INTERSECTIONS OF CLOSED BALLS AND GEOMETRY OF BANACH SPACES

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ABSTRACT. In section 1 we present definitions and basic results concerning the Mazur intersection property (MIP) and some of its related properties as the MIP* . Section 2 is devoted to renorming Banach spaces with MIP and MIP*. Section 3 deals with the connections between MIP, MIP* and differentiability of convex functions. In particular, we will focuss on Asplund and almost Asplund spaces. In Section 4 we discuss the interplay between porosity and MIP. Finally, in section 5 we are concerned with the stability of the (closure of the) sum of convex sets which are intersections of balls and with Mazur spaces.

1. THE MAZUR INTERSECTION PROPERTY AND ITS RELATIVES

It was Mazur [39] who first drew attention to the euclidean space property: *every bounded closed convex set can be represented as an intersection of closed balls.* He began the investigation to determine those normed linear spaces which posses this property, named after him the *Mazur intersection property* or MIP. He proved Theorem 1.1, whose proof is so nice and clear that it deserves to be the starting point for this survey. The following easy (and useful) fact will be used extensively throughout the rest of the paper: a closed, convex and bounded set C is an intersection of balls if and only if for every $x \notin C$, there is a closed ball containing the set but missing the point. Hence, the MIP can be regarded as a separation property by balls which is stronger than the classical separation property by hyperplanes. We denote by B and S the unit ball and unit sphere of a Banach space. Analogously, B^* and S^* will stand for the corresponding unit ball and unit sphere in the dual space.

Theorem 1.1. *If a norm $\|\cdot\|$ in a Banach space X is Fréchet differentiable, then $(X, \|\cdot\|)$ satisfies the Mazur intersection property.*

Proof. Consider a closed convex and bounded set C and assume that $0 \notin C$. We will find $x \in X$ and $r > 0$ such that $C \subset x + rB$ but $0 \notin (x + rB)$. Since $0 \notin C$, there is

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a norm one functional $f \in S^*$ such that $\inf f(C) > 0$. Using Bishop-Phelps theorem, we can find a norm-attaining functional $g \in S^*$ close enough to f so that $\inf g(C) > 0$. If we pick $x \in S$ satisfying $g(x) = 1$ then $g = \| \cdot \|'(x)$. The idea now is considering a ball big enough so that its boundary play the role of a separating hyperplane. To this end, put $\varepsilon = (\inf g(C))/2$ and, for $n \geq 2$, consider the ball $B_n = n\varepsilon x + (n-1)\varepsilon B$. Clearly, for every $n \geq 2$ we have $0 \notin B_n$. We will show that $C \subset B_n$ for some n. If this is not the case, for each $n \geq 2$ we can choose $x_n \in C \setminus B_n$. Then $\|x_n - n\varepsilon x\| > (n-1)\varepsilon$ and hence

$$\|x - (1/n\varepsilon)x_n\| > 1 - 1/n \tag{1.1}$$

Using that $\| \cdot \|$ is Fréchet differentiable at x and $g = \| \cdot \|'(x)$ we can write, for every $h \in X$,

$$\|x + h\| - \|x\| - g(h) = r(h), \qquad \text{where} \quad \lim_{h \to 0} r(h)/||h|| = 0. \tag{1.2}$$

Replacing now in the above equation h by $-(1/n\varepsilon)x_n$, using 1.1 and the equality $\varepsilon = \inf g(C)/2$, we obtain

$$r(-(1/n\varepsilon)x_n) = \|x - (1/n\varepsilon)x_n\| - 1 + g((1/n\varepsilon)x_n) > 1/n \ .$$

Hence, for $n \geq 2$,

$$\frac{r(-(1/n\varepsilon)x_n)}{\| - (1/n\varepsilon)x_n\|} \geq \frac{(1/n)}{\|(1/n)\varepsilon^{-1}x_n\|} \geq \frac{\varepsilon}{\sup_n\{\|x_n\|\}} \ . \tag{1.3}$$

which contradicts 1.2 since $\{x_n\} \subset C$, C is bounded and $\lim_n \|(1/n\varepsilon)x_n\| = 0$. □

Norm one functionals $f \in X^*$ satisfying that for every $\varepsilon > 0$ there exists a weak* slice $\mathcal{S} = \{x^* \in B^* : x^*(x) \geq 1 - \delta\}$ (where $x \in S$ and $\delta > 0$) such that $\operatorname{diam}(f \cup \mathcal{S}) < \varepsilon$ were introduced in [11] under the name of *semi-denting* points. When, in addition, we ask that $f \in \mathcal{S}$, then we recover the classical definition of weak* denting point. Semidenting points play an important role in questions related to the MIP because of the following key result, due to Chen and Lin, whose proof can be found in [11]. It is the key to the subsequent characterization of MIP, probably the most useful between the several characterizations known of this property [18].

Proposition 1.2. *A functional $f \in S^*$ is a semi-denting point of B^* if and only if for every closed convex and bounded set C and every $x \in X$, if f separates C and x then there is a ball D in X with $C \subset D$ and $x \notin D$.*

Proposition 1.3. *Given a Banach space, the following conditions are equivalent:*

(i) *The space has the Mazur intersection property.*
(ii) *There is a dense set of semi-denting points in S^*.*
(iii) *There is a dense set of weak* denting points in S^*.*

Proof. To prove the equivalence between (ii) and (iii), note that weak* denting points are semi-denting points so we only need to prove (ii) $\implies$ (iii). To this end, define F_n as the set of those norm one functionals lying in the (relative to S^*) interior of some $S^* \cap \mathcal{S}$ where $\mathcal{S}$ is a weak* slice of diameter less than $1/n$. Then F_n is open and, using (ii), dense in S^*. Therefore $F = \cap_n F_n$ is also dense in S^* (actually, F is a G_δ dense set). Note, finally, that F is the set of weak* denting points of S^*.

To prove that (i) implies (ii), we will use Proposition 1.2 to see that every norm one functional is a semidenting point. Indeed, consider $f \in S^*$, C a closed, convex and bounded set and, finally, $x \in X \setminus C$. Assume, for instance, that $f(x) > 0$ and $\sup f(C) < 0$ (otherwise we can consider a suitable translation $C - y$ and $x - y$). There is $\lambda > 0$ satisfying $C \subset \lambda M_f$ where $M_f = \{z \in B : f(z) \leq 0\}$. Now, since X has the MIP, M_f is an intersection of balls, thus implying the existence of a ball D containing M_f but missing x. The same ball D separates C from x.

The arguments to prove that (ii) implies (i) are quite similar. Let C be convex, bounded and closed and let $x \notin C$. By using (ii), we can find a semi-denting point $f \in S^*$ separating C from x, say for instance that $\sup f(C) < f(x)$. We may assume that $\sup f(C) < 0$ and $f(x) > 0$. Clearly, for enough big $n \in \mathbb{N}$, $C \subset nB$ and $x \in nB$. Using that f is semi-denting, it is not difficult to prove that M_f is an intersection of balls, and so it is nM_f. As a consequence, there is a ball containing M_f (hence C) that miss the point x, thus implying that C is also an intersection of balls. $\square$

Clearly, the set of semi-denting points is closed. Indeed, if $f \in S^*$ is not semi-denting, there is $\varepsilon > 0$ such that the set $B(f, \varepsilon) = \{x^* \in S^* : \|x^* - f\|^* < \varepsilon\}$ does not contain the intersection of S^* with a weak* slice and thus no point g of $B(f, \varepsilon)$ is semi-denting, either. As a consequence, condition (ii) of Proposition 1.3 easily implies that every norm-one functional is a semi-denting point. A weak* denting point is an extreme point. In a finite dimensional space, and extreme point is always a weak* denting point, so the classical Phelps' result is inmediate from the above proposition.

Corollary 1.4. [47] *A finite dimensional normed linear space X has the MIP if and only if the set of extreme points of B^* is dense in S^*.*

Since the weak* denting points of B^{**} must be points of X, we get easily the following consequence of Proposition 1.3. Besides, having in mind Proposition 1.1, note also that next corollary generalizes the well known result that X is reflexive if the norm of X^* is Fréchet differentiable.

Corollary 1.5. *A Banach space whose dual X^* satisfies the MIP is reflexive.*

There exist some other characterizations of spaces with MIP, in terms of the duality mapping, support mappings and points of ε-differentiability (see [18]), though probably the most useful is the one given in Proposition 1.3.

Among the several intersection properties that appeared as variations on the MIP, probably the most important is the *weak* Mazur intersection property* or MIP* introduced in [18]: a dual space satisfies the MIP* if every weak* compact convex set is an intersection of closed dual balls. In [18] it is shown that every result for MIP has an analogous formulation for MIP*. In particular, it is connected with convexity properties of the predual space:

Proposition 1.6. [18] *A dual space X^* has the MIP* if and only if the set of denting points of the predual unit ball is dense in its unit sphere.*

The nice piece of work contained in [18] was the culmination of previous results obtained, among others, by Phelps [47] and Sullivan [56]. Since these pioneering works, the investigation on different intersection properties has been slow but steady. Whitfield and Zizler studied in [60] the property that every compact convex set is an intersection of closed balls. Further research on this property was carried out later by Sersouri in [52] and [53] and later by J. Vanderwerff [59]. The corresponding intersection property for weakly compact and convex sets was investigated by Zizler in [65] and J. Vanderwerff in [59]. Finally, an uniform version of the MIP was considered in [61] by Whitfield and Zizler. A unified approach to different intersection properties is presented by Chen and Lin in [10]. Other authors have also contributed to the study of MIP and MIP* as Acosta and Galan in [1], P. Bandyopadhyaya and A. Roy in [3] and finally, P. Georgiev and P. S. Kenderov, whose results will be mentioned in the next sections.

2. Renorming Banach spaces with MIP or MIP*

Both MIP and MIP* are metric properties and hence invariant under isometries but not under isomorphisms. The question of whether a Banach space can be renormed with MIP or a dual space with MIP* has not an easy answer. Indeed, one might well ask how, when provided with a norm, one can construct an equivalent norm such that every closed convex body is an intersection of (new) closed balls. Zizler [65] realized that Troyansky renorming techniques for LUR norms ([9], Lemma 7.1.1) can be applied to study intersection properties. This fruitful idea turned out to be specially successful when applied first to MIP* [41] and later to MIP [31]. Recall that a biorthogonal system $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$ is *fundamental* provided $X = \overline{\operatorname{span}}\,(\{x_i\}_{i\in I})$.

Lemma 2.1. *Let X be a Banach space with a fundamental biorthogonal system $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$. Then, the subspace $Y = \operatorname{span}(\{x_i\}_{i\in I})$ admits a LUR norm.*

Theorem 2.2. *Let X be a Banach space with a fundamental biorthogonal system. Then X^* admits an equivalent norm with the MIP*.*

The above theorem applies to a fairly wide class of Banach spaces including, for instance, the dual of $\ell_\infty(\Gamma)$. This fact will be used later to prove that almost every norm (in the sense of Baire) in this space is Fréchet differentiable on a dense set. We only know few Banach spaces which admits no fundamental biorthogonal system. This is the case, for instance, of Kunen, Shelah and the space $\ell_\infty^c(\Gamma)$ (the subspace of all elements of $\ell_\infty(\Gamma)$ with countable support, being $\operatorname{card}\Gamma$ strictly bigger than the cardinal of the continuum), spaces that will appear later in this survey. Before to state the analogous versions of these results for the MIP let us mention that, once we know that there is an equivalent norm with MIP (or MIP*, if it is dual) in a Banach space, then there are many. In fact, Georgiev [16] proved that almost every norm (again in the Baire sense, that will be precised latter) satisfies this property provided there is one satisfying it.

Proposition 2.3. [16] *Given a Banach space X, the set of norms having the MIP is either empty or residual. Analogously, the set of dual norms having MIP* is either empty or residual (in the set of all dual norms).*

This result has many applications. For instance, it can be used together with the following proposition to show the density of norms which are Fréchet differentiable in open dense sets in spaces with MIP or MIP*. There exist even stronger results linking MIP, MIP* and differentiability that will be discussed later, in the section devoted to almost Asplund spaces.

Proposition 2.4. [41] *If X^* has MIP*, then the predual norm can be approximated by norms which are Fréchet differentiable on an open dense set. Also, if X has MIP, then the dual norm can be approximated by (dual) norms which are Fréchet differentiable on an open dense set.*

It was for long time an open problem to determine whether spaces with the MIP are Asplund spaces. Also, it was unknown if every Asplund space admits a norm with the MIP or, in particular, a Fréchet differentiable norm. The latter was shown in the negative by Haydon [28]. First and second problems were also answered in the negative in [31] using, together with Proposition 1.3, the following results.

Theorem 2.5. *Let $(X^*, \|\cdot\|^*)$ be a dual Banach space with a biorthogonal system $\{x_i, f_i\}_{i\in I} \subset X^* \times X$ and $X_0 = \operatorname{span}(\{x_i\}_{i\in I})$. Then, X^* admits an equivalent dual norm $|\cdot|^*$ which is locally uniformly rotund at the points of X_0. Then, if X_0 is dense in X^*, the Banach space X with the predual norm $|\cdot|$ has the Mazur Intersection property.*

Outline of the proof. We may assume that $\|f_i\| = 1$, for every $i \in I$ and let us consider $\Delta = \{0\} \cup \mathbb{N} \cup I$. Define the map T from X^* into $\ell_\infty(\Delta)$ as follows:

$$T(x)(\delta) = \begin{cases} \|x\|^* & \text{if} \quad \delta = 0 \\ 2^{-n} G_n(x) & \text{if} \quad \delta = n \in \mathbb{N} \\ f_i(x) & \text{if} \quad i \in I \end{cases}$$

for every $x \in X^*$ and $\delta \in \Delta$, where

$$F_A(x) = \sum_{i \in A} |f_i(x)|$$

$$E_A(x) = \operatorname{dist}\big(x, \operatorname{span}(\{x_i\}_{i \in A})\big) \quad A \subset I, \quad \operatorname{card} A < \infty$$

$$G_n(x) = \sup_{\operatorname{card} A \leq n} \{E_A(x) + nF_A(x)\}.$$

Clearly $T(X^*) \subseteq \ell_\infty(\Delta)$ and $T(X_0) \subseteq c_0(\Delta)$. On the other hand, since $2^{-n}(1+n^2) \leq 2$ for every $n \in \mathbb{N}$, we have $\|x\|^* \leq \|T(x)\|_\infty \leq 2\|x\|^*$.

For every $\delta \in \Delta$, consider the map $T_\delta(x) = T(x)(\delta)$, $x \in X^*$. Obviously, if $\delta \in I \cup \{0\}$ the map T_δ is weak*-l.s.c. . Moreover, the maps F_A and, the maps E_A are weak*-l.s.c., so T_δ is weak*-l.s.c. for every $\delta \in \Delta$.

Let p be the Day norm [9, p.69] in $\ell_\infty(\Delta)$, and consider in X^* the map $n(x) = p(T(x))$, $x \in X^*$. It can be easily proved that $n(\cdot)$ is an equivalent norm in X^*. The norm $n(\cdot)$ has the following expression:

$$n(x)^2 = \sup\{\sum_{i=1}^{n} \frac{|T_{\delta_i}(x)|^2}{4^i} : \ (\delta_1, \delta_2, \ldots, \delta_n) \subset \Delta, \ \delta_i \neq \delta_j, \ n \in \mathbb{N}\}$$

so $n(\cdot)$ is weak*-l.s.c. , that is, n is a dual norm $|\cdot|^*$. The norm p defined in $\ell_\infty(\Delta)$ is locally uniformly rotund at the points of $c_0(\Delta)$. It can be checked that the norm $|\cdot|^*$ is locally uniformly rotund at the points of X_0 [31]. Now, it is straightforward to verify that the points of $X_0 \cap S_{|\cdot|^*}$ are weak* denting points of $B_{|\cdot|^*}$. Finally, if the subspace X_0 is dense in X^*, by the Proposition 1.3, the space X endowed with the predual norm of $|\cdot|^*$ has the Mazur intersection property. □

Corollary 2.6. *Let X, Y be Banach spaces such that* $\operatorname{dens} X^* \leq \operatorname{dens} Y^*$. *Suppose that Y^* has a fundamental biorthogonal system $\{y_i, f_i\}_{i \in I} \subset Y^* \times Y$. Then, the Banach space $X \oplus Y$ admits an equivalent norm with the MIP.*

Proof. Let us consider $Z = X \oplus Y$ with the norm $||(x,y)||_Z = ||x||_X + ||y||_Y$. By Theorem 2.5 we need only to show that $Z^* \approx X^* \oplus Y^*$ has also a fundamental biorthogonal system in $Z^* \times Z$. An element $x^* + y^*$ of $X^* \oplus Y^*$ is considered an element of Z^* in the usual way: $(x^* + y^*)(x + y) = x^*(x) + y^*(y)$ for every $x \in X$, $y \in Y$. Relabel the fundamental biorthogonal system given in Y^* as $\{y_i^n, f_i^n\}_{i\in I, n\in\mathbb{N}}$. We may assume that $\|y_i^n\|_Y \le 1/n$ for every $i \in I$, $n \in \mathbb{N}$. Let us take a dense set $\{x_i\}_{i\in I}$ of X^*. Then, the system

$$S = \{x_i + y_i^n,\ f_i^n\}_{i\in I, n\in\mathbb{N}} \subset Z^* \times Z$$

is a fundamental biorthogonal system in Z^* and we conclude the proof. □

As a corollary, we get that every Banach space X can be embedded into a Banach space with the MIP: just consider $X \oplus \ell_2(\Gamma)$ with $\operatorname{card}\Gamma = \operatorname{dens} X^*$. Thus, for instance, the non-Asplund space $\ell_1 \oplus \ell_2(c)$ admits an equivalent norm with the MIP. We also obtain as an application of the above corollary the following result of Deville [6].

Corollary 2.7. [6] *For every ordinal η, the long James space $J(\eta)$, its predual $M(\eta)$ and every finite dual of $J(\eta)$ admit an equivalent norm with the Mazur intersection property.*

Proof. First, we need to observe that $\ell_2(\eta)$ can be complementably embedded into $J(\eta)$. Indeed, consider the subset

$$A = \{\alpha \in [0,\eta] : \alpha = 2n \text{ or } \alpha = \gamma + 2n \text{ , with } \gamma \text{ ordinal limit and } n \ge 1\ \}$$

and the subspace $H(\eta) = \{f \in J(\eta) : f(\alpha) = 0 \quad \text{if} \quad \alpha \notin A\}$. The subspace $H(\eta)$ is isomorphic to $\ell_2(A)$ and $\operatorname{card} A = \operatorname{card}\eta$. On the other hand, the projection $f \in J(\eta) \longrightarrow p(f) \in H(\eta)$ defined as

$$p(f)(\alpha) = \begin{cases} f(\alpha) - f(\alpha - 1) & \text{if} \quad \alpha \in A \\ 0 & \text{if} \quad \alpha \notin A \end{cases}$$

is continuous and, therefore, $H(\eta)$ is complemented in $J(\eta)$. Thus, we have that $J(\eta) \approx \ell_2(\eta) \oplus Y$ for a Banach space Y (which can be easily identified with $J(\eta)$) and $J(\eta)^* \approx \ell_2(\eta) \oplus Y^*$.

On the other hand, $M(\eta)$, $J(\eta)$ and every finite dual of $J(\eta)$ are Asplund spaces [12]. Consequently $\operatorname{dens}\ell_2(\eta) = \operatorname{card}\eta \ge \operatorname{dens} Y = \operatorname{dens} Y^* = \operatorname{dens} Y^{**}$ and, applying Corollary 2.6, we obtain that $J(\eta)$ and $J(\eta)^*$ admit a norm with the Mazur intersection property. The assertion for $M(\eta)$ and the dual spaces of $J(\eta)^*$ follows from the fact that $M(\eta)$ is isometric to $J(\eta)^*$ (cf. [12]). □

Consider the James'tree space JT. It is shown in [37] that JT^{**} is isomorphic to $JT \oplus \ell_2(\mathbb{R})$. Then, as a consequence of Corollary 2.6, we obtain that JT^{**} and finite even duals of JT^{**} admit an equivalent norm with the Mazur intersection property. On the other hand, notice that the space JT^* and finite odd duals of JT admit a Fréchet differentiable norm since their duals are WCG. We finish this section with the following consequences, the first one already mentioned.

Corollary 2.8. *(i) Every Banach space X can be almost isometrically complementably embedded into a Banach space with the Mazur intersection property.*
(ii) Every Banach space X may be isometrically embedded into a Banach space Z with the Mazur intersection property.

Proof. (i) Let us consider the Banach space $Z = X \oplus \ell_2(\Gamma)$ with $\operatorname{card}\Gamma = \operatorname{dens} X^*$. By Corollary 2.6, Z can be renormed with the MIP and a useful result of Georgiev [16] ensures that the set of equivalent norms with the Mazur intersection property in a Banach space is either empty or residual. In this case the set is residual and implies the assertion. Notice that $\operatorname{dens} Z = \operatorname{dens} X^*$. Clearly, this is sharp in the sense that, necessarily, if a Banach space Z has the MIP, $\operatorname{dens} Z = \operatorname{dens} Z^*$. In addition, if X is a subspace of Z, $\operatorname{dens} Z \geq \operatorname{dens} X^*$.

(ii) We denote by $\alpha = \operatorname{dens} X$, $\beta = \alpha^+ (= \min\{\gamma \text{ ordinal number} : \operatorname{card}\gamma > \alpha\})$, and the Banach space

$$m_\alpha(\beta) = \{\, x \in \ell_\infty(\beta) : \ \operatorname{supp} x \ \text{ has cardinality at most } \alpha \,\},$$

with the supremum norm $\|x\| = \sup_{\gamma<\beta} |x_\gamma|$. Obviously, X may be isometrically embedded into $(m_\alpha(\beta), \|\cdot\|)$. On the other hand, by Corollary 2.8, $m_\alpha(\beta)$ embeds into a Banach space $(Z, |\cdot|)$ with the Mazur intersection property and, by a result of Partington [46], $(m_\alpha(\beta), \|\cdot\|)$ embeds *isometrically* into $(m_\alpha(\beta), |\cdot|)$. Therefore, X embeds isometrically into $(Z, |\cdot|)$. Note that, with this argument, we have $\operatorname{dens} X^* < \operatorname{dens} Z^*$ □

We are concern now with the three-space problem for the MIP. The following result states that being isomorphic to a Banach space with the MIP is a three space property [51]. An application of this result states that every space of continuous functions over a tree can be renormed with the MIP [31].

Proposition 2.9. *Let X be a Banach space and Y be a closed subspace of X such that Y admits a norm with the Mazur Intersection Property and X/Y admits a norm with the Mazur intersection property. Then X admits a norm with the Mazur intersection property.*

Sketch of the Proof. M. Raja [51] proved the following renorming theorem: *Consider the set D of all weak*-denting points of the (dual) unit ball of a dual Banach space X^*. Then, X^* admits an equivalent* dual *norm which is locally uniformly rotund at every point of D.* Thus, we may assume that both Y^* and $(X/Y)^*$ admit equivalent dual norms with a (G_δ) dense set of LUR points. The existence of an equivalent dual norm in X^* with a (G_δ) dense set of LUR points follows by imitating the proof of the three-space property for locally uniform rotund renormings given in [21]: We consider, under the standard identifications, $(X/Y)^*$ to be the annihilator subspace $Y^\perp$ with the weak* topology in $(X/Y)^*$ being the same as the induced weak* topology which $Y^\perp$ inherites as a subspace of X^*. Then, we may assume that there is a norm on $Y^\perp$ which is $\sigma(Y^\perp, X)$-l.s.c. and has a G_δ dense set of locally uniformly rotund points. The subspace $Y^\perp$ is weak* closed so this norm can be extended to an equivalent dual norm $\|\cdot\|^*$ on X^*. Let $|\cdot|^*$ be an equivalent dual norm on Y^* which is locally uniformly rotund at a G_δ dense set. Consider the restriction map $Q: X^* \to Y^*$, which is weak*-weak* continuous and the Bartle-Graves continuous selection mapping $B: Y^* \to X^*$, which is bounded on bounded sets, $B(y^*) = |y^*|^* B(y^*/|y^*|^*)$ and $B(0) = 0$. For every $y^* \in S_{|\cdot|^*} = \{y^* \in Y^* : \ |y^*|^* = 1\}$, take $y \in Y$ such that $y^*(y) = 1$ and $|y| \le 2$. Define $P_{y^*}(x^*) = x^*(y)B(y^*)$, for $x^* \in X^*$, which is weak*-weak* continuous. The following family of weak* l.s.c. convex functions defined on X^*

$$\begin{aligned} \varphi_{y^*}(x^*) &= |Q(x^*) + y^*|^*, \\ \psi_{y^*}(x^*) &= ||x^* - P_{y^*}(x^*)||^*, \qquad y^* \in S_{||\cdot||^*}, \end{aligned}$$

is uniformly bounded on bounded sets. Therefore, if we consider

$$\phi_k(x^*) = \sup\{\varphi_{y^*}(x^*)^2 + \tfrac{1}{k}\psi_{y^*}(x^*)^2 : \ y^* \in S_{|\cdot|^*}\},$$

$$\phi(x^*) = \|x^*\|^{*2} + |Q(x^*)|^{*2} + \sum_k 2^{-k}\phi_k(x^*),$$

the Minkowski functional $|||\cdot|||^*$ of the set $\{x^* \in X^* : \phi(x^*) + \phi(-x^*) \le 4\}$ is an equivalent dual norm on X^*.

Consider the mapping (not necessarily linear) $S: X^* \to Y^\perp$, defined as $S(x^*) = x^* - B(Q(x^*))$. It is proved in [21] that x^* is a locally uniformly rotund point for $|||\cdot|||^*$ provided $Q(x^*)$ is locally uniformly rotund for $|\cdot|^*$ and $S(x^*)$ is locally uniformly rotund for $\|\cdot\|^*$ in $Y^\perp$. To conclude, observe that the mappings S and Q are continuous and open. Then, the sets

$$\begin{aligned} L_{|\cdot|^*} &= \{x^* \in X^* : \ |\cdot|^* \text{ is locally uniformly rotund at } Q(x^*)\}, \\ L_{\|\cdot\|^*} &= \{x^* \in X^* : \ S(x^*) \text{ is weak* denting of } \|\cdot\|^* \text{ in } Y^\perp\} \end{aligned}$$

and therefore $L = L_{|\cdot|^*} \cap L_{\|\cdot\|^*}$ are G_δ dense sets of X^*. Hence, the space $(X, |||\cdot|||)$ has the Mazur intersection property. $\square$

Haydon gave in [28] an example of an Asplund space admitting no equivalent Gâteaux differentiable norm, namely the space $C_0(L)$ of all continuous functions vanishing at the infinity over the following tree L: denote by ω_1 the smallest uncountable ordinal, α an ordinal number and consider $L = \bigcup_{\alpha<\omega_1} \omega_1^\alpha$ which is called the full uncountable branching tree of height ω_1. Therefore, it is a natural question to ask whether the space $C_0(L)$ admits an equivalent norm with the Mazur intersection property [9, Ch. VII]. The answer is affirmative. Moreover, for every tree T, the space $C_0(T)$ admits a norm with the Mazur intersection property.

Lemma 2.10. *Let K be a compact Hausdorff scattered space such that* $\operatorname{card} K = \operatorname{card} I$*, being I the set of isolated points of K. Then, the Banach space $C(K)$ admits an equivalent norm with the Mazur intersection property.*

Proof. The space $C(K)$ is an Asplund space, so its dual space is identifiable with $\ell_1(K)$. For every $\omega \in K' = K \setminus I$, we can consider disjoint subsets of different points $\{t_n^\omega\}_{n=1}^\infty \subset I$ and $A = I \setminus \{t_n^\omega : \ \omega \in K', \ n \in \mathbb{N} \ \}$. Denote by $\delta_t \in \ell_1(K)$ the evaluation at the point $t \in K$ and by χ_t the characteristic function at the point t. Clearly $\chi_t \in C(K)$ if and only if t is an isolated point in K. Let us consider the biorthogonal system $\{y_n^\omega, f_n^\omega\}_{n\in\mathbb{N},\,\omega\in K'} \subset C(K)^* \times C(K)$, where $y_n^\omega = (1/n)\delta_{t_n^\omega}$ and $f_n^\omega = n\chi_{t_n^\omega}$. Then, the system

$$\mathcal{S} = \{\delta_\omega + y_n^\omega, f_n^\omega\}_{n\in\mathbb{N},\,\omega\in K'} \cup \{\delta_t, \chi_t\}_{t\in A} \subset C(K)^* \times C(K)$$

is a fundamental biorthogonal system in $C(K)^*$. We apply now Corollary 2.6 to finish the proof. □

Remark 2.11. The above tree $L = \bigcup_{\alpha<\omega_1} \omega_1^\alpha$ equipped with the order topology is a locally compact scattered Hausdorff space such that the cardinal of its isolated points is equal to $\operatorname{card}(L)$. Hence, its Alexandrof compactification αL is a compact Hausdorff scattered space such that $\operatorname{card}(\alpha L) = \operatorname{card}(I)$, being I the set of isolated points of αL. So, the Banach space $C(\alpha L)$ verifies Lemma 2.10. As $C_0(L)$ is isomorphic to $C(\alpha L)$, $C_0(L)$ also verifies this Lemma.

Every tree T equipped with the order topology is a locally compact scattered Hausdorff space with $\operatorname{card}(T) \geq \operatorname{card}(I)$, being I the set of isolated points of T. When $\operatorname{card}(T) > \operatorname{card}(I)$ we cannot apply Lemma 2.10 but, in spite of this fact, next proposition shows that $C_0(T)$ admits an equivalent norm with the MIP.

Proposition 2.12. *The Banach space $C_0(T)$ admits a norm with the Mazur intersection property whenever T is a tree space.*

Proof. For any $t \in T$ we denote by t^+ the set of immediate successors of t and consider the subset of T

$$H = \{t \in T' \ : \ t^+ = \emptyset\},$$

where T' is the set of all accumulation points of T and the closed subspace of $C_0(T)$

$$Y = \{f \in C_0(T) \,:\, f(t) = 0, \quad \text{if} \;\; t \in H\}.$$

The space $T \setminus H$ is locally compact, Hausdorff, scattered and verifies that the cardinal of its isolated points is equal to $\operatorname{card}(T \setminus H)$. Hence, the Alexandrov compactification $\alpha(T \setminus H)$ of $T \setminus H$ is scattered and verifies that $\operatorname{card}(\alpha(T \setminus H)) = \operatorname{card}(I)$, I being the set of isolated points of $\alpha(T \setminus H)$. Observe that $Y \approx C_0(T \setminus H)$ is isomorphic to the space of all continuous functions on $\alpha(T \setminus H)$. Then, by Lemma 2.10, we obtain a norm on Y such that its dual norm has a dense set of locally uniformly rotund points. On the other hand, it can be easily verified using the fact that H is an antichain and the Tietze's extension theorem that $C_0(T)/Y$ is isomorphic to $c_0(H)$, and then, $C_0(T)/Y$ admits a norm such that its dual norm has a dense set of locally uniformly rotund points. Now the assertion follows from Proposition 2.9. □

3. MIP, MIP*, Asplund and almost Asplund spaces

The results obtained in the previous section provide a wide range of Banach spaces with an equivalent MIP norm. This could induce to think that this class of Banach spaces is larger than the class of Asplund spaces. This is not the case. There are Asplund spaces which cannot be renormed with the MIP ([31] and [22]). An example to this assertion is the Kunen space [35], a $C(K)$ Banach space where K is a scattered compact set (and thus $C(K)$ is Asplund) constructed assuming the continuum hypothesis. The Kunen space is a non-separable Asplund space satisfying that for every uncountable set $\{x_i\}_{i\in I}$ in the space, there exists $i_0 \in I$ such that

$$x_{i_0} \in \overline{\operatorname{conv}} \left(\{x_i\}_{I\setminus\{i_0\}}\right). \tag{3.1}$$

The first example of a non-separable Banach space satisfying (3.1) was constructed by Shelah assuming the diamond principle for $\aleph_1$ [54].

Proposition 3.1. *The Kunen and Shelah spaces do not admit an equivalent norm with the Mazur intersection property. Analogously, the duals of the previous spaces do not admit a dual norm with the MIP*.*

Proof. First, if a Banach space X with a norm $|\cdot|$ has the Mazur intersection property, then, by Proposition 1.3(iii), the dual norm $|\cdot|^*$ has a dense set of weak* denting points in its unit sphere. Consider $0 < \delta < 1$ and find a family of weak* denting points $(f_\alpha)_{\alpha\in I} \subset S_{|\cdot|^*}$ with $\operatorname{card} I = \operatorname{dens} X^* = \operatorname{dens} X$ such that

$$|f_\alpha - f_\beta| \geq \delta, \qquad \text{for} \;\; \alpha \neq \beta. \tag{3.2}$$

Then, there is a family of slices $S(B_{|\cdot|^*},\ y_\alpha,\ \rho_\alpha)$, for $\alpha \in I$, with $|y_\alpha| = 1,\ f_\alpha(y_\alpha) > \rho_\alpha > 0$, and

$$S(B_{|\cdot|^*},\ y_\alpha,\ \rho_\alpha) \cap S(B_{|\cdot|^*},\ y_\beta,\ \rho_\beta) = \emptyset, \qquad \text{for} \ \ \alpha \neq \beta. \tag{3.3}$$

We denote $x_\alpha = (1/\rho_\alpha)y_\alpha$ for every $\alpha \in I$. It follows from (3.3) that $f_\alpha(x_\alpha) > 1$ and $|f_\alpha(x_\beta)| \leq 1$ for $\alpha, \beta \in I,\ \beta \neq \alpha$. Consequently,

$$x_\alpha \notin \overline{\text{conv}}\big(\{x_\beta\}_{\beta \in I \setminus \{\alpha\}}\big). \tag{3.4}$$

Therefore, if X is a non-separable Banach space with the MIP, there is an uncountable subset $\{x_\alpha\}_{\alpha \in I} \subset X$ satisfying (3.4). This implies that that the Kunen and Shelah spaces does not admit an equivalent norm with the MIP.

For the second assertion, consider the Banach space $(X^*, |\cdot|^*)$ with the weak* Mazur intersection property. Then, by Proposition 1.6, the norm $|\cdot|$ has a dense set of denting points in its unit sphere. Take $0 < \delta < 1$ and find a family of denting points $(x_\alpha)_{\alpha \in I}$ in X, $|x_\alpha| = 1$, with card $I = \text{dens}\, X$ such that

$$|x_\alpha - x_\beta| \geq \delta, \qquad \text{for} \ \ \alpha \neq \beta. \tag{3.5}$$

¿From the fact that the points $(x_\alpha)_{\alpha \in I}$ are denting in $B_{|\cdot|}$ and condition (3.5), we get that, for every α, $x_\alpha \notin \overline{\text{conv}}\left(\{x_\beta\}_{\beta \in I \setminus \{\alpha\}}\right)$. Thus, the duals of the Kunen and Shelah spaces do not admit an equivalent dual norm with MIP*. □

The property exhibited in (3.1) shared by the spaces contructed by Shelah and Kunen, that is, for every uncountable family of points in the space there is one point in the closed convex hull of the rest, has been extensively studied in [22]. Let us denote this property by KS. The following result was proved for the Kunen space in [31] and for the general case in [22].

Theorem 3.2. *Let X be a Banach space. The following assertions are equivalent:*
(i) X has the KS property.
(ii) Every weak-closed convex subset $K \subset X^*$ is weak*-separable.*
(iii) Every convex subset $K \subset X^*$ is weak-separable.*

Let us mention that there are still a number of open problems concerning the MIP, as the existence of points of Fréchet differentiability in spaces with this property. While spaces with Fréchet differentiable norm satisfy the Mazur intersection property, it is unknown if it is also the case of spaces with a (Fréchet) differentiable bump function. In this setting, it was proved in [8] the following result.

Theorem 3.3. [8] *If a Banach space has the Radon-Nikodým property and a Fréchet differentiable bump function, then it has an equivalent norm with the MIP.*

We are concerned now with the connections between Mazur intersection property on X or weak* Mazur intersection property on X^* and the generic differentiability of "most" equivalent (dual) norms defined on X^* or X, respectively. Let F be the space of all sublinear, positively homogeneous, continuous functionals on a Banach space X, furnished with the metric ρ associated to the uniform convergence on bounded sets. Analogously, let F^* be the space of all sublinear, positively-homogeneous, continuous and w^*-lower semicontinuous functionals on X^*. The spaces (F, ρ) and (F^*, ρ) are complete metric spaces and thus Baire spaces.

A Banach space X (resp. the dual X^* of a Banach space X) is called *almost Asplund* (resp. *almost weak* Asplund*) space, if there exists a dense G_δ subset F_0 of F (resp. F_0^* of F^*) such that every $f \in F_0$ (resp. every $f^* \in F_0^*$) is Fréchet differentiable on a dense G_δ subset of X (resp. of X^*). The first author to consider this class of Banach spaces was P. Georgiev [15]. He proved that MIP in X and MIP* in X^* imply that X is almost Asplund and X^* is almost weak* Asplund. More connections between differentiability of convex functions and Mazur (weak* Mazur) intersection properties were investigated by Kenderov and Giles [34] and J. P. Moreno [41], among others. Later on, following the ideas of [15], it was proved in [17] that the dual of a Banach space with the MIP is a *almost weak* Asplund* space and, analogously, the predual of a dual space with the MIP* intersection property is an *almost Asplund* space. We will focus here on this last result and its geometrical derivations.

Some interesting consequences are obtained by considering *norms* instead of sublinear functionals. Among them, we can mention that "almost all in the Baire sense" (we shall detail this later) equivalent norms on a Banach space with a fundamental biorthogonal system are Fréchet differentiable on a dense G_δ subset. This is the case, for instance, of spaces $\ell_1(\Gamma)$ and $\ell_\infty(\Gamma)$, for every Γ, whose bad differentiability behavior is well known. Moreover, there are only few examples of spaces without fundamental biorthogonal system ([49], [44]) so this result applies for most Banach spaces.

Denote by $\mathcal{H}_X$, or just $\mathcal{H}$ if there is no ambiguity on the space we are considering, the set of all bounded, closed, convex and nonempty subsets of a real Banach space X. The Hausdorff distance between $C_1, C_2 \in \mathcal{H}$ is given by

$$d(C_1, C_2) = \inf\{\varepsilon > 0 : C_1 \subset C_2 + \varepsilon B,\ C_2 \subset C_1 + \varepsilon B\},$$

where B is the unit ball of X. The space $(\mathcal{H}, d)$ is a complete metric space [36] and, hence, a Baire space. Denote by $\mathcal{H}^*$ the elements of $\mathcal{H}_{X^*}$ which are weak* closed. The space $(\mathcal{H}^*, d)$ is also a complete metric space. The mappings $I : (\mathcal{H}, d) \to (F^*, \rho)$, where $I(K) :=$ σ_K the support functional on K: $\sigma_K(x^*) = \sup_{x\in K}\langle x, x^*\rangle$, and $\hat{I} : (\mathcal{H}^*, d) \to (F, \rho)$, where $\hat{I}(K^*) := \sigma_{K^*}$, the support functional on K^* defined on X, $\sigma_{K^*}(x) = \sup_{x^*\in K^*}\langle x, x^*\rangle$, are homeomorphisms. The existence of the homeomorphisms I and $\hat{I}$ and the duality between Fréchet differentiability and strong exposition can be tied together in the following Lemma 3.4 whose proof is omitted.

Lemma 3.4. *A Banach space X is almost Asplund if and only if there is a dense G_δ subset $\mathcal{H}_0^* \subset \mathcal{H}^*$ such that every element of $\mathcal{H}_0^*$ has a dense G_δ set of weak*-strongly exposing functionals in X. A dual Banach space X^* is almost weak* Asplund if and only if there is a dense G_δ subset $\mathcal{H}_0 \subset \mathcal{H}$ such that every element of $\mathcal{H}_0$ has a dense G_δ set of strongly exposing functionals in X^*.*

Let $b : X \to S^*$ be a selection of the subdifferential mapping of the norm, i.e. $\langle x, b(x)\rangle = \|x\|$ for every $x \in X$. Given $C \subset X$, $f \in X^*$ and $\alpha > 0$, we will denote by $S(C, f, \alpha)$ the slice $\{x \in C : f(x) > \sup f(C) - \alpha\}$. The following lemma is a key tool in the proof of the result cited above. There is an analogous version for a dual Banach space with the weak* Mazur Intersection Property.

Lemma 3.5. *Let X be an infinite dimensional Banach space with the Mazur intersection property. Then, for every $n \geq 2$, there is a subset $X_n \subset X$ such that:*

(i) $\cup_{n=2}^\infty b(X_n)$ *is dense in* S^*,

(ii) $\langle x, b(x)\rangle > \sup_{z\in X_n\setminus\{x\}}\langle b(x), z\rangle$, *for every* $x \in X_n$,

(iii) $\|b(x) - b(y)\| > \frac{1}{n}$, *for every* $x, y \in X_n$, $x \neq y$.

Proof. By Proposition 1.3, the dual norm has a dense set X_0^* of weak* denting points in its unit sphere. Consider for every $n \geq 2$, a maximal subset $X_n^* \subset X_0^*$ satisfying $\|x^* - y^*\| > 2/n$, for every $x^*, y^* \in X_n^*$, $x^* \neq y^*$. Then, $F_0^* = \cup_{n=2}^\infty X_n^* \subset X_0^*$ is dense in S^*, and for every $x^* \in X_n^*$ there is a slice $S(B^*, y_n(x^*), \gamma_n(x^*))$, $y_n(x^*) \in B^*$ and $\gamma_n(x^*) \in (0, \frac{1}{n})$ so that,

$$x^* \in S(B^*, y_n(x^*), \gamma_n(x^*)), \qquad \operatorname{diam} S(B^*, y_n(x^*), \gamma_n(x^*)) < \frac{1}{2n}$$

and

$$S(B^*, y_n(x^*), \gamma_n(x^*)) \cap S(B^*, y_n(z^*), \gamma_n(z^*)) = \emptyset \tag{3.6}$$

for every $x^*, z^* \in X_n^*$, $x^* \neq z^*$. By (3.6) it follows that $y_n(x_1^*) \neq y_n(x_2^*)$ for $x_1^* \neq x_2^*$, i.e. the mapping $y_n : X_n^* \to S$ is an injection. We have $\|x^* - b(y_n(x^*))\| < \frac{1}{2n}$, for every $x^* \in X_n^*$ and

$$\|b(y_n(x_1^*)) - b(y_n(x_2^*))\| > \frac{1}{n}$$

for each $x_1^*, x_2^* \in X_n^*,\ x_1^* \neq x_2^*$. If we define $X_n = \{\frac{y_n(x^*)}{1-\gamma_n(x^*)} : x^* \in X_n^*\}$, then it is easy to check the conditions (i), (ii), (iii) and the proof is completed. $\square$

Theorem 3.6. *Consider a Banach space X with dual X^*.*

(i) *If X has the Mazur intersection property then X^* is almost weak* Asplund.*
(ii) *If X^* has the weak* Mazur intersection property then X is almost Asplund.*

Sketch of the proof. The idea of the proof is contained in Theorem 4 of [15]. In order to prove (i), it is enough to show the existence of a dense G_δ subset $\mathcal{B}_0 \subset \mathcal{H}$ such that every element of $\mathcal{B}_0$ has a dense G_δ set of strongly exposing functionals in X^*. Let $\{X_n\}_{n\geq 2}$ be the sequence we have found in Lemma 3.5 and for every $x \in X_n$ define:

$$\alpha_n(x) = \langle x, b(x)\rangle - \sup_{y \in X_n \setminus \{x\}} \langle y, b(x)\rangle,$$

For integers $n \geq 2$ and $m \geq 1$ denote:

$$H_{n,m} = \{x \in X_n : \alpha_n(x) > \frac{1}{m}\}$$

and define $\mathcal{B}_{n,m,k}$ as the set of all $Z \in \mathcal{H}$ for which there are $\alpha > 0$ and $\gamma > 0$ such that $\operatorname{diam} S(Z, b(x), \alpha) < \frac{1}{k} - \gamma$ for each $x \in H_{n,m}$ if $H_{n,m} \neq \emptyset$ and $\mathcal{B}_{n,m,k} = \mathcal{H}$ if $H_{n,m} = \emptyset$. It can be proved that $\mathcal{B}_{n,m,k}$ is a dense and open subset of $\mathcal{H}$ for every $n \geq 2$ and $m, k \in \mathbb{N}$. We omit the rather technical and cumbersome proof that can be found in [17]. Finally, it is easy to see that every element of $\mathcal{B}_0 := \cap_{n,m,k}\mathcal{B}_{n,m,k}$ is strongly exposed by each $x^* \in M$, being $M = \cup_{n,m}\{b(w) : w \in H_{n,m}\}$. By the Baire category theorem, $\mathcal{B}_0$ is dense G_δ in $\mathcal{H}$. Since M is dense in S^* and since the strongly exposing functionals form a G_δ subset, the proof is completed. The proof of (ii) is similar. $\square$

An interesting corollary is now at hand, as a direct consequence of the above result and the results in section 2.

Corollary 3.7. *Consider a Banach space X with dual X^*.*

(i) *If X has a fundamental biorthogonal system then X is almost Asplund.*
(ii) *If X^*has a fundamental biorthogonal system $\{x_i, x_i^*\}_{i\in I} \subset X^* \times X$ then X^* is almost weak* Asplund.*

Let N be the set of all equivalent norms on a Banach space X furnished with the metric ρ, defined in this way,

$$\rho(n_1, n_2) = \sup\{|n_1(x) - n_2(x)|; x \in B_{||\cdot||}\}, \qquad \text{where } n_1,\, n_2 \in N,$$

and N^* the set of all equivalent dual norms on X^*. Since N is an open subset of the complete metric space of all continuous seminorms on X under the distance ρ and the map $\pi : \|\cdot\| \to \|\cdot\|^*$ is an homeomorphism between N and N^*, both are Baire spaces. If the space $\mathcal{H}$ ($\mathcal{H}^*$) is replaced by the set of all unit balls of equivalent norms (dual norms, respectively), we obtain analogous results replacing F (F^*) by N (N^*).

There are few known Banach spaces without fundamental biorthogonal systems. In fact, the question whether every Banach space is almost Asplund remains open. According to Corollary 3.7, a possible counterexample should have no fundamental biorthogonal system. This is the case of Kunen space mentioned above, but it is Asplund. On the other hand, it is worth to mention that the duals of the Kunen and Shelah spaces are not almost weak* Asplund. In fact, there is no equivalent dual norm being Fréchet differentiable on a dense set in the preceding spaces. Otherwise, the unit ball of the associated (predual) norm in the Kunen or Shelah spaces would be the closed convex hull of its strongly exposed points. This would produce in the Kunen and Shelah spaces, by imitating the proof of Proposition 3.1, an uncountable family satisfying the separation property given in (3.4), thus a contradition. Plichko proved that $\ell_\infty^c(\Gamma)$ (being $\operatorname{card}\Gamma$ strictly bigger than the cardinal of the continuum) does not admit a fundamental biorthogonal system. We do not know if this space and the Shelah space are almost Asplund.

Next theorem illustrates, under a different point of view, the relationship between convexity and Mazur intersection properties. As an application, analogies and differences between these properties and the Radon-Nikodým property are exhibited. Our aim here is to point out that Mazur intersection properties seem to be a good alternative to Radon-Nikodým property when some convexity conditions are required [17], [23] and [29]. Recall that a Banach space X is said to have the Radon-Nikodým property if every element of $\mathcal{H}$ is the closed convex hull of its strongly exposed points. A Banach space X is Asplund if and only if X^* has the Radon-Nikodým property.

Theorem 3.8. *(A) Let X be a Banach space whose dual X^* has the weak* Mazur intersection property. Then*

- (i) *there exists a dense G_δ subset $\mathcal{B}_0 \subset \mathcal{H}$ such that every element of $\mathcal{B}_0$ is the closed convex hull of its strongly exposed points.*
- (ii) *there exists a dense G_δ subset $\mathcal{B}_0^* \subset \mathcal{B}_{X^*}^*$ such that every element of $\mathcal{B}_0^*$ is the weak* closed convex hull of its weak* strongly exposed points.*

(B) Let X be a Banach space with the Mazur intersection property. Then there is $\mathcal{B}_0$ satisfying (i) and there exists a dense G_δ subset $\mathcal{B}_0^* \subset \mathcal{B}_{X^*}^*$ such that every element of $\mathcal{B}_0^*$ is the weak closed convex hull of its weak* denting points.*

4. Intersection of closed balls and porosity

4.1. **Distance of two sets.** Given a normed space X, and two closed and bounded subsets $C, D \subset X$, denote by $\varrho(C, D) = \inf\{\|x - y\| : x \in A, y \in B\}$. F. Hausdorff calls $\varrho(A, B)$ the lower distance between A and B, though it is clear that it is not a metric, since the triangle inequality is not fulfilled. How to define then a distance between closed and bounded sets? Here is the most accepted formula, namely the *Hausdorff distance*, that we have already used in section 3:

$$\begin{aligned} d(C, D) &= \sup\{\varrho(x, D), \varrho(y, C) : x \in C, y \in D\} \\ &= \inf\{\varepsilon > 0 : C \subset D + \varepsilon B \quad \text{and} \quad D \subset C + \varepsilon B\} \end{aligned}$$

being B the unit ball. A well known theorem of H. Hahn establishes that the family of all closed and bounded sets of X, endowed with the Haudorff distance, is a complete metric space when X is complete [36]. Recall that $\mathcal{H}_X$ (or simply by $\mathcal{H}$, when it causes no confusion) denotes the family of all closed, bounded and convex subsets of X. To prove that $\mathcal{H}$ is also a complete metric space with the Hausdorff metric, when X is complete, it just suffices to prove that, given a convergent sequence $\{C_n\} \subset \mathcal{H}$, the limit C also is a convex set. We may assume that $d(C_n, C) < 1/n$, for every n. Defining $D_n = \overline{C_n + (1/n)B}$, we know that $C \subset D_n$ and $d(D_n, C) < 2/n$, so $\lim_n\{D_n\} = C$. Now, take $x, y \in C$ and suppose that z lies in the segment $[x, y] = \{tx + (1-t)y : t \in [0, 1]\}$. If $z \notin C$, there is $m \in \mathbb{N}$ satisfying $(z + (2/m)B) \cap C = \emptyset$. This implies that $z \notin D_m$, which contradicts the fact $z \in D_n$, for every n. Thus $z \in C$ and C is convex. Therefore, $\mathcal{H}$ endowed with the Hausdorff distance is a complete metric space and hence a Baire space.

4.2. **Porous sets.** Motivated by problems in Real Analysis and, especially, in differentiation theory, several authors considered what came to be known as *porosity*, a notion which concerns the size of holes of a set near a point. Topologically speaking, porous sets are smaller than merely being a countable union of nowhere dense closed sets [62]. Consequently, porosity has been usually used to describe smallness in a topological sense. Precisely, let M be a metric space, P a subset of M, $B(x, R)$ the closed ball centered at x with radius R and $\gamma(x, R, P)$ the supremum of all r for which there exists $y \in M$ such that $B(y, r) \subset B(x, R) \setminus P$. The number

$$\rho(x, P) = 2 \limsup_{R \to 0} \frac{\gamma(x, R, P)}{R}$$

is called the porosity of P at x. We say that P is porous at x whenever $\rho(x, P) > 0$ and, when P is porous at every point of M, we simply say that P is a porous set. If there is $\varepsilon > 0$ satisfying $\rho(x, P) > \varepsilon$ for every $x \in M$, then P is said to be uniformly porous. Finally, replacing "lim sup" by "lim inf" in the above definition, we encounter the notions of *very porosity* and *very porous set*, respectively. The unit sphere of a normed linear space is an easy example of an uniformly very porous set.

In convex geometry, the use of porosity received in recent years a great deal of attention. Several topics as smoothness, strict convexity, diameters, nearest points and others have been investigated by using porosity. We refer to the works of Zamfirescu [63], [64] and Gruber [25], [26] for more information about this rich line of research.

In Banach space theory, porosity has been used to describe topological properties of the set of points of Frechet nondifferentiability [48], [50] and also in relation with questions of best approximation [5] and variational principles [7]. For these and other applications of porosity, we refer to Zajicek's survey [62] and Phelps' book [48].

Let $\mathcal{M}$ be the collection of all intersections of balls, considered as a subset of $\mathcal{H}$ furnished with the Hausdorff metric. The space has the Mazur intersection property or MIP if $\mathcal{M} = \mathcal{H}$ [39]. We will prove that $\mathcal{M}$ is uniformly very porous if and only if the space fails the MIP. To this end, we need a handy description of the elements of $\mathcal{H} \setminus \mathcal{M}$, obtained as a consequence of Proposition 4.1, whose proof is partially based in Proposition 1.3. The only difficulty lies in (iii) implies (i) (see [30] for the details of the proof). In what follows, given $f \in X^*$, we denote $K_f = \ker f \cap B$, $L_f = \{x \in B : f(x) \geq 0\}$ and $M_f = \{x \in B : f(x) \leq 0\}$.

Proposition 4.1. *Given a Banach space, the following conditions are equivalent:*

(i) *The space has the Mazur intersection property.*
(ii) *There is a dense set $F \subset S^*$ satisfying $M_f \in \mathcal{M}$ ($L_f \in \mathcal{M}$) for each $f \in F$.*
(iii) *There is a dense set $F \subset S^*$ satisfying $K_f \in \mathcal{M}$ for each $f \in F$.*

Theorem 4.2. *The set $\mathcal{M}$ is uniformly very porous if and only if the space fails the Mazur Intersection Property.*

Proof. We find it convenient to isolate from the argument the following observation: *consider $C \in \mathcal{H}$ and $\lambda > 0$ so that $D = \{x \in C : d(x, \partial C) \geq \lambda\} \neq \emptyset$; every set $E \in \mathcal{H}$ with $d(C, E) < \lambda$ contains also D.* The proof is fairly easy: if $x \in D \setminus E$, there is a norm one functional f separating x and E. Say, for instance, that $f(x) > \sup f(E)$. Clearly, $\sup f(C) \geq f(x) + \lambda > \sup f(E) + \lambda$, so $d(C, E) > \lambda$, a contradiction.

By Proposition 4.1, if X fails the Mazur Intersection Property there is a norm one functional f such that $M_f \notin \mathcal{M}$. It means that there is also $x_0 \in B \setminus M_f$ such that every ball containing M_f contains also x_0. Denote by $\alpha = f(x_0) > 0$ and consider an arbitrary subset $C \in \mathcal{B}$. We will prove that

$$\rho(C, \mathcal{M}) = 2 \liminf_{R \to 0} \frac{\gamma(C, R, \mathcal{M})}{R} \geq \frac{\alpha}{1+\alpha} .$$

and the proof will be accomplished by looking at two cases.

Case 1. The functional f attains its maximum over C, say at $y_0 \in C$. Define the sets $C_R = \overline{C + RB}$ and $D_R = \{x \in C_R : f(x) \leq \sup f(C)\}$. Notice that $D_R \notin \mathcal{M}$ since D_R contains $y_0 + RM_f$ and misses the point $y_0 + Rx_0$. However, we do not know the existence of $r > 0$ such that $B_d(D_R, r) \subset \mathcal{H} \setminus \mathcal{M}$, which is necessary to compute the porosity of C. It is then convenient to select a suitable modification of D_R, namely the set $E_R = \overline{D_R + \frac{\alpha R}{2} B}$. We claim that the ball $B_d(E_R, \alpha R/2 - 1/n)$ satisfies

$$B_d(E_R, \alpha R/2 - 1/n) \cap \mathcal{M} = \emptyset$$

for $n \in \mathbb{N}$ large enough so that $\alpha R/2 - 1/n > 0$. Indeed, if $G \in \mathcal{H}$ and $d(G, E_R) \leq \alpha R/2 - 1/n$ then $y_0 + Rx_0 \notin G$ but, due to the first remark, $y_0 + RM_f \subset G$ so every ball containing G should contain also $y_0 + Rx_0$.

Now, since $d(E_R, C) \leq R + R\alpha/2$, then $B_d(E_R, \alpha R/2 - 1/n) \subset B(C, R + R\alpha)$. It means that $\gamma(C, R + R\alpha, \mathcal{M}) \geq \alpha R/2 - 1/n$, for n large enough, so $\gamma(C, R + R\alpha, \mathcal{M}) \geq \alpha R/2$, thus implying that

$$2 \liminf_{R \to 0} \frac{\gamma(C, R + R\alpha, \mathcal{M})}{R + R\alpha} \geq \liminf_{R \to 0} \frac{\alpha R}{R + R\alpha} = \frac{\alpha}{1+\alpha} .$$

Case 2. The functional f does not attain its maximum over C. Given $R > 0$, we take y_m so that $f(y_m) = \sup f(C)$ and $d(y_m, C) < R/m$. Consider now $C_m = \overline{\text{conv}}\,(\{y_m \cup C\})$. Since C_m satisfies the condition of Case 1, $\gamma(C_m, R + R\alpha, \mathcal{M}) \geq \alpha R/2$ and, consequently, $\gamma(C, R + R\alpha + R/m, \mathcal{M}) \geq \alpha R/2$. Therefore

$$2 \liminf_{R \to 0} \frac{\gamma(C, R + R\alpha + R/m, \mathcal{M})}{R + R\alpha + R/m} \geq \liminf_{R \to 0} \frac{\alpha R}{R + R\alpha + R/m} = \frac{\alpha}{1 + \alpha + 1/m}$$

for every $m \in \mathbb{N}$ and the theorem is proved. $\square$

Notice that, if $C \notin \mathcal{M}$, then $x + \lambda C \notin \mathcal{M}$ for every $x \in X$ and $\lambda \in \mathbb{R}$. It means that $\mathcal{M}$ is porous in a much stronger sense than stated in Theorem 4.2, and close to the notions of cone meager and angle-smallness introduced by Preiss and Zajicek (see [50] and [48]).

5. Stability of the sum in $\mathcal{M}$.

Two of the most important ways of combining two convex sets C, D to produce a third one are the vector sum $C + D$ and the convex hull $\operatorname{conv}(C \cup D)$, together with the operations $C \hat{+} D = \overline{(C + D)}$ and $\overline{\operatorname{conv}}\,(C \cup D)$ of forming the respective closures. The stability of $\mathcal{M}$ with respect to the usual set operations is very easy to check: $\mathcal{M}$ is stable under translations, dilations and intersections and it is not stable under unions, convex hulls and the closure of convex hulls. For instance, if you consider in $\mathbb{R} \oplus_\infty \mathbb{R}$ the sets $C = \{(0,0)\}$ and $D = \{(1,1)\}$, then $\operatorname{conv}(C \cup D)$ is not an intersection of balls. However, the situation with respect to the sum and the closure of the sum seems to be more complicated. The present note is concerned with the extent to which the property of being an intersection of balls is preserved by the operations $+$ and $\hat{+}$. We will concentrate our attention also in a modest but quite relevant question: let B be the unit ball of X, $\lambda > 0$ and $C \in \mathcal{M}$; is it true that $C + \lambda B \in \mathcal{M}$? An affirmative answer to this question would provide the following topological consequence for $\mathcal{M}$.

Proposition 5.1. *The set $\mathcal{M}$ is a closed subset of $\mathcal{H}$ provided $C \hat{+} \lambda B \in \mathcal{M}$ for every $C \in \mathcal{M}$ and each $\lambda > 0$.*

Proof. Let $\{C_n\}$ be a sequence in $\mathcal{M}$ and let $C \in \mathcal{H}$ be such that $\lim_n d(C_n, C) = 0$. To prove that $C \in \mathcal{M}$, take $x \notin C$ and let $\delta = \operatorname{dist}(x, C) > 0$. We may assume that $d(C_n, C) < \delta/4$, for every $n \in \mathbb{N}$. On the one hand, $C \subset C_n \hat{+} 2d(C_n, C)B$ and, on the other hand, $x \notin C_n \hat{+} 2d(C_n, C)B$. Now, as the set $C_n \hat{+} 2d(C_n, C)B$ is an intersection of balls, there is a ball D such that $x \notin D$ and $C \subset C_n \hat{+} 2d(C_n, C)B \subset D$. □

The stability of $\mathcal{M}$ under the operation $+$ implies, in particular, that $C + D$ is a closed set whenever $C, D \in \mathcal{M}$. Therefore, in this case, $C \hat{+} \lambda B = C + \lambda B \in \mathcal{M}$ and, by the above proposition, $\mathcal{M}$ is closed. Incidentally, let us mention that the stability under $\hat{+}$ does not imply the stability under the vector sum, as the following remark shows. Recall that many non-reflexive Banach spaces can be renormed to satisfy the MIP. The space $c_0(\mathbb{N})$ is the simplest example since every separable space with separable dual admits a Fréchet differentiable (and thus MIP) norm [9].

Remark 5.2. When X is a nonreflexive Banach space with the MIP and $C \in \mathcal{M}$, the set $C + \lambda B$ need not be closed. Consequently, $\mathcal{M}$ need not be stable under vector sums, even if it is stable under $\hat{+}$.

Detail. Indeed, when X is nonreflexive, there is a functional $f \in S^*$ which does not attain its norm. Since X has the MIP and $\mathcal{M} = \mathcal{H}$, the set $C = \{x \in B : f(x) \leq 0\}$ is an intersection of balls. However, this is not the case for $C + \lambda B$ because this set is not closed when $0 < \lambda < 1/2$. Indeed, there is $x \in (1/2)B$ for which $f(x) = \lambda$. Hence for all n with $1/n < 1/2 - \lambda$ we have $x + (\lambda + 1/n)B \subset B$. Then $\emptyset \neq D_n = (x + (\lambda + 1/n)B) \cap f^{-1}((-\infty, 0]) \subset C$. If $x_n \in D_n$, clearly $(x_n + \lambda B) \cap (x + (1/n)B) \neq \emptyset$ and so x is in the closure of $C + \lambda B$. However, $C \cap f^{-1}(\lambda) = \emptyset$ and so $x \notin C + \lambda B$.

5.1. **The binary intersection property.** When B is the unit ball and $C = \cap_i B_i$ is an intersection of balls, it is tempting to write

$$C + \lambda B = \cap_i B_i + \lambda B = \cap_i (B_i + \lambda B) \tag{5.1}$$

and, as a consequence, to conclude that $C + \lambda B \in \mathcal{M}$. However, (5.1) is false in general. To be convinced of this, consider $(\mathbb{R}^2, \|\cdot\|_2)$ and define B_1 as the Euclidean unit ball, $B_2 = B_1 + (2, 0)$ and take $\lambda = 1$.

As an easy example, notice that (5.1) holds in $(\mathbb{R}^n, \|\cdot\|_\infty)$. Sine [55] proved that (5.1) is satisfied in those normed spaces with the so called *binary intersection property* (BIP): every collection of mutually intersecting closed balls has nonempty intersection. However, we will prove in Section 5.2 that the validity of (5.1) for every $\lambda > 0$ does not characterizes spaces with the BIP. This property plays a major role in questions of extendability of general continuous linear maps, as proved by Nachbin and Goodner (see [45] and references therein). We note that normed spaces with the BIP are complete. Moreover, a Banach space X has the BIP if and only if $X = C(K, \mathbb{R})$ with the supremum norm, where K is a extremally disconnected, or *Stonean*, compact Hausdorff space (Nachbin [43], Goodner [27] and Kelley [32]). The following proposition improves that above mentioned result of Sine.

Proposition 5.3. *If a normed space X has the BIP then every (nonempty) $C = \cap_i B_i \in \mathcal{M}$ and $D \in \mathcal{M}$ satisfy $\cap_i B_i + D = \cap_i (B_i + D)$.*

Proof. Recall that, as noted above, we have $X = C(K, \mathbb{R})$. Given an extreme point e of the unit ball of X, there is only one way of making X into a complete vector lattice having e as an order unit such that the norm deduced from the order relation and e is identical to the sup norm [43]. For instance, we can choose $e = \mathbf{1}_K$ if the canonical order induced by $\mathbb{R}$ in $C(K, \mathbb{R})$ is desired. Every closed ball is identical to a segment and, in particular, $B_i = B(x_i, r_i) = [x_i - r_i e, x_i + r_i e]$. Therefore,

$$C = \cap_i B_i = [\sup_i \{x_i - r_i e\}, \inf_i \{x_i + r_i e\}]$$

and, analogously, $D = [\alpha, \beta]$. Indeed, given any bounded family $\{f_i\} \subset C(K)$ both $\inf_i f_i$ and $\sup_i f_i$ (taken in the order of $C(K)$) are continuous functions on K (see [38], Prop.

1.a.4). Consequently,

$$\begin{aligned}
\cap_i B_i + D &= [\sup_i\{x_i - r_i e\}, \inf_i\{x_i + r_i e\}] + [\alpha, \beta] \\
&= [\sup_i\{x_i - r_i e\} + \alpha, \inf_i\{x_i + r_i e\} + \beta] \\
&= [\sup_i\{\alpha + x_i - r_i e\}, \inf_i\{\beta + x_i + r_i e\}] \\
&= \cap_i (B_i + D)
\end{aligned}$$

□

5.2. **The case of $c_0(I)$.** The geometry of the unit ball of the space $\ell_\infty(I)$ is quite close to that of the unit ball of $c_0(I)$. Thus, it seems quite natural to ask about the stability of $\mathcal{M}$ in this latter space. (Recall that for a (not necessarily countable) set I, a point $x = (x_i)$ is in $c_0(I)$ provided $x_i \to 0$ in the sense that for any $\epsilon > 0$, there are only finitely many indices $i \in I$ for which $|x_i| > \epsilon$.) First of all, we must try to obtain an easy-to-use description of sets which are intersection of balls. Denote by $\{e_i\}$ and $\{f_i\}$ the canonical basis of $c_0(I)$ and the associated functionals, respectively. Since the unit ball for the supremum norm on $c_0(I)$ is $B = \cap_i f_i^{-1}([-1, 1])$ it is easy to show that B' is a closed ball with radius $\lambda > 0$ if and only if it has the form $B' = \cap_i f_i^{-1}([a_i, b_i])$, where $a_i \to -\lambda$, $b_i \to \lambda$ and $b_i - a_i = 2\lambda$ for all $i \in I$. Consequently, if $\{B_\alpha = \cap_i f_i^{-1}([a_{\alpha i}, b_{\alpha i}])\}$ is a collection of closed balls with nonempty intersection, we have

$$\begin{aligned}
\cap_\alpha B_\alpha &= \cap_\alpha \cap_i f_i^{-1}([a_{\alpha i}, b_{\alpha i}]) \\
&= \cap_i \cap_\alpha f_i^{-1}([a_{\alpha i}, b_{\alpha i}]) \\
&= \cap_i f_i^{-1}([\sup_\alpha\{a_{\alpha i}\}, \inf_\alpha\{b_{\alpha i}\}])\,. \qquad (5.2)
\end{aligned}$$

Moreover, fixing an index α_0, for each i we have $a_{\alpha_0 i} \le \sup_\alpha a_{\alpha i} \le \inf_\alpha b_{\alpha i} \le b_{\alpha_0 i}$ and, as a consequence, there exists $k > 0$ such that $-k \le \sup_\alpha a_{\alpha i} \le \inf_\alpha b_{\alpha i} \le k$ for every $i \in I$. Conversely, *a set $C = \cap_i f_i^{-1}[a_i, b_i]$ is an intersection of balls provided there exists $k > 0$ such that $-k \le a_i \le b_i \le k$ for all i.* To see this, let $x \notin C$ and suppose, for instance, that $f_{i_0}(x) < a_{i_0}$. We claim that the ball $(a_{i_0} + k)e_{i_0} + kB$ contains C but not x. To this end, note first that $f_{i_0}(x - (a_{i_0} + k)e_{i_0}) < -k$ so $x \notin (a_{i_0} + k)e_{i_0} + kB$. Clearly, $C \subset kB \cap f_{i_0}^{-1}([a_{i_0}, b_{i_0}])$ and, also,

$$kB \cap f_{i_0}^{-1}([a_{i_0}, b_{i_0}]) \subset (a_{i_0} + k)e_{i_0} + kB\,.$$

Indeed, if $y \in kB \cap f_{i_0}^{-1}([a_{i_0}, b_{i_0}])$, then $f_{i_0}(y - (a_{i_0} + k)e_{i_0}) = f_{i_0}(y) - (a_{i_0} + k) \ge -k$ and also $f_{i_0}(y) - (a_{i_0} + k) \le b_{i_0} - a_{i_0} - k \le k - k - a_{i_0} = -a_{i_0} \le k$. For any other index $i \ne i_0$, we have $|f_i(y - (a_{i_0} + k)e_{i_0})| = |f_i(y)| \le k$. We are ready now to state the next proposition.

Proposition 5.4. *Given C and D two (nonempty) intersections of balls in $c_0(I)$, the set $C+D$ is also an intersection of balls. Precisely, if $C = \cap_i f_i^{-1}[a_i, b_i]$ and $D = \cap_i f_i^{-1}[c_i, d_i]$, then $C + D = \cap_i f_i^{-1}[a_i + c_i, b_i + d_i]$.*

Proof. The inclusion $C + D \subset \cap_i f_i^{-1}[a_i + c_i, b_i + d_i]$ is straightforward. To prove the reverse inclusion, we will assume that $0 \in [c_i, d_i]$ for every $i \in I$ (otherwise, we would replace C and D by $C' = C - u$ and $D' = D - u$ for some $u \in D$). Let $z = \sum_i z_i e_i \in \cap_i \{f_i^{-1}([a_i + c_i, b_i + d_i])\}$. We want $x = \sum_i x_i e_i \in C$ and $y = \sum_i y_i e_i \in D$ such that $z_i = x_i + y_i$ for every $i \in I$. Since $a_i + c_i \le a_i \le b_i \le b_i + d_i$, each $i \in I$ falls into one (and only one) of the following subsets:

$$\begin{aligned} I_1 &= \{i \in I : a_i + c_i \le z_i < a_i\} \\ I_2 &= \{i \in I : a_i \le z_i \le b_i\} \\ I_3 &= \{i \in I : b_i < z_i \le b_i + d_i\} \ . \end{aligned}$$

We define $x_i = a_i$ in case $i \in I_1$, $x_i = z_i$ in case $i \in I_2$ and $x_i = b_i$ in case $i \in I_3$. Obviously, $a_i \le x_i \le b_i$ and $c_i \le y_i = z_i - x_i \le d_i$. Since $|x_i| \le |z_i|$ for all $i \ge m$ for some $m \in \mathbb{N}$, we are assured that x (and hence y) is an element of $c_0(I)$. □

Corollary 5.5. *If $C = \cap_\alpha B_\alpha$ is a nonempty intersection of balls in $(c_0, \|\cdot\|_\infty)$ and $\lambda > 0$ then $\cap_\alpha B_\alpha + \lambda B = \cap_\alpha (B_\alpha + \lambda B)$. Consequently, the validity of* (5.1) *does not characterizes the BIP.*

Proof. Since $B_\alpha = \cap_i f_i^{-1}([a_{\alpha i}, b_{\alpha i}])$ and $\lambda B = \cap_i f_i^{-1}([-\lambda, \lambda])$, Proposition 5.4 implies that $B_\alpha + \lambda B = \cap_i f_i^{-1}([a_{\alpha i} - \lambda, b_{\alpha i} + \lambda])$. As a consequence, using again Proposition 5.4 we obtain

$$\begin{aligned} \cap_\alpha B_\alpha + \lambda B &= \cap_i f_i^{-1}([\sup_\alpha\{a_{\alpha i}\}, \inf_\alpha\{b_{\alpha i}\}]) + \cap_i f_i^{-1}([-\lambda, \lambda]) \\ &= \cap_i f_i^{-1}([\sup_\alpha\{a_{\alpha i}\} - \lambda, \inf_\alpha\{b_{\alpha i}\} + \lambda]) \\ &= \cap_i f_i^{-1}([\sup_\alpha\{a_{\alpha i} - \lambda\}, \inf_\alpha\{b_{\alpha i} + \lambda\}]) \\ &= \cap_i \cap_\alpha f_i^{-1}([a_{\alpha i} - \lambda, b_{\alpha i} + \lambda]) \\ &= \cap_\alpha \cap_i f_i^{-1}([a_{\alpha i} - \lambda, b_{\alpha i} + \lambda]) \\ &= \cap_\alpha (B_\alpha + \lambda B)\,. \end{aligned} \tag{5.3}$$

□

5.3. **Polyhedral norms.** Recall that a Banach space is *polyhedral* [33] if the unit ball of any of its finite dimensional subspaces is a polyhedron. The typical example of a polyhedral space is c_0 endowed with the usual supremum norm. Is it true that $\mathcal{M}$ is stable under vector sums in every polyhedral space? We will answer this question in the negative, despite the fact that the geometry of the unit ball of these spaces is quite close to that of $(c_0, \|\cdot\|_\infty)$.

Most of the knowledge that we have about polyhedral spaces is due to the work of V. Fonf (see [13] and [14]). Among many other things he proved that, given a polyhedral Banach space X with unit ball B, there is a set (not necessarily countable) $\{f_i\}_{i\in I}$ of norm–one functionals such that:

$$\text{For every } x \in X, \text{ there is } i_0 \in I \text{ such that } \quad \|x\| = f_{i_0}(x) \tag{5.4}$$

$$\text{For every } i \in I, \quad f_i^{-1}(\{1\}) \cap B \quad \text{has nonempty (relative) interior in} \quad f_i^{-1}(\{1\}) \tag{5.5}$$

With this tool in our hands, we easily obtain a description of the sets in $\mathcal{M}$ which is just a generalization of the one obtained for $c_0(\Gamma)$. In the following proposition, we keep the above notation (see [24].

Proposition 5.6. *A bounded convex set C in a polyhedral Banach space is an intersection of balls if and only if $C = \cap_i f_i^{-1}([\inf f_i(C), \sup f_i(C)])$.*

The Proposition above implies that in a finite dimensional Banach space with polyhedral norm, every set in $\mathcal{M}$ is a *finite* intersection of balls. The first question pertaining to the stability of $\mathcal{M}$ in a polyhedral space is whether, given two sets $C = \cap_i f_i^{-1}[a_i, b_i]$ and $D = \cap_i f_i^{-1}[c_i, d_i]$, one has

$$C + D = \cap_i f_i^{-1}[a_i + c_i, b_i + d_i] \ . \tag{5.6}$$

As the next proposition shows, the answer to this question can be negative, even if we reformulate the question in a slightly different way: Is (5.6) true if we assume, in addition, that $a_i = \inf f_i(C)$, $b_i = \sup f_i(C)$, $c_i = \inf f_i(D)$ and $d_i = \sup f_i(D)$? The answer is again *no*, since a positive answer would imply the stability of $\mathcal{M}$ under vector sums in every polyhedral Banach space, and this is not the case even in finite dimensional spaces.

Proposition 5.7. *The set $\mathcal{M}$ is not stable under vector sums in $(\mathbb{R}^n, \|\cdot\|_1)$, $n > 3$ or in $\ell_1(I)$.*

Proof. The segment C joining the point $(1/2, 1/2, 0)$ with $(-1/2, -1/2, 0)$ is an intersection of exactly two balls of radius 1. This is also the case of the segment D joining the point $(-1/2, 1/2, 0)$ with $(1/2, -1/2, 0)$. However, the set $C + D$ is not an intersection of balls. Indeed, denote by $\{f_1, f_2, f_3, f_4\}$ the norm one functionals satisfying (5.4) and (5.5) and by B the unit ball. Since $C + D = B \cap \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, we have that $\inf f_i(C + D) = -1$ and $\sup f_i(C + D) = 1$ for every $i = 1, ..., 4$. According to Corollary

6.4, if $C+D$ were an intersection of balls then

$$C+D=\cap_i f_i^{-1}([\inf f_i(C+D),\sup f_i(C+D)])=\cap_i f_i^{-1}([-1,1])=B$$

which is a contradiction.

The spaces $(\mathbb{R}^n,\|\cdot\|_1)$ and $\ell_1(I)$ are particular cases of $X=Y\oplus_1 Z$ where $Y=(\mathbb{R}^3,\|\cdot\|_1)$ and $\oplus_1$ denotes that the sum is endowed with the ℓ_1-norm. The intersection of every ball in X with the subspace Y is an ℓ_1-ball. As a consequence, if a closed, bounded and convex subset of Y is an intersection of X-balls, it is also an intersection of Y-balls. Finally, the sets C and D considered in the above paragraph are intersection of X-balls but this is not the case for the set $C+D$. For instance, to see that D is the intersection of the two balls $B_1=(1/2,1/2,0)+B$ and $B_2=(-1/2,-1/2,0)+B$, just take into account that, for every $x=(x_1,x_2,x_3)+z\in B_1\cap B_2$ we have $|x_1-1/2|+|x_2-1/2|+|x_3|+\|z\|\le 1$ and $|x_1+1/2|+|x_2+1/2|+|x_3|+\|z\|\le 1$. Consequently,

$$\begin{aligned}|x_1-1/2|+|x_2-1/2|&\le 1-|x_3|-\|z\|\\|x_1+1/2|+|x_2+1/2|&\le 1-|x_3|-\|z\|\end{aligned}$$

and the only solution is when $|x_3|=\|z\|=0$ and $(x_1,x_2)\in D$. □

It has been proved in [24] that in $(\mathbb{R}^3,\|\cdot\|_1)$ the family $\mathcal{M}$ is stable under adding balls. As a consequence, we get that this property is different from being stable under the closure of vector sums. Though the result is also true for $(\mathbb{R}^n,\|\cdot\|_1)$ with $n>3$, the arguments of the proof are those of the tridimensional case, which has the advantage of great simplicity.

In Remark 5.2, we observed the existence of spaces for which $\mathcal{M}$ is not stable by adding balls. However, *we have no example of a normed space for which $\mathcal{M}$ is not stable under the operation $C\hat{+}\lambda B$, $C\in\mathcal{M}$ and $\lambda>0$.* On the other hand, the set of norms for which $\mathcal{M}$ is stable under vector sums is not closed in the space of all equivalent norms, endowed with the uniform metric. Indeed, in a finite dimensional Banach space, the set of norms with the Mazur intersection property is dense.

6. Mazur sets and Mazur spaces

As we mentioned in the introduction, a set C is an intersection of balls if it satisfies the following separation property: For every $x\notin C$, there is a closed ball B such that $C\subset B$ but $x\notin B$. This property can be strengthened by simply replacing the point x by a hyperplane. We say that C is a **Mazur set** if given any hyperplane H with $\mathrm{dist}(C,H)>0$, there is a ball D such that $C\subset D$ and $D\cap H=\emptyset$. Note that this is equivalent to saying that C *is a Mazur set if given* $f \in X^*$ *with* $\sup f(C) < \lambda$, *then there exists a ball* D *such that* $C \subset D$ *and* $\sup f(D) < \lambda$. (Consider the hyperplane $H = f^{-1}(\lambda)$). Denote by $\mathcal{P}$ the collection of all Mazur sets of a normed space.

By the separation theorem, every Mazur set is an intersection of balls and so $\mathcal{P} \subset \mathcal{M} \subset \mathcal{H}$. However, we will show that the converse is not always true, even if the norm is Fréchet differentiable. There are mainly two reasons connecting Mazur sets with the subject of this paper: On the one hand, $\mathcal{P}$ is always stable under (the closure) of vector sums; on the other hand, sometimes $\mathcal{P} = \mathcal{M} \neq \mathcal{H}$.

Proposition 6.1. *Given two Mazur sets* C *and* D, *the set* $C \hat{+} D$ *is always a Mazur set. However,* $C \cap D$ *is not necessarily a Mazur set. Consequently,* $\mathcal{P}$ *is always stable under the closure of vector sums but it is not necessarily stable with respect to intersections.*

Proof. Let C and D be two Mazur subsets of a Banach space X. Consider a functional $f \in X^*$ and $\lambda \in \mathbb{R}$ such that $\sup f(C \hat{+} D) < \lambda$. Denote by $\alpha = \sup f(C)$ and $\beta = \sup f(D)$. Clearly, $\sup f(C \hat{+} D) = \sup f(C) + \sup f(D)$ and so $\alpha + \beta < \lambda$. Therefore, there are two real numbers α' and β' satisfying $\alpha < \alpha'$, $\beta < \beta'$ and $\alpha' + \beta' < \lambda$. Now, since C and D are Mazur sets, there are two closed balls B_1 and B_2 such that $C \subset B_1$ and $D \subset B_2$ satisfying $\sup f(B_1) < \alpha'$ and $\sup f(B_2) < \beta'$. The sum of the two balls B_1 and B_2 is again a ball B_3 that obviously contains $C \hat{+} D$ and satisfies

$$\sup f(B_3) = \sup f(B_1) + \sup f(B_2) < \alpha' + \beta' < \lambda.$$

Since we know that there exist Banach spaces for which $\mathcal{M}$ is not stable under the closure of vector sums (we proved that $(\mathbb{R}^3, \|\cdot\|_1)$ is such an example), the first part of this proposition implies that $\mathcal{P}$ can actually be different from $\mathcal{M}$. The two segments C and D of Proposition 5.7 are the intersection of two balls (which are, obviously, Mazur sets) but they themselves are not Mazur sets. □

Definition 6.2. Spaces in which every element of $\mathcal{M}$ is a Mazur set ($\mathcal{P} = \mathcal{M}$) will be called *Mazur spaces.*

In an analogous way, we can define a subset C of a dual Banach space X^* to be a weak* Mazur set if it can be separated by balls from weak* closed hyperplanes H with $\text{dist}(C, H) > 0$. We can denote the family of all weak* Mazur sets by $\mathcal{P}^*$ and we can say that X^* is a weak* Mazur space if $\mathcal{P}^* = \mathcal{M}$. Proposition 6.1 can be formulated for weak* Mazur sets and proved in essentially the same way. We do not know, however, an example of a weak* Mazur set which is not a Mazur set. Therefore, we know no example of a weak* Mazur *space* which is not a Mazur space (that is, a dual space for which $\mathcal{P} \subsetneq \mathcal{P}^* = \mathcal{M}$). Going back to Mazur spaces, the next proposition shows that the case $\mathcal{P} = \mathcal{M} = \mathcal{H}$ has a nice geometric characterization, in terms of weak* denting points of the dual unit ball. Recall that a Banach space satisfies the MIP if and only if the set of weak* denting points of the dual ball is a residual set of the dual sphere [18] (see also Proposition 1.3).

Proposition 6.3. *A Mazur space X satisfies the Mazur intersection property if and only if every norm one functional in X^* is a weak* denting point of B^*.*

Proof. Chen and Lin proved in [10] that f is a weak* denting point of the dual unit ball B^* if, and only if, for every bounded subset $A \subset X$ with $\inf f(A) > 0$ there is a ball D containing A such that $\inf f(D) > 0$. Suppose that $\mathcal{P} = \mathcal{H}$ and consider $f \in B^*$ and a bounded subset A such that $\inf f(A) > 0$. Then $C \equiv \overline{\operatorname{conv}}(A) \in \mathcal{P}$ and thus there is a ball D satisfying $A \subset C \subset D$ with $\inf f(D) > 0$. Conversely, let $C \in \mathcal{H}$ and H be a closed hyperplane such that $\operatorname{dist}(C, H) > 0$. We may assume that H is the kernel of a norm-one functional $f \in B^*$ and $\inf f(C) > 0$. The existence of the desired ball is due to the fact that f is a weak* denting point. □

In Remark II.7.6 of [9], there is an example of a dual norm on $\ell_1(\mathbb{N})$ with the property that every point of the unit sphere is a weak* denting point. Consequently, Mazur spaces with the MIP need not be reflexive, although they are certainly Asplund spaces. Indeed, their dual spaces admit dual LUR norms [51] and, therefore, they admit Fréchet differentiable norms. Spaces for which every point of the unit sphere is a denting point can be characterized as those satisfying a weaker notion of local uniform rotundity introduced by Troyanski in [58] and called *average locally uniform rotundity* (see also [9]). On the other hand, there is a wide family of Banach spaces which are not Asplund spaces, even though they can be renormed to satisfy the MIP [31]. Obviously, these (renormed) spaces cannot be Mazur spaces. The next corollary contains an example of an Asplund space satisfying the MIP but failing to be a Mazur space.

Corollary 6.4. *A reflexive space with a Fréchet differentiable norm is always a Mazur space. However, spaces with Fréchet differentiable norms need not be Mazur spaces. Finally, Mazur spaces with the MIP are always smooth spaces.*

Proof. In a reflexive space with a Fréchet differentiable norm, every norm one functional of the dual is the differential of the norm at some point. Consequently, it is a weak* strongly exposed point (and thus a weak* denting point) of the dual unit ball.

On the other hand, it is well known that there is only a partial duality between smoothness and convexity. As a matter of fact, from the pioneering results about renormings on spaces of continuous functions on scattered compact spaces due to Talagrand [57], we know that there are spaces with Fréchet differentiable norm whose dual space admits no rotund norm. This is the case, for instance, for $C([0, \omega_1])$. Since every weak* denting point is also an extreme point, the proposition above implies that the dual norm of a Fréchet norm in a Mazur space must be rotund. As a consequence, $C([0,\omega_1])$, endowed with an equivalent Fréchet differentiable norm is not a Mazur space. The previous proposition shows that, in particular, a Mazur space with the MIP has a dual rotund norm and thus the norm of the space itself is Gâteaux differentiable. $\square$

To finish our discussion on Mazur spaces and the MIP, notice that the condition of Fréchet differentiability in Corollary 6.4 is essential. Indeed, there are even finite dimensional Banach spaces with the MIP which are not Mazur spaces. Take, for instance, a norm in $\mathbb{R}^3$ with a dense set of denting points which contains a segment in its unit sphere. The predual norm has the MIP but $\mathbb{R}^3$ endowed with this predual norm is not a Mazur space.

6.1. **Examples of Mazur spaces.** This section is devoted to presenting some examples of Mazur spaces which are not merely reflexive spaces with a Fréchet differentiable norm. We will prove that this is the case for $c_0(I)$ and $\ell_\infty(I)$ with their usual norms. These spaces are natural candidates to be Mazur spaces in view of the results obtained in Sections 2 and 3. It is a bit surprising that every two dimensional space is a Mazur space. This result distinguishes dimension $d \leq 2$ from dimension $d \geq 3$: Note that $(\mathbb{R}^3, \|\cdot\|_1)$ is not a Mazur space, since $\mathcal{M}$ is not stable under vector sums (Proposition 5.7).

Proposition 6.5. *For every set I, the space $(c_0(I), \|\cdot\|_\infty)$ is a Mazur space.*

Proof. Consider $C = \cap_i f_i^{-1}([a_i, b_i])$, a norm one functional $f = \sum_i y_i e_i^* \in \ell_1$ and two real numbers $\alpha > \beta$ such that $\inf f(C) = \alpha > \beta$. There is no loss in generality in assuming that $0 \in C$. We must find a ball D such that $C \subset D$ and $\inf f(D) > \beta$. We know from Section 5.2 that $D = \cap_i f_i^{-1}([c_i, d_i])$ is a ball of radius $\lambda > 0$ if and only if $c_i \to -\lambda$, $d_i \to \lambda$ and $d_i - c_i = 2\lambda$. Since we want $C \subset D$, we need $[a_i, b_i] \subset [c_i, d_i]$ for every $i \in I$ and accordingly we choose $\lambda = \sup\{\max\{|a_i|, |b_i|\}\}$. The strategy will be to define $c_i = -\lambda$ and $d_i = \lambda$ except for a finite number of coordinates. More precisely, let $F \subset I$ be a finite set such that $\sum_{i \notin F} |y_i| < (\alpha - \beta)/2\lambda$. For every $i \in F$, we define

$$c_i = \begin{cases} a_i & \text{if} \quad y_i > 0 \\ b_i - 2\lambda & \text{if} \quad y_i \leq 0 \end{cases} \qquad\qquad d_i = \begin{cases} a_i + 2\lambda & \text{if} \quad y_i > 0 \\ b_i & \text{if} \quad y_i \leq 0 \end{cases}$$

and, for every $i \notin F$, take $c_i = -\lambda$ and $d_i = \lambda$. It is easy to check that $D = \cap_i f_i^{-1}([c_i, d_i])$ is a ball and that $C \subset D$. We just need to compute $\inf f(D)$. Let $F^+ = \{i \in F : y_i > 0\}$.

For every $x \in D$ we have

$$\begin{aligned} f(x) = \sum_{i \in I} x_i y_i &= \sum_{i \in F^+} x_i y_i + \sum_{i \in F \setminus F^+} x_i y_i + \sum_{i \notin F} x_i y_i \\ &\geq \sum_{i \in F^+} a_i y_i + \sum_{i \in F \setminus F^+} (b_i - 2\lambda) y_i - \sum_{i \notin F} \lambda |y_i| \\ &\geq \sum_{i \in F^+} a_i y_i + \sum_{i \in F \setminus F^+} b_i y_i - (\alpha - \beta)/2 \\ &\geq \alpha - (\alpha - \beta)/2 = (\alpha + \beta)/2 > \beta \end{aligned}$$

since the point $\sum_{i \in F^+} a_i e_i + \sum_{i \in F \setminus F^+} b_i e_i$ is an element of $C = \cap_i f_i^{-1}([a_i, b_i])$. Indeed, $a_i, b_i \in [a_i, b_i]$ when $i \in F$ and, for the rest of coordinates, 0 always belongs to $[a_i, b_i]$ since $0 \in C$. □

Proposition 6.6. *Let K be a Stonean compact Hausdorff space. Then $C(K)$ is a Mazur space.*

We finish this section with a result that distinguishes dimension $d \leq 2$. Indeed, we will see later that for normed linear space X with dimension greater than 2 there is an equivalent norm $\|\cdot\|$ for which $(X, \|\cdot\|)$ is not a Mazur space.

Theorem 6.7. *Every two dimensional normed linear space is a Mazur space.*

The following lemma is a key tool in proving Theorem 6.7. We will denote by B^* the dual unit ball of B. As usual, $\operatorname{ext} C$ stands for the collection of all extreme points of C.

Lemma 6.8. *Suppose that $C \in \mathcal{M}$, $x \in \partial C$ and that there exists $f \in \partial B^* \setminus \operatorname{ext} B^*$ satisfying $f(x) = \sup f(C)$. Then there is $y \in B$ with $f(y) = \sup f(B)$ such that any $g \in \partial B^*$ with $g(y) = \sup g(B)$ satisfies $g(x) = \sup g(C)$.*

Proof. Since f is not an extreme point of B^*, there is a vertex $y \in \partial B$ such that $f(y) = 1$. Suppose that there is $g \in \partial B^*$ with $g(y) = 1$ but $g(x) < \sup g(C)$. Choose $h \in \partial B^*$ with $h(y) = 1$ such that f lies in the interior of the segment defined by h and g. Let x' be the intersection of the lines $\{s \in \mathbb{R}^2 : h(s) = h(x)\}$ and $\{s \in \mathbb{R}^2 : g(s) = \sup g(C)\}$. Since $x' \notin C$, the proof of the lemma will be accomplished by showing that x' is in every ball containing C, which provides a contradiction.

Indeed, let $a + \lambda B$ be a ball such that $C \subset a + \lambda B$. Consider a point $z \in C$ satisfying $g(z) = \sup g(C)$. Necessarily

$$g(a + \lambda y) \geq g(a + \lambda B) \geq \sup g(C) = g(z)$$

and, analogously, $h(a+\lambda y) \geq h(x)$. Hence we have

$$x' \in \operatorname{conv}\{x, z, a+\lambda y\}$$

which implies $x' \in C$. □

Notice that the condition $f \notin \operatorname{ext} B^*$ was essential in the above lemma. In fact, the statement is not true for extreme points. Suppose, for instance, that D is the euclidean ball in $\mathbb{R}^2$ and $B = \{(2,0)+3D\} \cap \{(0,2)+3D\} \cap \{(-2,0)+3D\} \cap \{(0,-2)+3D\}$. Let $\{y\} = \partial B \cap \{(t,t), t>0\}$ and let f be the unique functional supporting $(0,-2)+3D$ at y. Define $C = \{(x_1,x_2) \in \mathbb{R}^2 : -1 \leq f(x_1,x_2) \leq 1, -1 \leq x_1 \leq 1\}$ and let $\{x\} = \partial C \cap \{(0,t), t>0\}$. The only support point of f in B is y and B has many other support functionals at y, while f is the only functional supporting C at x.

On the other hand, the lemma is not valid for higher dimensional spaces. Consider, for instance, the space $(\mathbb{R}^3, \|\cdot\|_1)$, the set $C = \{(t,t,0), -1 \leq t \leq 1\}$, the point $x = (0,0,0) \in C$ and the functional $f(x_1,x_2,x_3) = x_3$ which is not an extreme point of the dual unit ball.

Proof of Theorem 6.7 It is enough to show that for any $C \in \mathcal{M}$, $f \in \partial B^*$ and $\varepsilon > 0$, there is a closed ball B_ε containing C and satisfying $\sup f(B_\varepsilon) = \sup f(C) + \varepsilon$. We split the proof into two cases.

Case 1: $f \in \operatorname{ext} B^*$. There exists $y \in \partial B$ such that $f(y) = 1$ and the line $L = \{s \in \mathbb{R}^2 : f(s) = 1\}$ is (at least) a one-sided tangent to B at y. Since y defines two sides in L, it is convenient to fix one which is tangent to B and call it the *positive* side (with respect to y). Let $L_C = \{x \in \mathbb{R}^2 : f(x) = \sup f(C)\}$ and $L_\varepsilon = \{x \in \mathbb{R}^2 : f(x) = \sup f(C) + \varepsilon\}$. We fix a point $z \in L_\varepsilon$ satisfying, first, that $\{z + sy : s \in \mathbb{R}\} \cap C = \emptyset$ and second, that the set $\partial C \cap \{x \in \mathbb{R}^2 : f(x) = \sup f(C)\}$ lies in the positive side of L_C with respect to the point

$$z' = \{z + sy : s \in \mathbb{R}\} \cap L_C \ .$$

Finally, for every $\lambda > 0$, we consider the point $a_\lambda = z - \lambda y$ and the ball $a_\lambda + \lambda B$. We just need to show that there is $\lambda_C > 0$ such that $C \subset a_{\lambda_C} + \lambda_C B$. To do that, we first choose a point b in the positive side of L_C with respect to z' and $\lambda_0 > 0$ such that

$$C \subset \operatorname{conv}\{b,\ a_{\lambda_0},\ z = a_{\lambda_0} + \lambda_0 y\} \,.$$

We need only find $\lambda_C > 0$ satisfying $\lambda_C \geq \lambda_0$ and $b \in a_{\lambda_C} + \lambda_C B$. Consider the point $b' = L_\varepsilon \cap \{a_{\lambda_0} + s(b - a_{\lambda_0}) : s \in \mathbb{R}\}$ and define the sequence $\{x_n = z + (b'-z)/n\}$. Let y_n be the corresponding point of $\partial(a_1 + B)$ such that the segment joining x_n and y_n is orthogonal (in the euclidean sense) to L_ε. If $x_n \in \partial(a_1 + B)$, in this case we define

$y_n = x_n$. Notice that y_n is well defined for n sufficiently large. Since the positive side of L_ε is tangent to $a_1 + B$ at z, we have

$$\|y_n - x_n\| \; \|x_n - z\|^{-1} \;=\; n\,\|y_n - x_n\| \; \|b' - z\|^{-1} \;\overset{n\to\infty}{\longrightarrow}\; 0\,.$$

Therefore, there is an n_0 such that

$$n\,\|y_n - x_n\| \; \|b' - z\|^{-1} \;<\; \varepsilon\,\|b' - z\|^{-1}$$

for every $n \geq n_0$. As a consequence, $n\,\|y_n - x_n\| \;<\; \varepsilon$ and hence $\partial(a_n + nB) \cap [b', b] \neq \emptyset$ for $n \geq n_0$. This implies that $b \in a_n + nB$ for $n \geq n_0$. To finish the proof of Case 1, define $\lambda_C = \max\{\lambda_0, n_0\}$.

Case 2: $f \notin \operatorname{ext} B^*$. Let $\phi, \psi \in \operatorname{ext} B^*$ be such that f lies in the interior of the segment $[\phi, \psi] \subset \partial B^*$. Let $y \in B$ be such that $f(y) = 1$. We have $\psi(y) = \phi(y) = 1$, since $\psi(y) \leq 1$, $\phi(y) \leq 1$ and there is $0 < t < 1$ satisfying $1 = f(y) = t\phi(y) + (1-t)\psi(y)$. Consider now $x \in C$ satisfying $f(x) = \sup f(C)$. By Lemma 6.8 we know that $\psi(x) = \sup \psi(C)$ and, analogously, $\phi(x) = \sup \phi(C)$. As in the preceding case, we will consider balls $a_\lambda + \lambda B$ for which $a_\lambda + \lambda y = x + \varepsilon y$. Now pick $z, w \in \mathbb{R}^2$ with $\phi(z) = \phi(x)$ and $\psi(w) = \psi(x)$ satisfying $C \subset \operatorname{conv}\{z, w, x\}$. The only question is whether there is $\lambda > 0$ so that $z, w \in a_\lambda + \lambda B$. The existence of such a λ can be proved using an argument of differentiability, as in *Case 1*, since ψ and ϕ are extreme points of B^*.

Corollary 6.9. *A Banach space has dimension less than three if and only if is a Mazur space with respect to every equivalent norm.*

Proof. It is clear that one dimensional spaces are always Mazur spaces and Theorem 6.7 states that this is also the case of two dimensional spaces. To prove the reverse, suppose that the Banach space X contains a three–dimensional subspace Y, which can be assumed (after renorming) to be $(\mathbb{R}^3, \|\cdot\|_1)$. Letting Z (in its inherited norm) be the complement of Y in X, so we can assume that X is the ℓ_1–sum $Y \oplus_1 Z$. We proved in Proposition 5.7 that in this case $\mathcal{M}$ is not stable under the closure of vector sums and hence X with this norm is not a Mazur space. □

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