x)|| \leq & ||g'(x)|| \, ||h_A(x)|| + ||h_A'(x)|| \, ||g(x)|| \\ \leq & C_0 \operatorname{Lip}(f) + 2C_0 \operatorname{Lip}(f)^{1/2}(C_0 \operatorname{Lip}(f)^{1/2} + R + \varepsilon) \\ \leq & C_0((1 + 2C_0) \operatorname{Lip}(f) + 2(R + \varepsilon) \operatorname{Lip}(f)^{1/2}). \end{aligned}$$

Thus, $\operatorname{Lip}(h) \leq C_0((1 + 2C_0) \operatorname{Lip}(f) + 2(R + \varepsilon) \operatorname{Lip}(f)^{1/2})$. □

Lemma 3.6. *Let (X,Z) be a pair of Banach spaces with property (*). Then, for every subset $A \subset X$, every continuous mapping $F : X \to Z$ such that $F_{|_A}$ is Lipschitz, and every $\varepsilon > 0$, there exists a C^1 smooth mapping $G : X \to Z$ such that*

(i) $||F(x) - G(x)|| < \varepsilon$ *for all* $x \in X$,
(ii) $\operatorname{Lip}(G_{|_A}) \leq C_0 \operatorname{Lip}(F_{|_A})$. *Moreover,* $||G'(y)|| \leq C_0 \operatorname{Lip}(F_{|_A})$ *for all* $y \in A$, *where C_0 is the constant given by property (*).*
(iii) *In addition, if F is Lipschitz, then there exists a constant $C_1 \geq C_0$ depending only on X and Z, such that the mapping G can be chosen to be Lipschitz on X and* $\operatorname{Lip}(G) \leq C_1 \operatorname{Lip}(F)$.

Sketch of the proof. The proof is similar to the real-valued case (see [12, Lemma 2.3]). Let us outline the Lipschitz case. Let us apply property (*) to F and $F_{|_A}$ to obtain C^1 smooth and Lipschitz mappings $g,\ h : X \to Z$ such that

(a) $||F(x) - g(x)|| < \varepsilon/4$ for all $x \in A$,
(b) $||F(x) - h(x)|| < \varepsilon$ for all $x \in X$,
(c) $\operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(F_{|_A})$ and $\operatorname{Lip}(h) \leq C_0 \operatorname{Lip}(F)$.

There is a C^1 smooth and Lipschitz function $u : X \to [0,1]$ such that $u(x) = 1$ whenever $||F(x) - g(x)|| \leq \varepsilon/4$ and $u(x) = 0$ whenever $||F(x) - g(x)|| \geq \varepsilon/2$, with $\operatorname{Lip}(u) \leq \frac{9C_0(\operatorname{Lip}(F) + C_0 \operatorname{Lip}(F_{|_A}))}{\varepsilon}$ (see [12] for details). Then, the mapping $G : X \to Z$ defined as $G(x) = u(x)g(x) + (1-u(x))h(x)$ for every $x \in X$ is the required mapping with $C_1 := \frac{C_0}{2}(29 + 27C_0)$. □

Lemma 3.7. *Let (X,Z) be a pair of Banach spaces with property (*), a closed subset $A \subset X$ and a mapping $f : A \to B_Z(0,R)$ (with $R \in (0,\infty]$) satisfying the mean value condition for a map $D : A \to \mathcal{L}(X,Z)$. Then, for every $\varepsilon > 0$ there exists a C^1 smooth mapping $h : X \to B_Z(0, R+\varepsilon)$ such that*

(i) $||f(y) - h(y)|| < \varepsilon$ *for all* $y \in A$,
(ii) $||D(y) - h'(y)|| < \varepsilon$ *for all* $y \in A$, *and*
(iii) $\operatorname{Lip}(f - h_{|_A}) < \varepsilon$.

Proof. Since A is closed, by a vector-valued version of the Tietze theorem (see for instance [7, Theorem 6.1]) there is a continuous extension $F : X \to B_Z(0,R)$ of f. Since X is a Banach space, $A \subset X$ is a closed subset and f satisfies the mean value condition for $D : A \to \mathcal{L}(X,Z)$ on A, there exists $\{B(y_\gamma, r_\gamma)\}_{\gamma\in\Gamma}$ a covering of A by open balls of X, with centers $y_\gamma \in A$, such that

$$(3.1)\quad ||D(y) - D(y_\gamma)|| \leq \frac{\varepsilon}{8C_0} \quad \text{and} \quad ||f(z) - f(w) - D(y_\gamma)(z-w)|| \leq \frac{\varepsilon}{8C_0}||z-w||,$$

for every $y,z,w \in B_\gamma \cap A$, where $B_\gamma := B(y_\gamma, r_\gamma)$ and C_0 is the constant given by property (*).

Let us define $T_\gamma : X \to Z$ by $T_\gamma(x) = f(y_\gamma) + D(y_\gamma)(x - y_\gamma)$, for every $x \in X$. Notice that T_γ satisfies the following properties:

(B.1) T_γ is C^∞ smooth on X,
(B.2) $T'_\gamma(x) = D(y_\gamma)$ for all $x \in X$, and
(B.3) $\operatorname{Lip}((T_\gamma - F)|_{B_\gamma \cap A}) \leq \frac{\varepsilon}{8C_0}$, since for all $z,w \in B_\gamma \cap A$,

$$||(T_\gamma - F)(z) - (T_\gamma - F)(w)|| = ||f(w) - f(z) - D(y_\gamma)(w-z)|| \leq \frac{\varepsilon}{8C_0}||z-w||.$$

Recall that if (X,Z) has property (*), X does too and thus X admits C^1 smooth partitions of unity (see for instance [11] and [12]). Thus, since $F: X \to B_Z(0,R)$ is a continuous mapping, there is a C^1 smooth mapping $F_0 : X \to Z$ such that $||F(x) - F_0(x)|| < \frac{\varepsilon}{2}$ for every $x \in X$.

Let us denote $B_0 := X \setminus A$, $\Sigma := \Gamma \cup \{0\}$ (we assume $0 \notin \Gamma$), and $\mathcal{C} := \{B_\beta : \beta \in \Sigma\}$, which is a covering of X. By [25] and [12, Lemma 2.2], there are an open refinement $\{W_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ of $\mathcal{C} = \{B_\beta : \beta \in \Sigma\}$ and a C^1 smooth and Lipschitz partition of unity $\{\psi_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ satisfying:

(P1) $\operatorname{supp}(\psi_{n,\beta}) \subset W_{n,\beta} \subset B_\beta$;
(P2) $\operatorname{Lip}(\psi_{n,\beta}) \le C_0 2^5 (2^n - 1)$ for every $(n,\beta) \in \mathbb{N} \times \Sigma$; and
(P3) for each $x \in X$ there is an open ball $B(x, s_x)$ of X with center x and radius $s_x > 0$, and a natural number n_x such that
  (1) if $i > n_x$, then $B(x,s_x) \cap W_{i,\beta} = \emptyset$ for every $\beta \in \Sigma$,
  (2) if $i \le n_x$, then $B(x,s_x) \cap W_{i,\beta} \ne \emptyset$ for at most one $\beta \in \Sigma$.

Let us define $L_{n,\beta} := \max\{\operatorname{Lip}(\psi_{n,\beta}), 1\}$ for every $n \in \mathbb{N}$ and $\beta \in \Sigma$. Now, for every $n \in \mathbb{N}$ and $\gamma \in \Gamma$, we apply Lemma 3.6 to $T_\gamma - F$ on $B_\gamma \cap A$ to obtain a C^1 smooth mapping $\delta_{n,\gamma} : X \to Z$ so that

$$||T_\gamma(x) - F(x) - \delta_{n,\gamma}(x)|| < \frac{\varepsilon}{2^{n+2} L_{n,\gamma}} \quad \text{for every } x \in X, \tag{C.1}$$

$$\|\delta'_{n,\gamma}(y)\| \le \frac{\varepsilon}{8} \quad \text{for every } y \in B_\gamma \cap A \tag{C.2}$$

and

$$\operatorname{Lip}((\delta_{n,\gamma})_{|_{B_\gamma \cap A}}) \le \frac{\varepsilon}{8}. \tag{C.3}$$

From inequality (3.1), (B.2), (C.2) and (C.3), we have, for all $y \in B_\gamma \cap A$,

$$\|T'_\gamma(y) - D(y) - \delta'_{n,\gamma}(y)\| \le \|T'_\gamma(y) - D(y)\| + \|\delta'_{n,\gamma}(y)\| \le \frac{\varepsilon}{4},$$

and

$$\operatorname{Lip}((T_\gamma - F - \delta_{n,\gamma})_{|_{B_\gamma \cap A}}) \le \frac{\varepsilon}{4}.$$

Let us define $\Delta^n_\beta : X \to Z$,

$$\Delta^n_\beta(x) = \begin{cases} F_0(x) & \text{if } \beta = 0, \\ T_\beta(x) - \delta_{n,\beta}(x) & \text{if } \beta \in \Gamma. \end{cases} \tag{3.2}$$

Thus, $||F(x) - \Delta^n_\beta(x)|| < \frac{\varepsilon}{2}$ whenever $n \in \mathbb{N}$, $\beta \in \Sigma$ and $x \in X$. Now, we define

$$h(x) = \sum_{(n,\beta)\in\mathbb{N}\times\Sigma} \psi_{n,\beta}(x) \Delta^n_\beta(x).$$

Since $\{\psi_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ is locally finitely nonzero, h is C^1 smooth. Now, if $x \in X$, then

$$||F(x) - h(x)|| \le \sum_{(n,\beta)\in\mathbb{N}\times\Sigma} \psi_{n,\beta}(x) ||F(x) - \Delta^n_\beta(x)|| \le \sum_{(n,\beta)\in\mathbb{N}\times\Sigma} \psi_{n,\beta}(x) \frac{\varepsilon}{2} < \varepsilon.$$

Therefore, $||h(x)|| < R + \varepsilon$ for all $x \in X$ (recall that $||F(x)|| < R$ for all $x \in X$). Following the proof of [12, Theorem 3.3], it can be cheked that

$$\|D(y) - h'(y)\| < \varepsilon \text{ for all } y \in A \text{ and } \operatorname{Lip}(f - h_{|_A}) < \varepsilon.$$

□

Lemma 3.8. *Let (X,Z) be a pair of Banach spaces with property (*), a closed subset $A\subset X$ and a Lipschitz mapping $f:A\to B_Z(0,R)$ (with $R\in(0,\infty]$) satisfying the mean value condition for a bounded map $D:A\to\mathcal{L}(X,Z)$ with $M=\sup\{||D(y)||:y\in A\}<\infty$. Then, for every $\varepsilon>0$ there is a C^1 smooth and Lipschitz mapping $g:X\to Z$ such that*

(i) $||f(y)-g(y)||<\varepsilon$ *for every $y\in A$,*
(ii) $||D(y)-g'(y)||<\varepsilon$ *for every $y\in A$,*
(iii) $\operatorname{Lip}(f-g_{|A})<\varepsilon$,
(iv) $||g(x)||<C_0\operatorname{Lip}(f)^{1/2}+R+\varepsilon$ *for every $x\in X$,*
(v) $\operatorname{Lip}(g)\le C_0((1+2C_0)\operatorname{Lip}(f)+2(R+\varepsilon)\operatorname{Lip}(f)^{1/2}+M)+\varepsilon$ *whenever $R<\infty$, and $\operatorname{Lip}(g)\le C_0(M+\operatorname{Lip}(f))+\varepsilon$ whenever $R=+\infty$; where C_0 is the constant given by property (*).*

Proof. **A. Construction of g.** For the construction of g we will use again the partitions of unity given in [12, Lemma 2.2]. Let us take $0<3\varepsilon'<\varepsilon$. By Lemma 3.7 there is a C^1 smooth mapping $h:X\to B_Z(0,R+\varepsilon')$ such that

(i) $||f(y)-h(y)||<\varepsilon'$ for all $y\in A$,
(ii) $||D(y)-h'(y)||<\varepsilon'$ for all $y\in A$, and
(iii) $\operatorname{Lip}(f-h_{|A})<\min\{\frac{\varepsilon'}{C_0(1+2C_0)},(\frac{\varepsilon'}{2C_0(R+2\varepsilon')})^2\}$ whenever $R<\infty$, and $\operatorname{Lip}(f-h_{|A})<\frac{\varepsilon'}{C_0}$ whenever $R=\infty$.

Since h is C^1 smooth on X, there exists $\{B(y_\gamma,r_\gamma)\}_{\gamma\in\Gamma}$ a covering of A by open balls of X, with centers $y_\gamma\in A$ such that

$$||h(y)-h(y_\gamma)||\le\frac{\varepsilon'}{8C_0}\quad\text{and}\quad ||h'(y)-h'(y_\gamma)||\le\frac{\varepsilon'}{8C_0},\ \text{for every } y\in B_\gamma, \tag{3.3}$$

where $B_\gamma:=B(y_\gamma,r_\gamma)$ and C_0 is the constant given by property (*) (which depends only on X and Z). Let us define T_γ by $T_\gamma(x)=h(y_\gamma)+h'(y_\gamma)(x-y_\gamma)$, for $x\in X$. Notice that T_γ satisfies the following properties:

(B.1) T_γ is C^∞ smooth on X,
(B.2) $T'_\gamma(x)=h'(y_\gamma)$ for all $x\in X$,
(B.3) $\operatorname{Lip}((T_\gamma-h)|_{B_\gamma})\le\frac{\varepsilon'}{8C_0}$, and
(B.4) $||T'_\gamma(x)||=||h'(y_\gamma)||\le||D(y_\gamma)||+\varepsilon'\le M+\varepsilon'$ for every $x\in X$.

Let us define $B_0:=X\setminus A$, $\Sigma:=\Gamma\cup\{0\}$ (we assume $0\notin\Gamma$), and $\mathcal{C}:=\{B_\beta:\beta\in\Sigma\}$, which is an open covering of X. Following the proof of Lemma 3.7, we obtain an open refinement $\{W_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ of $\mathcal{C}=\{B_\beta:\beta\in\Sigma\}$ and a C^1 smooth and Lipschitz partition of unity $\{\psi_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ satisfying conditions (P1), (P2) and (P3).

Let us define $L_{n,\beta}:=\max\{\operatorname{Lip}(\psi_{n,\beta}),1\}$ for every $n\in\mathbb{N}$ and $\beta\in\Sigma$. Now, for every $n\in\mathbb{N}$ and $\gamma\in\Gamma$, we apply property (*) to $T_\gamma-h$ on B_γ in order to obtain a C^1 smooth mapping $\delta_{n,\gamma}:X\to Z$ so that

$$||T_\gamma(x)-h(x)-\delta_{n,\gamma}(x)||<\frac{\varepsilon'}{2^{n+2}L_{n,\gamma}}\qquad\text{for every } x\in B_\gamma\quad\text{and} \tag{C.1}$$

$$\operatorname{Lip}(\delta_{n,\gamma})\le C_0\operatorname{Lip}((T_\gamma-h)_{|B_\gamma})\le\frac{\varepsilon'}{8}. \tag{C.2}$$

In particular,

$$(3.4)\qquad ||T_\gamma(x)-\delta_{n,\gamma}(x)||<||h(x)||+\frac{\varepsilon'}{2^{n+2}L_{n,\gamma}}<R+2\varepsilon'\qquad \text{for every } x\in B_\gamma.$$

From inequality (3.3), (B.2) and (C.2) and for every $y\in B_\gamma$, we have

$$\|T'_\gamma(y)-h'(y)-\delta'_{n,\gamma}(y)\|\le\|T'_\gamma(y)-h'(y)\|+\|\delta'_{n,\gamma}(y)\|\le\frac{\varepsilon'}{4}.$$

Therefore,

$$\operatorname{Lip}((T_\gamma-h-\delta_{n,\gamma})_{|B_\gamma})\le\frac{\varepsilon'}{4}.$$

Now, for $R<\infty$, since $\operatorname{Lip}(h_{|A})\le\operatorname{Lip}(f)+(\varepsilon'/C_0)^2$, let us apply Lemma 3.5 to $h_{|A}:A\to B_Z(0,R+\varepsilon')$ to obtain C^1 smooth and Lipschitz mappings $F_0^n:X\to B_Z(0,C_0\operatorname{Lip}(f)^{1/2}+R+3\varepsilon')$ such that $||h(z)-F_0^n(z)||<\frac{\varepsilon'}{2^{n+2}L_{n,0}}$ for all $z\in A$ and $n\in\mathbb{N}$, and

$$\operatorname{Lip}(F_0^n)\le C_0((1+2C_0)\operatorname{Lip}(h_{|A})+2(R+2\varepsilon')\operatorname{Lip}(h_{|A})^{1/2}).$$

From condition (iii) above, we deduce

$$\operatorname{Lip}(F_0^n)\le C_0((1+2C_0)\operatorname{Lip}(f)+2(R+2\varepsilon')\operatorname{Lip}(f)^{1/2})+2\varepsilon'.$$

If $R=+\infty$, we apply property (*) to $h_{|A}:A\to Z$ in order to obtain C^1 smooth mappings $F_0^n:X\to Z$ such that $||h(x)-F_0^n(x)||<\frac{\varepsilon'}{2^{n+2}L_{0,\beta}}$ on A, and $\operatorname{Lip}(F_0^n)\le C_0\operatorname{Lip}(h_{|A})\le C_0\operatorname{Lip}(f)+\varepsilon'$.

Finally, let us define $\Delta_\beta^n:X\to Z$ and $g:X\to Z$ as

(3.5)

$$\Delta_\beta^n(x)=\begin{cases}F_0^n(x) & \text{if } \beta=0,\\ T_\beta(x)-\delta_{n,\beta}(x) & \text{if } \beta\in\Gamma,\end{cases}\qquad\text{and}\qquad g(x)=\sum_{(n,\beta)\in\mathbb{N}\times\Sigma}\psi_{n,\beta}(x)\Delta_\beta^n(x).$$

B. The function g satisfies (i)-(v). Since $\{\psi_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ is locally finitely nonzero, the mapping g is C^1 smooth. It is clear that

$$||g(x)||\le\sum_{(n,\beta)\in\mathbb{N}\times\Sigma}\psi_{n,\beta}(x)||\Delta_\beta^n(x)||<C_0\operatorname{Lip}(f)^{1/2}+R+\varepsilon\qquad\text{for all } x\in X.$$

The proofs of $||h(y)-g(y)||<\varepsilon'$, $||h'(y)-g'(y)||<\varepsilon'$ for all $y\in A$ and $\operatorname{Lip}((h-g)_{|A})<\varepsilon'$ follow along the same lines as [12, Theorem 3.3]. Thus, $||f(y)-g(y)||<\varepsilon$ for all $y\in A$, $||D(y)-g'(y)||<\varepsilon$ for $y\in A$, and $\operatorname{Lip}(f-g_{|A})<\varepsilon$.

In addition, since $||T'_\gamma(x)||+||\delta'_{n,\gamma}(x)||\le M+9\varepsilon'/8$, we have, for $R<\infty$,

$$\begin{aligned}||(\Delta_\beta^n)'(x)||\le&\max\{C_0((1+2C_0)\operatorname{Lip}(f)+2(R+2\varepsilon')\operatorname{Lip}(f)^{1/2})+2\varepsilon',M+9\varepsilon'/8\}\\ \le& C_0((1+2C_0)\operatorname{Lip}(f)+2(R+2\varepsilon')\operatorname{Lip}(f)^{1/2}+M)+2\varepsilon',\end{aligned}$$

and for $R=\infty$,

$$||(\Delta_\beta^n)'(x)||\le\max\{C_0\operatorname{Lip}(f)+\varepsilon',M+9\varepsilon'/8\}\le C_0(\operatorname{Lip}(f)+M)+9\varepsilon'/8.$$

Let us check that g is Lipschitz. From the fact that $\sum_{(n,\beta)\in F_x} \psi'_{n,\beta}(x) = 0$ for all $x \in X$, where $F_x := \{(n,\beta) \in \mathbb{N} \times \Sigma : x \in \operatorname{supp}(\psi_{n,\beta})\}$ and the fact (P3) we deduce that, for $R < \infty$,

$$||g'(x)|| \leq \sum_{(n,\beta)\in F_x} ||\psi'_{n,\beta}(x)|| \, ||h(x) - \Delta^n_\beta(x)|| + \sum_{(n,\beta)\in F_x} \psi_{n,\beta}(x)||(\Delta^n_\beta)'(x)||$$

$$\leq \sum_{\{n:(n,\beta(n))\in F_x\}} L_{n,\beta(n)} \, \frac{\varepsilon'}{2^{n+2} L_{n,\beta(n)}}$$

$$+ \sum_{\{n:(n,\beta(n))\in F_x\}} \psi_{n,\beta(n)}(x)(C_0((1+2C_0)\operatorname{Lip}(f) + 2(R+2\varepsilon')\operatorname{Lip}(f)^{1/2} + M) + 2\varepsilon')$$

$$< C_0((1+2C_0)\operatorname{Lip}(f) + 2(R+2\varepsilon')\operatorname{Lip}(f)^{1/2} + M) + 3\varepsilon',$$

for all $x \in X$, where $\beta(n)$ is the only index β (if it exists) satisfying condition (P3)-(2) for x. Thus, for $R < \infty$, $\operatorname{Lip}(g) \leq C_0((1+2C_0)\operatorname{Lip}(f) + 2(R+\varepsilon)\operatorname{Lip}(f)^{1/2} + M) + \varepsilon$. (Recall, that here we do not assume $\varepsilon < \operatorname{Lip}(f)$.)

Similarly, for $R = \infty$, it can be checked that $||g'(x)|| \leq C_0(\operatorname{Lip}(f) + M) + \varepsilon$ for every $x \in X$. □

Proofs of Theorems 3.1 and 3.2. Let us assume that the mapping $f : A \to Z$ satisfies the mean value condition and consider $0 < \varepsilon < 1$. Then, by Lemma 3.7 there exists a C^1 smooth mapping $G_1 : X \to Z$ such that if $g_1 := G_{1|_A}$, then

(i) $||f(y) - g_1(y)|| < \frac{\varepsilon}{2^4 C_0}$ for every $y \in A$,
(ii) $||D(y) - G'_1(y)|| < \frac{\varepsilon}{2^4 C_0}$ for every $y \in A$, and
(iii) $\operatorname{Lip}(f - g_1) < \min\{\frac{\varepsilon}{2^4 C_0(1+2C_0)}, (\frac{\varepsilon}{2^4 C_0})^2\}$.

Notice that the mapping $f - g_1$ satisfies the mean value condition for the bounded map $D - G'_1 : A \to \mathcal{L}(X, Z)$ with $\sup\{||D(y) - G'_1(y)|| : y \in A\} \leq \frac{\varepsilon}{2^4 C_0}$. Let us apply Lemma 3.8 to $f - g_1$ to obtain a C^1 smooth mapping $G_2 : X \to Z$ such that if $g_2 := G_{2|_A}$, then

(i) $||(f - g_1)(y) - g_2(y)|| < \frac{\varepsilon}{2^5 C_0}$ for every $y \in A$,
(ii) $||D(y) - (G'_1(y) + G'_2(y))|| < \frac{\varepsilon}{2^5 C_0}$ for every $y \in A$,
(iii) $\operatorname{Lip}(f - (g_1 + g_2)) < \min\{\frac{\varepsilon}{2^5 C_0(1+2C_0)}, (\frac{\varepsilon}{2^5 C_0})^2\}$,
(iv) $||G_2(x)|| \leq C_0 \frac{\varepsilon}{2^4 C_0} + \frac{\varepsilon}{2^4 C_0} + \frac{\varepsilon}{2^5 C_0} \leq \frac{\varepsilon}{2^2}$ for all $x \in X$, and
(v) $\operatorname{Lip}(G_2) \leq C_0((1+2C_0)\frac{\varepsilon}{2^4 C_0(1+2C_0)} + 2(\frac{\varepsilon}{2^4 C_0} + \frac{\varepsilon}{2^5 C_0})\frac{\varepsilon}{2^4 C_0} + \frac{\varepsilon}{2^4 C_0}) + \frac{\varepsilon}{2^5 C_0} \leq \frac{\varepsilon}{2^2}$.

By induction, we find a sequence $G_n : X \to Z$ of C^1 smooth mappings satisfying for $n \geq 2$, where $g_n := G_{n|_A}$,

(i) $||(f - \sum_{i=1}^{n-1} g_i)(y) - g_n(y)|| < \frac{\varepsilon}{2^{n+3} C_0}$ for every $y \in A$,
(ii) $||D(y) - \sum_{i=1}^{n} G'_i(y)|| < \frac{\varepsilon}{2^{n+3} C_0}$ for every $y \in A$,
(iii) $\operatorname{Lip}(f - \sum_{i=1}^{n} g_i) < \min\{\frac{\varepsilon}{2^{n+3} C_0(1+2C_0)}, (\frac{\varepsilon}{2^{n+3} C_0})^2\}$,
(iv) $||G_n(x)|| \leq \varepsilon/2^n$ for all $x \in X$, and
(v) $\operatorname{Lip}(G_n) \leq \varepsilon/2^n$.

It can be cheked as in [2] and [12] that the mapping $G : X \to Z$ defined as $G(x) := \sum_{n=1}^{\infty} G_n(x)$ is C^1 smooth and it is an extension of f to X.

Let us now consider $f : A \to Z$ a Lipschitz mapping satisfying the mean value condition for a bounded map $D : A \to \mathcal{L}(X, Z)$ with $M := \sup\{||D(y)|| : y \in A\} < \infty$. We can assume that $\varepsilon \leq \frac{16(M+\mathrm{Lip}(f))}{9}$ (if $\mathrm{Lip}(f) = 0$, the extension is obvious). By Lemma 3.8, there exists a C^1 smooth mapping $G_1 : X \to Z$ such that if $g_1 := G_{1|_A}$, then

(i) $||f(y) - g_1(y)|| < \frac{\varepsilon}{2^4 C_0}$ for every $y \in A$,
(ii) $||D(y) - G_1'(y)|| < \frac{\varepsilon}{2^4 C_0}$ for every $y \in A$,
(iii) $\mathrm{Lip}(f - g_1) < \min\{\frac{\varepsilon}{2^4 C_0(1+2C_0)}, (\frac{\varepsilon}{2^4 C_0})^2\}$, and
(iv) $\mathrm{Lip}(G_1) \leq C_0(M + \mathrm{Lip}(f)) + \frac{\varepsilon}{2^4}$.

The mappings $G_n : X \to Z$ for $n \geq 2$ are defined as in the general case. It can be checked that the mapping $G : X \to Z$ defined as $G(x) := \sum_{n=1}^{\infty} G_n(x)$ is C^1 smooth, is an extension of f to X and

$$\mathrm{Lip}(G) \leq C_0(M + \mathrm{Lip}(f)) + \frac{\varepsilon}{2^4} + \sum_{n=2}^{\infty} \frac{\varepsilon}{2^n} \leq (1 + C_0)(M + \mathrm{Lip}(f)).$$

□

Let us now give an application to C^1 Banach manifolds.

Definition 3.9. *Let us consider X and Z Banach spaces, a C^1 Banach manifold M modeled on the Banach space X, a subset A of M and a continuous map $f : M \to Z$. We say that the mapping $f : A \to Z$ satisfies* **the mean value condition** *on A if for every $x \in A$, there is (equivalently, for all) C^1 smooth chart $\varphi : U \to X$ with U an open subset of M, $x \in U$, such that the mapping $f \circ \varphi^{-1}$ satisfies the mean value condition on $\varphi(A) \cap \varphi(U)$.*

The proof of the following corollary is similar to that given in the real-valued case [13] (for a detailed proof, see also [26]). Recall that a paracompact C^1 manifold M modeled on a Banach space X admits C^1 smooth partitions of unity whenever the Banach space X does.

Corollary 3.10. *Let X and Z Banach spaces and M a paracompact C^1 Banach manifold modeled on the Banach space X. Assume that the pair (X, Z) satisfies property (*). Let A be a closed subset of M and $f : A \to Z$ a mapping. Then, f satisfies the mean value condition on A if and only if there is a C^1 smooth extension $G : M \to Z$ of f.*

4. Smooth extension from subspaces

Finally, let us make a brief comment on the extension of C^1 smooth mappings defined on a subspace. Let X and Z be Banach spaces and Y a closed subspace of X. If every C^1 smooth mapping $f : Y \to Z$ can be extended to a C^1 smooth mapping $G : X \to Z$, then for every bounded and linear operator $T : Y \to Z$ there is a bounded and linear operator $\widetilde{T} : X \to Z$ such that $\widetilde{T}_{|_Y} = T$. Moreover, assume that every C^1 smooth and Lipschitz mapping $f : Y \to Z$ can be extended to a C^1 smooth and Lipschitz mapping $G : X \to Z$ with $\mathrm{Lip}(G) \leq C\,\mathrm{Lip}(f)$ with C depending only on X and Z. Then, for every bounded and linear operator $T : Y \to Z$ there is a bounded and linear operator $\widetilde{T} : X \to Z$ such that $\widetilde{T}_{|_Y} = T$ and

$||\widetilde{T}||_{\mathcal{L}(X,Z)} \leq C||T||_{\mathcal{L}(Y,Z)}$. Indeed, it is enough to consider $\widetilde{T} = G'(0)$, where G is the extension mapping of T given by the assumptions.

Definition 4.1. *We say that the pair of Banach spaces (X, Z) satisfies the **linear extension property** if there is $\lambda \geq 1$, which depends only on X and Z, such that for every closed subspace $Y \subset X$ and every bounded and linear operator $T : Y \to Z$, there is a bounded and linear operator $\widetilde{T} : X \to Z$ such that $\widetilde{T}_{|_Y} = T$ and $||\widetilde{T}||_{\mathcal{L}(X,Z)} \leq \lambda||T||_{\mathcal{L}(Y,Z)}$.*

Examples 4.2.
- (i) *Maurey's extension theorem* [22] *asserts that the pair of Banach spaces (X, Z) satisfies the linear extension property whenever X has type 2 and Z has cotype 2. Therefore, (L_2, L_p) for $1 < p < 2$ and (L_p, L_2) for $2 < p < \infty$ satisfy the linear extension property (recall that L_p has type 2 for $2 \leq p < \infty$ and cotype 2 for $1 < p \leq 2$, see* [1]*).*
- (ii) *For every compact metric space K, every non-empty set Γ and $1 < p < \infty$, the pairs $(c_0(\Gamma), C(K))$ and $(\ell_p(\mathbb{N}), C(K))$ satisfy the linear extension property (*[20, Theorem 3.1], [16] *and* [15, Chapter 40]*).*
- (iii) *For every compact metric space K, the pair $(X, C(K))$ satisfies the linear extension property whenever X is an Orlicz space with a separable dual* [17].
- (iv) *The pair $(X, c_0(\mathbb{N}))$ satisfies the linear extension property whenever X is a separable Banach space* [27] *(see also* [15, Chapter 40]*).*

We shall prove the following useful proposition.

Proposition 4.3. *Let (X, Z) be a pair of Banach spaces satisfying the linear extension property and Y a closed subspace of X. If $f : Y \to Z$ is a C^1 smooth mapping (C^1 smooth and Lipschitz mapping), then f satisfies the mean value condition (mean value condition for a bounded map, respectively) on Y.*

Proof. First, let us give the following lemma.

Lemma 4.4. *Let (X, Z) be a pair of Banach spaces satisfying the* linear extension property *and Y a closed subspace of X. Then there is a constant $\eta \geq 1$ and there is a continuous map $B : \mathcal{L}(Y, Z) \to \mathcal{L}(X, Z)$ such that $B(f)_{|_Y} = f$ and $||B(f)||_{\mathcal{L}(X,Z)} \leq \eta||f||_{\mathcal{L}(Y,Z)}$ for every $f \in \mathcal{L}(Y, Z)$.*

The proof of this lemma follows the lines of the real-valued case [2, Lemma 2]. Indeed, let us take $W = \mathcal{L}(X, Z)$, $V = \mathcal{L}(Y, Z)$ and $T : W \to V$ the bounded and linear map given by the restriction to Y, $T(f) = f_{|_Y}$. By assumption, the map T is onto. Thus, we apply the Bartle-Graves's theorem (see [6, Lemma VII 3.2]) in order to find the map B.

Now, if $f : Y \to Z$ is a C^1 smooth mapping (C^1 smooth and Lipschitz mapping), we consider the mapping $D : Y \to \mathcal{L}(X, Z)$ defined as $D(y) := B(f'(y))$ for every $y \in Y$. Then, f satisfies the mean value condition for D (the mean value condition for the bounded map D, respectively). □

Now, we can apply Theorems 3.1 and 3.2 to obtain the following result on C^1 smooth extensions and C^1 smooth and Lipschitz extensions to X of C^1 smooth mappings defined on Y whenever (X, Z) satisfies property (*) and the linear extension property.

Corollary 4.5. *Let (X, Z) be any of the following pairs of Banach spaces:*

(i) (L_p, L_2), $2 < p < \infty$,
(ii) $(c_0(\Gamma), C(K))$, Γ *is a non-empty set and* K *is a compact metric space,*
(iii) $(\ell_p(\mathbb{N}), C(K))$, $1 < p < \infty$ *and* K *is a compact metric space,*
(iv) $(X, C(K))$, X *is an Orlicz space with separable dual and* K *is a compact metric space,*
(v) $(X, c_0(\mathbb{N}))$, X *with separable dual,*
(vi) $(X, \mathbb{R})$, *such that there is a set* $\Gamma \neq \emptyset$ *and there is a bi-Lipschitz homeomorphism* $\varphi : X \to c_0(\Gamma)$ *with* C^1 *smooth coordinate functions (for instance, when* X^* *is separable).*

Let Y *be a closed subspace of* X*. Then, every* C^1 *smooth mapping* $f : Y \to Z$ *has a* C^1 *smooth extension to* X*.*

Moreover, there is $C \geq 1$*, which depends only on* X *and* Z*, such that every Lipschitz and* C^1 *smooth mapping* $f : Y \to Z$ *has a Lipschitz and* C^1 *smooth extension* $F : X \to Z$ *to* X *with* $\operatorname{Lip}(F) \leq C \operatorname{Lip}(f)$*.*

Let us now consider the following definition.

Definition 4.6. *Let* X *and* Z *be Banach spaces and* Y *a closed subspace of* X*. We say that the pair* (Y, Z) *has the* ***linear*** X***-extension property*** *if there is* $\lambda \geq 1$*, which depends on* X*,* Y *and* Z*, such that for every bounded and linear map* $T : Y \to Z$ *there is a bounded and linear extension* $\widetilde{T} : X \to Z$ *with* $||\widetilde{T}||_{\mathcal{L}(X,Z)} \leq \lambda ||T||_{\mathcal{L}(Y,Z)}$*.*

By Theorem 3.1, Theorem 3.2 and a slight modification of Proposition 4.3, we obtain the following corollary.

Corollary 4.7. *Let* (X, Z) *be a pair of Banach spaces with property (*). Let* Y *be a closed subspace of* X *such that the pair* (Y, Z) *has the linear* X*-extension property. Then, every* C^1 *smooth mapping* $f : Y \to Z$ *has a* C^1 *smooth extension to* X*.*

Moreover, there is $C \geq 1$*, which depends on* X*,* Y *and* Z*, such that every Lipschitz and* C^1 *smooth mapping* $f : Y \to Z$ *has a Lipschitz and* C^1 *smooth extension* $F : X \to Z$ *to* X *with* $\operatorname{Lip}(F) \leq C \operatorname{Lip}(f)$*.*

We conclude this note with some considerations on extension morphisms of C^1 smooth mappings. Let X and Z be Banach spaces and consider the Banach space

$$C^1_L(X, Z) := \{f : X \to Z \ : \ f \text{ is } C^1 \text{ smooth and Lipschitz}\},$$

with the norm $||f||_{C^1_L} := ||f(0)|| + \operatorname{Lip}(f)$. We write $C^1_L(X) := C^1_L(X, \mathbb{R})$.

Definition 4.8. *Let* X *and* Z *be Banach spaces and* Y *a closed subspace of* X*. We say that a bounded and linear mapping* $T : C^1_L(Y, Z) \to C^1_L(X, Z)$ *(*$T : Y^* \to X^*$*) is an extension morphism whenever* $T(f)_{|_Y} = f$ *for every* $f \in C^1_L(Y, Z)$ *(for every* $f \in Y^*$*, respectively).*

Lemma 4.9. *Let* X *be a Banach space and* Y *a closed subspace of* X*. If there exists an extension morphism* $T : C^1_L(Y) \to C^1_L(X)$*, then there exists an extension morphism* $S : Y^* \to X^*$*.*

Proof. Let $T : C^1_L(Y) \to C^1_L(X)$ be an extension morphism and define $D : C^1_L(X) \to X^*$ as $D(f) = f'(0)$ for every $f \in C^1_L(X)$. The mapping D is linear, bounded and $||D|| \leq 1$. Thus, $D \circ T : C^1_L(Y) \to X^*$ is linear and bounded. Also, $((D \circ T)(f))_{|_Y} =$

$(T(f)'(0))_{|Y} = f'(0) \in \mathcal{L}(Y,Z)$. Now, let us take the restriction $S := D \circ T_{|Y^*} : Y^* \to X^*$, which is linear, bounded,

$$S(\varphi)_{|Y} = (T(\varphi))'(0)_{|Y} = \varphi'(0) = \varphi, \text{ and}$$
$$||S(\varphi)||_{X^*} = ||D \circ T(\varphi)||_{X^*} \leq ||T(\varphi)||_{C^1_L} \leq ||T||\,||\varphi||_{C^1_L},$$

for every $\varphi \in Y^*$. □

The above lemma and the results given by H. Fakhoury in [8] provide the following characterizations.

Proposition 4.10. *Let X and Z be Banach spaces. The following statements are equivalent:*

(i) *There is a constant $M > 0$ such that for every closed subspace $Y \subset X$, there exists an extension morphism $P : C^1_L(Y,Z) \to C^1_L(X,Z)$ with $||P|| \leq M$.*
(ii) *There is a constant $M > 0$ such that for every closed subspace $Y \subset X$, there exists an extension morphism $T : C^1_L(Y) \to C^1_L(X)$ with $||T|| \leq M$.*
(iii) *There is a constant $M > 0$ such that for every closed subspace $Y \subset X$, there exists an extension morphism $S : Y^* \to X^*$ with $||S|| \leq M$.*
(iv) *X is isomorphic to a Hilbert space.*

Proof. The equivalence between (iii) and (iv) was established in [8, Théorème 3.7].

$(i) \Rightarrow (ii)$ Let us take $z \in S_Z$ and $\varphi \in S_{Z^*}$ (where S_Z and S_{Z^*} denote the unit spheres of Z and Z^*, respectively) such that $\varphi(z) = 1$. Let Y be a closed subspace of X and $P : C^1_L(Y,Z) \to C^1_L(X,Z)$ an extension morphism with $||P|| \leq M$. For every $f \in C^1_L(Y)$, let us consider the Lipschitz and C^1 smooth mapping $f_z : Y \to Z$ defined as $f_z(y) = zf(y)$ for every $y \in Y$. Let us define $R_f := P(f_z) \in C^1_L(X,Z)$. Then, the mapping $T : C^1_L(Y) \to C^1_L(X)$ defined as $T(f) = \varphi \circ R_f$, where $f \in C^1_L(Y)$ is a linear extension mapping with $||T|| \leq M$.

$(ii) \Rightarrow (iii)$ is given by Lemma 4.9 and $(iv) \Rightarrow (i)$ follows from the result that every closed subspace of a Hilbert space H is complemented in H [21]. □

Recall that X is a $\mathscr{P}_\lambda$-space if X is a complemented subspace of every Banach superspace W of X (see [5, p. 95]).

Corollary 4.11. *Let X and Z be Banach spaces. The following statements are equivalent:*

(i) *There is a constant $M > 0$ such that for every Banach superspace W of X, there exists an extension morphism $P : C^1_L(X,Z) \to C^1_L(W,Z)$ with $||P|| \leq M$.*
(ii) *There is a constant $M > 0$ such that for every Banach superspace W of X, there exists an extension morphism $T : C^1_L(X) \to C^1_L(W)$ with $||T|| \leq M$.*
(iii) *There is a constant $M > 0$ such that for every Banach superspace W of X, there exists an extension morphism $S : X^* \to W^*$ with $||S|| \leq M$.*
(iv) *X is a $\mathscr{P}_\lambda$-space.*

Proof. The equivalence between (iii) and (iv) was established in [8, Corollaire 3.3]. The rest of the proof is similar to that of Proposition 4.10. □

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