CONVEX SETS IN BANACH SPACES AND A PROBLEM OF ROLEWICZ

A. S. GRANERO, M. JIMÉNEZ SEVILLA, AND J. P. MORENO

Abstract. Let $\mathcal{B}_X$ be the set of all closed, convex and bounded subsets of a Banach space X equipped with the Hausdorff metric. In the first part of this work we study the density character of $\mathcal{B}_X$ and investigate its connections with the geometry of the space, in particular with a property shared by the spaces of Shelah and Kunen. In the second part we are concerned with the problem of Rolewicz, namely the existence of support sets, for the case of spaces $C(K)$.

1. Introduction

In this paper we discuss some topics concerning the set $\mathcal{B}_X$ of all bounded, closed, convex and nonempty subsets of a real Banach space X. The Hausdorff metric between $C_1, C_2 \in \mathcal{B}_X$ is given by

$$d(C_1, C_2) = \inf\{\varepsilon > 0 : C_1 \subset C_2 + \varepsilon B_{\|\cdot\|},\ C_2 \subset C_1 + \varepsilon B_{\|\cdot\|}\},$$

where $B_{\|\cdot\|}$ is the unit ball of X. It is well known that $(\mathcal{B}_X, d)$ is a complete metric space [11] and, hence, a Baire space.

The first part is devoted to study the density character of $\mathcal{B}_X$ and its interplay with different geometrical properties. These properties are property α, the (weak*) Mazur intersection property and the following cornerstone property, which we shall name after the *Kunen-Shelah* property: among any uncountable family of elements of X, there is one that belongs to the closed convex hull of the rest. Shelah [23] (assuming the diamond principle for $\aleph_1$) and Kunen [18] (assuming the continuum hypothesis) constructed Banach spaces $\mathcal{S}$ and $\mathcal{K}$ respectively with the above property. Most of our interest in Section 2 will tend to emphasize the effects of Kunen-Shelah property for the topological properties of $\mathcal{B}_X$. For instance, we prove here that, in many cases, spaces enjoying this property satisfy $\operatorname{dens} X = \operatorname{dens} \mathcal{B}_X$

1991 *Mathematics Subject Classification.* 46B20.
Supported in part by DGICYT grant PB 94–0243 and PB 93–0452.

while, in general, $\operatorname{dens}\mathcal{B}_X = 2^{\operatorname{dens}X}$. Moreover, assuming $\mathfrak{c} < 2^{\omega_1}$ (where ω_1 is the first uncountable ordinal), an Asplund space X with $\operatorname{dens}X = \mathfrak{c}$ enjoys the Kunen-Shelah property if and only if $\operatorname{dens}X = \operatorname{dens}\mathcal{B}_X$. These conditions are satisfied, for instance, by $\mathcal{K}$ and $\mathcal{S}$.

Section 3 is related to the problem posed by Rolewicz on the existence of a support set in any nonseparable Banach space. We shall restrict our attention here to $C(K)$ spaces. Lazar [14] proved that for every compact Hausdorff space K, such that either K is not hereditarily Lindelöf or K is not hereditarily separable, then $C(K)$ has a support set. Recently, Borwein and Vanderwerff proved in [1] the existence of such a set in the Kunen space $\mathcal{K}$ (which is of the form $\mathcal{K} = C(K)$ with K scattered compact) solving a problem posed by Finet and Godefroy [3]. We begin by observing that the result of Borwein and Vanderwerff can be extended to the class of all nonseparable Asplund spaces $C(K)$. Then, we characterize Hausdorff compacta K which are not hereditarily Lindelöf in terms of the existence of a positive semibiorthogonal system (see definition in section 4). As a consequence, we obtain that the existence of a support set in a Banach lattice does not imply the existence of a positive semibiorthogonal system. Thus, the characterization obtained in [1] cannot be strengthened in this direction. Finally, we prove that every Banach space $C(K)$ where K is not measure separable (denoted by $K \notin MS$) has a support set.

2. Density character of $\mathcal{B}_X$

In this section, the density character of $\mathcal{B}_X$ is investigated. Recall that, if Ω is a topological space, the density character of Ω (denoted by $\operatorname{dens}\Omega$) is the smallest cardinal number of a dense subset of Ω. Given a Banach space X, a first easy estimate of $\operatorname{dens}\mathcal{B}_X$ is

$$\operatorname{dens}X \leq \operatorname{dens}\mathcal{B}_X \leq 2^{\operatorname{dens}X} \tag{2.1}$$

and our aim consists in determining necessary and sufficient conditions ensuring either left or right equalities, and their consequences for the geometry of the space.

There is a general argument to estimate $\operatorname{dens}\mathcal{B}_X$: the existence of an "almost biorthogonal" system in $X \times X^*$ like the one appearing in the following definition. A Banach space X is said to have property α [21] if there exist $0 \leq \lambda < 1$ and a family $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$ with $||x_i|| = ||x_i^*||^* = x_i^*(x_i) = 1$ such that: (1) for

$j \neq i$, $|x_i^*(x_j)| \le \lambda$ and (2) $B_{\|\cdot\|} = \overline{\mathrm{co}}\,(\{\pm x_i\}_{i\in I})$. Next, we denote by $\operatorname{card} I$ the cardinality of the set I.

Proposition 2.1. *Let X be a Banach space with a family $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$, such that $||x_i|| \le M$ and $||x_i^*||^* \le M$ for some constant $M > 0$, $x_i^*(x_i) = 1$ and $|x_i^*(x_j)| \le \lambda$, for $j \neq i$ and some $0 \le \lambda < 1$. Then* $\operatorname{dens} \mathcal{B}_X \ge 2^{\operatorname{card} I}$. *In particular, if X has property α, then* $\operatorname{dens} \mathcal{B}_X = 2^{\operatorname{dens} X}$.

Proof. For each subset $J \subseteq I$, we consider $C_J = \overline{\mathrm{co}}\,(\{x_i\}_{i\in J}) \in \mathcal{B}_X$. Then, if $J \neq J'$ and $i_0 \in J \setminus J'$, we have

$$||x_{i_0} - y|| \ge x_{i_0}^*(x_{i_0} - y)/||x_{i_0}^*||^* \ge (1-\lambda)/||x_{i_0}^*||^* \ge (1-\lambda)/M$$

for every $y \in C_{J'}$. Hence

$$d(C_J, C_{J'}) \ge \operatorname{dist}(x_{i_0}, C_{J'}) = \inf\{||x_{i_0} - y|| : \ y \in C_{J'}\} \ge (1-\lambda)/M\,,$$

and the proof is finished. □

Some consequences can be now deduced from the preceding proposition. First, a Banach space X with a biorthogonal system $S = \{x_i, x_i^*\}_{i\in I} \subset X \times X^*$ satisfies $\operatorname{dens} \mathcal{B}_X \ge 2^{\operatorname{card} I}$. Moreover, if S is a *long biorthogonal system* (that is, if $\operatorname{card} I = \operatorname{dens} X$) then $\operatorname{dens} \mathcal{B}_X = 2^{\operatorname{dens} X}$. Indeed, by using a result of Plichko [19] (as in [7]) we obtain a *bounded* biorthogonal system with the same cardinality as the given one. The assertion then follows from the preceding proposition. The preceding arguments lead us directly to the characterization of finite dimensional Banach spaces as those spaces satisfying $\operatorname{dens} \mathcal{B}_X = \aleph_0$.

If η is a cardinal, recall that the cofinality of η (denoted by $\operatorname{cf}(\eta)$) is the smallest cardinal β such that there exists a sequence of ordinals $\{\beta_i\}_{i<\beta}$ strictly less than η satisfying $\eta = \sup\{\beta_i : \ i < \beta\}$. As another application of Proposition 2.1, we derive an estimate of $\operatorname{dens} \mathcal{B}_X$ when either X has the Mazur intersection property, or X^* has the weak* Mazur intersection property. The major impact of this estimate occurs when the cofinality of $\operatorname{dens} X$ is not countable. Recall that a Banach space X is said to have the *Mazur intersection property* ([16], [4]) if every element of $\mathcal{B}_X$ is the intersection of closed balls. If Y is a dual Banach space, denote by $\mathcal{B}_Y^*$ the set of all weak*-compact convex and nonempty subsets of Y. The dual Banach space

Y is said to have the weak* Mazur intersection property [4] if every element of $\mathcal{B}_Y^*$ is the intersection of closed dual balls.

Proposition 2.2. *Let X be an infinite dimensional Banach space. If either X has the Mazur intersection property or X^* has the weak* Mazur intersection property then $\operatorname{dens}\mathcal{B}_X > 2^\alpha$ for every $\alpha < \operatorname{dens} X$. Moreover, if $\operatorname{cf}(\operatorname{dens} X)$ is not countable, then $\operatorname{dens}\mathcal{B}_X = 2^{\operatorname{dens} X}$.*

Proof. Under these hypothesis, it is proved in [9] (see also [10]) the existence of a bounded family $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$ and a set $\{\lambda_i\}_{i\in I} \subset (0,1)$ satisfying $\operatorname{card}(I) = \operatorname{dens} X$ such that $x_i^*(x_i) = 1$ and $x_i^*(x_j) \leq 1 - \lambda_i$ for every $j \neq i$. Since

$$I = \cup_{n=1}^{\infty} \{i \in I : \lambda_i > 1/n\}$$

then, for each $\alpha < \operatorname{dens} X$, there is $n_\alpha \in \mathbb{N}$ such that $\operatorname{card}(\{i \in I : \lambda_i > 1/n_\alpha\}) > \alpha$. Similarly, if $\operatorname{cf}(\operatorname{dens} X)$ is not countable, there exists $n_0 \in \mathbb{N}$ with $\operatorname{card}(\{i \in I : \lambda_i > 1/n_0\}) = \operatorname{dens} X$. The remainder of the proof follows easily by using Proposition 2.1. □

So far we have shown the existence of a wide class of Banach spaces satisfying $\operatorname{dens}\mathcal{B}_X = 2^{\operatorname{dens} X}$ since every "reasonable" Banach space has a long biorthogonal system. On the other hand, the only examples exhibited satisfying

$$\operatorname{dens}\mathcal{B}_X = \operatorname{dens} X \,, \tag{2.2}$$

are finite dimensional spaces. It is now natural to inquire whether there exists an infinite dimensional space X for which equality (2.2) holds. Needless to say, we are looking for a Banach space admitting no equivalent norm with property α. This is the case, as it was observed in [7], of any Banach space X with the Kunen-Shelah property: given an uncountable subset B of X, there is $x \in B$ such that $x \in \overline{\operatorname{co}}(B \setminus x)$. The first known space with this property was constructed by Shelah [23] using the diamond principle for $\aleph_1$. Later on, assuming only the continuum hypothesis, Kunen [18] provided a second example. These spaces where used to answer some long standing questions in the theory of Banach spaces (see for instance [1],[3], [7], [8], [9]). Both examples are Asplund and, in addition, Kunen space $\mathcal{K}$ is of the form $\mathcal{K} = C(K)$ being K a scattered compact. As the reader probably guesses, these spaces are our candidates to fulfill equation (2.2).

Let us fix some notation. It will be convenient to consider the following subsets of $\mathcal{B}_X$: $\mathcal{U}_X = \{C \in \mathcal{B}_X : \operatorname{int} C \neq \emptyset\}$ and $\mathcal{O}_X = \{C \in \mathcal{B}_X : 0 \in \operatorname{int} C\}$. It is clear that both $\mathcal{O}_X$ and $\mathcal{U}_X$ are open and $\mathcal{U}_X$ is dense in $\mathcal{B}_X$. Similarly, consider the following subsets of $\mathcal{B}^*_Y$: $\mathcal{U}^*_Y = \{C \in \mathcal{B}^*_Y : \operatorname{int} C \neq \emptyset\}$ and $\mathcal{O}^*_Y = \{C \in \mathcal{B}^*_Y : 0 \in \operatorname{int} C\}$. Given the Banach space X and a subset $C \in \mathcal{O}_X$, recall that $C^o = \{f \in X^* : f(x) \leq 1, x \in C\}$ is the polar set of C. In turn, given $C \in \mathcal{O}^*_{X^*}$, the set C_o is $C^o \cap X$. Recall that $f \in C^o$ is a weak* denting point if, for every $\varepsilon > 0$, there exist $x \in X$ and $\delta > 0$ with $\operatorname{diam}\{g \in C^o : g(x) > f(x) - \delta\} < \varepsilon$.

Proposition 2.3. *Let X be an Asplund space satisfying the Kunen-Shelah property. Then every subset of $\mathcal{B}_X$ is a countable intersection of closed half-spaces. If, in addition,* $\operatorname{dens} X = \mathfrak{c}$ *then* $\operatorname{dens} \mathcal{B}_X = \operatorname{dens} X$.

Proof. Let us consider $C \in \mathcal{B}_X$ and assume, without loss of generality, that $0 \in \operatorname{int} C$. Then, $C^o \subset X^*$ is a weak* compact and convex set with nonempty interior. Since X is an Asplund space, X^* has the Radon Nikodým property and $C^o = \overline{\mathrm{co}}^{w^*}(\mathcal{D}_*)$, being $\mathcal{D}_*$ the set of all weak* denting points of C^o. By [9], there exists a countable set $\mathcal{D}'_* \subset \mathcal{D}_*$ such that $\mathcal{D}_* \subset \overline{\mathcal{D}'_*}$, thus implying $C^o = \overline{\mathrm{co}}^{w^*}(\mathcal{D}'_*)$ and $C = \{x \in X : x^*(x) \leq 1, x^* \in \mathcal{D}'_*\}$. For the second assertion, given a dense subset $\{x^*_i\}_{i\in I}$ of X^* with $\operatorname{card} I = \operatorname{dens} X^* = \operatorname{dens} X$ and $J \subset I$, denote by $C_J = \{x \in X : \ x^*_i(x) \leq 1, \ i \in J\}$. Then

$$\mathcal{O}_X \subset \mathcal{F} = \{C_J \in \mathcal{B}_X : \ J \subset I, \quad J \ \text{ countable}\}.$$

Since $\operatorname{dens} \mathcal{B}_X = \operatorname{dens} \mathcal{O}_X$ and $\operatorname{dens} X = (\operatorname{dens} X)^{\aleph_0} = \operatorname{card} \mathcal{F}$, the proof is complete. Notice that, in fact, we obtain $\operatorname{card} X = \operatorname{dens} X = \operatorname{dens} \mathcal{B}_X = \operatorname{card} \mathcal{B}_X$. □

We can paraphrase the result obtained in the preceding proposition by stating that spaces under consideration are extremely "poor" in convex sets. We shall see later that a certain converse may be deduced with additional hypothesis. Recall that a family $\{x_\alpha : \alpha < \omega_1\}$ in a Banach space X is *weakly right-separated* if $x_\alpha \notin \overline{\{x_\beta : \beta > \alpha\}}^w$ for all $\alpha < \omega_1$. It can be easily seen that the lack of weakly right-separated families implies the Kunen-Shelah property [18]. As far as we know, it is an open problem whether or not the converse holds. The Kunen space has not weakly right separated families; in fact, it is hereditarily Lindelöf in the weak topology. It seems to be unknown whether or not the Shelah space has weakly right separated families. We are now interested in the question if having no weakly right separated families implies (2.2). The answer, in some cases, is affirmative.

Proposition 2.4. *Let X be a Banach space with no weakly right separated family. Then:*

(i) *Every subset $C \in \mathcal{B}^*_{X^*}$ is w^*-separable.*
(ii) *Every subset $C \in \mathcal{B}_X$ is a countable intersection of closed half-spaces.*
(iii) *If $\operatorname{dens} X = \operatorname{dens} X^* = c$, then $\operatorname{dens} X = \operatorname{dens} \mathcal{B}_X$.*

Proof. (i) Let $C \in \mathcal{B}^*_{X^*}$ and assume that C is not w^*-separable. Then we can construct, in a standard way, a family $\{C_\alpha\}_{\alpha<\omega_1} \subset \mathcal{B}^*_{X^*}$ of w^*-separable subsets of C such that $C_\alpha \subsetneq C_{\alpha+1}$, for all $\alpha < \omega_1$. If we take $f_\alpha \in C_{\alpha+1} \setminus C_\alpha$ and $x_\alpha \in X$ satisfying

$$\sup\{f(x_\alpha) : f \in C_\alpha\} < 1 < f_\alpha(x_\alpha),$$

then $f_\alpha(x_\beta) < 1$ whenever $\alpha < \beta$ and thus $x_\alpha \notin \overline{\{x_\beta : \beta > \alpha\}}^w$ for all $\alpha < \omega_1$. Hence, $\{x_\alpha\}_{\alpha<\omega_1}$ is a weakly right-separated family, which is a contradiction. (ii) Given $C \in \mathcal{O}_X$, there is a sequence $\{x^*_n\} \subset C^o$ such that $C^o = \overline{\{x^*_n\}}^{w^*}$. Therefore, $C = \{x \in X : x^*_n(x) \leq 1,\ n \in \mathbb{N}\}$. The statement of (ii) can be extended to every element of $\mathcal{B}_X$ by using the density of $\mathcal{U}_X$. (iii) It can be proved as in Proposition 2.3. $\square$

Recall that Sersouri [22] proved the equivalence between the next properties:

(1) X has the *strong Kunen-Shelah property*: For every family $\{x_\alpha\}_{\alpha<w_1} \subset X$, there exists an α_0 such that

$$x_{\alpha_0} \in \overline{\operatorname{co}}\left(\{x_\beta : \beta > \alpha_0\}\right).$$

(2) Every family $(C_\alpha)_{\alpha<w_1}$ of closed convex subsets of X which is decreasing (i.e. $C_{\alpha+1} \subseteq C_\alpha$) is stationary (i.e. there is $\alpha_0 < w_1$ such that $C_\alpha = C_{\alpha_0}$ for every $\alpha \geq \alpha_0$).

By using arguments similar to those exhibited in the preceding proof, we can show that for an Asplund space X any of the above properties is equivalent to

(3) Every element $C \in \mathcal{B}^*_{X^*}$ is weak* separable.

(4) X has the Kunen-Shelah property.

It is worth to mention that condition (2) implies clearly the Corson's property $(\mathcal{C})$ that is, every collection of closed convex subsets of X with empty intersection contains a countable subcollection with empty intersection. On the other hand, we do not know if the Kunen-Shelah property implies the strong Kunen-Shelah property for a general Banach space.

The connections between equation (2.2) and the Kunen-Shelah property become clearer when $\operatorname{dens} X = \mathfrak{c}$ as it occurs with the Kunen and Shelah spaces. For this purpose, we find it necessary to assume that $\mathfrak{c} < 2^{\omega_1}$, a condition which is weaker than the continuum hypothesis.

Proposition 2.5. *Assume that $\mathfrak{c} < 2^{\omega_1}$ and consider a Banach space X such that* $\operatorname{dens} X = \mathfrak{c}$.

(i) *If* $\operatorname{dens} \mathcal{B}_X = \operatorname{dens} X$, *then X enjoys the Kunen-Shelah property.*
(ii) *If, in addition, X is an Asplund space, then* $\operatorname{dens} \mathcal{B}_X = \operatorname{dens} X$ *iff X has the Kunen-Shelah property.*

Proof. (i) If X lacks the Kunen-Shelah property, we can obtain, as in Proposition 2.2, a bounded uncountable family $\{x_\alpha\}_{\alpha<\omega_1}$ and $\varepsilon > 0$ such that $\operatorname{dist}(x_\alpha, \overline{\operatorname{co}}(\{x_\beta : \beta \neq \alpha\})) \geq \varepsilon > 0$. Then, Proposition 2.1 yields the estimate $\operatorname{dens} \mathcal{B}_X \geq 2^{\omega_1} > \operatorname{dens} X$, which is a contradiction. (ii) One implication is already proved in part (i) and the other follows from Proposition 2.3. □

Let us make some remarks about the material in this section. A first comment should be made on the density character of $\mathcal{B}^*_{X^*}$. An estimate analogous to the one given in (2.1) is

$$\operatorname{dens} \mathcal{B}_X = \operatorname{dens} \mathcal{B}^*_{X^*} \leq \operatorname{dens} \mathcal{B}_{X^*} .$$

The equality follows from the facts that the sets $\mathcal{O}_X$ and $\mathcal{O}^*_{X^*}$ are homeomorphic, $\operatorname{dens} \mathcal{B}_X = \operatorname{dens} \mathcal{O}_X$ and $\operatorname{dens} \mathcal{B}^*_{X^*} = \operatorname{dens} O^*_{X^*}$. It frequently occurs that $\operatorname{dens} \mathcal{B}^*_{X^*} < \operatorname{dens} \mathcal{B}_{X^*}$. For instance, when $X = \ell_1$ since ℓ_∞ has a long biorthogonal system ($\ell_1(\mathbb{R})$ embeds isometrically into ℓ_∞). Also, it is the case when $X = \mathcal{K}$ since $\mathcal{K}^* = \ell_1(\mathbb{R})$ and, consequently, $\operatorname{dens} \mathcal{B}_{\mathcal{K}^*} = 2^{\mathfrak{c}}$.

The second observation concerns the concept of nicely smooth norm introduced by Godefroy in [5]. Note that Proposition 2.4 implies that non separable Banach spaces with no weakly right-separated family do not admit an equivalent nicely smooth norm. Indeed, every closed dual ball is w^*-separable. This is somehow a generalization of a result given in [9], which provides an alternative proof to the fact that no nicely smooth norm exist in the Kunen and Shelah spaces.

The third and final remark pertains to Proposition 2.4 and can be expressed with the following open question raised by Godefroy: is every w^*-closed (not necessarily convex) subset of the dual of Kunen space w^*-separable?

3. On the Rolewicz problem

A point x of a set $C \in \mathcal{B}_X$ is called a *support point* for C if there is a functional $x^* \in X^*$ such that $x^*(x) = \inf_C x^* < \sup_C x^*$. We will say that $C \in \mathcal{B}_X$ is a *support set* if C contains only support points. Rolewicz proved in [20] that a support set must be nonseparable and asked whether such a set exists in every nonseparable space. Later on, some authors directed their attention to the study of this suggestive problem. Kutzarova [12], Lazar [14] and Montesinos [17] proved the existence of support sets in certain nonseparable spaces. The underlying idea is that such sets can be constructed from an uncountable biorthogonal system. We shall restrict our attention in this section to $C(K)$ spaces, where K is a compact Hausdorff space. Our aim is to examine topological conditions on K which ensure the existence in $C(K)$ of a support set. Lazar found in [14] two of such conditions: K contains a non-G_δ closed subset (equivalently, K is not hereditarily Lindelöf) or K is not hereditarily separable. Recently, Borwein and Vanderwerff solved a problem by Finet and Godefroy [3], namely the existence of a support set in the Kunen space. In the first result of this section we observe, using Lazar's arguments, that this fact can be extended to every space $C(K)$ where K is an uncountable scattered compact. Recall that the α^{th}–derivative of a topological set K is usually denoted by $K^{(\alpha)}$.

Proposition 3.1. *Every nonseparable Banach space $C(K)$, with K Hausdorff and scattered compact, has a support set.*

Proof. If $K^{(w_1)} \neq \emptyset$, then $\{K \setminus K^{(\alpha)}\}_{\alpha<\omega_1}$ is an open covering of $K \setminus K^{(\omega_1)}$ without a countable subcovering. Otherwise, there is $\alpha_0 < \omega_1$ such that $K^{(\alpha)} \setminus K^{(\alpha+1)}$ is uncountable. In any case, K is not hereditarily Lindelöf. □

Observe that, in the case $K^{(\alpha)} = \emptyset$ for some $\alpha < \omega_1$, then $C(K)$ has an uncountable biorthogonal system.

The existence of a support set in the Kunen space (which has no uncountable biorthogonal systems) and the subsequent necessity to clarify the interplay between both concepts, moved Borwein and Vanderwerff [1] to recapture the following weakening considered by Lazar. The family $\{x_\alpha, f_\alpha\}_{\alpha<w_1} \subset X \times X^*$ is called a *semibiorthogonal system* if $f_\beta(x_\alpha) = 0$ for all $\alpha < \beta$, $f_\alpha(x_\alpha) > 0$ and $f_\beta(x_\alpha) \geq 0$ for all α. It is proved in [1] that a Banach space has a support set if, and only if, there exists a semibiorthogonal system. Since every $C(K)$ space is a Banach lattice, we can adopt this new point of view to examine the above characterization. The notation and terminology used in the remainder for Banach lattices can be found in [15]. Given a Banach lattice, denote by X^+ its positive cone. Let us say that a semibiorthogonal system $\{x_\alpha, f_\alpha\}_{\alpha<w_1}$ is *positive* whenever $x_\alpha \in X^+$ and $f_\alpha \in (X^*)^+$ for every α. A natural question arises: there exists a positive semibiorthogonal system in every Banach lattice with a support set? We shall answer this question in the negative as a consequence of the following characterization.

We say that a family of sets $\{C_\alpha\}_{\alpha<\omega_1}$ is *expansive* (*contractive*) whenever $C_\alpha \subsetneq C_{\alpha+1}$ (respectively $C_{\alpha+1} \subsetneq C_\alpha$) for every $\alpha < \omega_1$.

Proposition 3.2. *Let K be a compact Hausdorff space and let $X = C(K)$. The following are equivalent:*

(i) *K is not hereditarily Lindelöf.*
(ii) *X has an expansive family of closed ideals.*
(iii) *There is a positive semibiorthogonal system in $X \times X^*$.*
(iv) *There is a positive semibiorthogonal system in $X \times \{\delta_k : k \in K\}$.*

Proof. (i)⇒(ii). If K is not hereditarily Lindelöf, there exists an uncountable family $\{U_\alpha\}_{\alpha<\omega_1}$ of open subsets of K such that $U = \cup_{\alpha<\omega_1} U_\alpha$ cannot be covered with any countable subfamily of $\{U_\alpha\}_{\alpha<\omega_1}$. We may assume that $U_\alpha \subsetneq U_{\alpha+1}$ for every $\alpha < \omega_1$. Then, the family of compacts $\{K_\alpha = K \setminus U_\alpha\}_{\alpha<\omega_1}$ is contractive and the associated family of closed ideals $I_\alpha = \{x \in C(K) : x|_{K_\alpha} \equiv 0\}$ is expansive.

(ii)⇒ (iii) Take $x_\alpha \in I_{\alpha+1} \setminus I_\alpha$, $x_\alpha > 0$ and $g_\alpha \in I_\alpha^\perp \subset X^*$ with $g_\alpha(x_\alpha) > 0$. As $I_\alpha^\perp$ is a sublattice, we have that $f_\alpha = g_\alpha^+ \in I_\alpha^\perp$ and $f_\alpha(x_\alpha) \geq g_\alpha(x_\alpha) > 0$. Then $\{x_\alpha, f_\alpha\}_{\alpha<w_1}$ is a positive semibiorthogonal system.

(iii)⇒(iv). Let $\{x_\alpha, f_\alpha\}_{\alpha<w_1} \subset X \times X^*$ a positive semibiorthogonal system. Recall that f_α is a regular measure defined on the Borel subsets of K. If we denote $G_\alpha = \{k \in K : x_\alpha(k) > 0\}$, then $G_\beta \cap \operatorname{supp} f_\alpha = \emptyset$ for $\beta < \alpha$. We can pick $k_\alpha \in \operatorname{supp} f_\alpha$ such that $x_\alpha(k_\alpha) > 0$. With this choice, we obtain that $\{x_\alpha, \delta_{k_\alpha}\}_{\alpha<\omega_1}$ is a positive semibirthogonal system.

(iv)⇒(i). Consider the positive semibiorthogonal system $\{x_\alpha, \delta_{k_\alpha}\}_{\alpha<\omega_1}$. Then the family $\{U_\alpha = \{k \in K : x_\alpha(k) > 0\}\}_{\alpha<\omega_1}$ is an open covering of the set $A = \{k_\alpha\}_{\alpha<\omega_1}$ which does not admit any countable subcovering. □

As it is noted in [1], the double arrow space K is hereditarily Lindelöf and hereditarily separable, while $C(K)$ has an uncountable biorthogonal system and hence a support set. This example provides a negative answer to the question preceding Proposition 3.2.

A support set $C \subset X$ (not necessarily bounded) is Y*–supported* [1], where $Y \subset X^*$, if for every $x \in C$ there is $f \in Y$ with $f(x) = \inf_C f < \sup_C f$. A *support cone* is a support set $C \subset X$ such that $\lambda x \in X$ whenever $\lambda \geq 0$ and $x \in C$.

Proposition 3.3. *Let X be a Banach lattice. The following are equivalent:*

(i) *X has an expansive family of closed ideals.*
(ii) *There is a positive semibiorthogonal system in $X \times X^*$.*
(iii) *There is a $(X^*)^+$–supported cone $C \subset X^+$.*
(iv) *There is a $(X^*)^+$–supported set $C \subset X^+$ with $0 \in C$.*

Proof. (i)⇒(ii) is already done in Proposition 3.2. (ii)⇒(i). Let us denote by $\{x_\alpha, f_\alpha\}_{\alpha<w_1}$ the positive semibiorthogonal system. Consider for every $\alpha < w_1$ the smallest closed ideal I_α containing $\{x_\beta\}_{\beta<\alpha}$. Then we shall deduce that $f_\alpha \in I_\alpha^\perp$. Indeed, it is easily proved from the fact that $f_\alpha > 0$, $x_\beta > 0$ for $\beta < w_1$, and

$$I_\alpha = \overline{\Big\{x \in X : |x| \leq |u| \text{ for some } u = \sum_{i\in F} \lambda_i x_i \text{ and a finite set } F \subset \alpha\Big\}}.$$

If $u = \sum_{i\in F} \lambda_i x_i$, $F \subset \alpha$ a finite set, it is clear that $u^+ \leq a = \sum_{i\in F^+} \lambda_i x_i$ and $u^- \leq b = \sum_{i\in F^-} (-\lambda_i) x_i$, where $F^+ = \{i \in F : \lambda_i \geq 0\}$ and $F^- = \{i \in F : \lambda_i \leq 0\}$. Thus, for $0 \leq x \leq |u|$ we have that $0 \leq f_\alpha(x) \leq f_\alpha(|u|) = f_\alpha(u^+) + f_\alpha(u^-) \leq$

$f_\alpha(a) + f_\alpha(b) = 0$. On the other hand, $f_\alpha \notin I_{\alpha+1}^{\perp}$, so $I_\alpha \subsetneq I_{\alpha+1}$. The rest of the proof follows easily from the arguments exhibited in [1]. □

We do not know if there is a positive semibiorthogonal system in X whenever X has a $(X^*)^+$-supported set $C \subset X^+$.

The next result concerns the class MS of compacta K, those spaces for which every regular measure on K is separable. This class contains, for instance, compact ordered spaces, compact scattered spaces and Eberlein compacta. The class MS has been recently studied by Džamonja and Kunen in [2].

Proposition 3.4. *If K is a compact Hausdorff space and $K \notin MS$ then $C(K)$ has a support set.*

Proof. Since $K \notin MS$, there exists a regular measure μ on K which is non-separable. From Maharam-Stone theorem [13, p. 122] we obtain that $L_1(\{0,1\}^{\omega_1}, \nu) \hookrightarrow L_1(\mu)$ where ν is the Haar probability measure on $\{0,1\}^{\omega_1}$. It means that $L_1(\{0,1\}^{\omega_1}, \nu) \hookrightarrow C(K)^*$. Taking into account that $\ell_2(\omega_1) \hookrightarrow L_1(\{0,1\}^{\omega_1}, \nu)$, there is a quotient map $q : C(K) \to \ell_2(\omega_1)$ and, in consequence, an uncountable biorthogonal system in $C(K)$. □

On the other hand, if K is any non-metrizable Rosenthal compact then $C(K)$ contains an uncountable biorthogonal system and the class of Rosenthal compact have property MS (see [6]).

We finish this section by mentioning that the Rolewicz problem is still open in spaces $C(K)$. If there exists a compact Hausdorff space K for which $C(K)$ has no support set, then K should be hereditarily Lindelöf, hereditarily separable and $K \in MS$. The "two arrows" space K satisfies these conditions but it cannot be a counterexample. Recall that Borwein and Vanderwerff proved that if X has the strong Kunen-Shelah property (as for instance, the Kunen and Shelah spaces), then X^* has no norm closed weak*–supported set.

References

[1] J. M. Borwein and J. D. Vanderwerff, *Banach spaces that admit support sets*, Proc. Amer. Math. Soc. **124**(3) 1996, 751–755.
[2] M. Džamonja and K. Kunen, *Properties of the class of measure separable compact spaces*, Fund. Math. **147** (1995), 261–277.

[3] C. Finet and G. Godefroy, *Biorthogonal systems and big quotient spaces*, Contemporary Math. vol. 85 (1989), 87–110.

[4] J. R. Giles, D. A. Gregory, and B. Sims, *Characterization of normed linear spaces with Mazur's intersection property*, Bull. Austral. Math. Soc. **18** (1978), 471–476.

[5] G. Godefroy, *Nicely smooth Banach spaces*, The University of Texas at Austin, Functional Analysis Seminar, 1984–1985

[6] G. Godefroy, *Compacts de Rosenthal*, Pac. J. Math. **91**(2), 1980, 293–306.

[7] B. V. Godun and S. L. Troyanski, *Renorming Banach spaces with fundamental biorthogonal systems*, Contemporary Math. **144** (1993), 119–126.

[8] M. Jiménez Sevilla and J.P. Moreno, *The Mazur intersection property and Asplund spaces*, C.R. Acad. Sci. Paris, Série I, **321** (1995), 1219–1223.

[9] M. Jiménez Sevilla and J.P. Moreno, *Renorming Banach spaces with the Mazur intersection property*, J. Funct. Anal. **144** (2) (1997), 486–504.

[10] M. Jiménez Sevilla and J.P. Moreno, *On denseness of certain norms in Banach spaces*, Bull. Austral. Math. Soc. **54** (1996), 183–196.

[11] K. Kuratowski, *Topology I*, Academic Press, New York and London, 1966.

[12] D. N. Kutzarova, *Convex sets containing only support points in Banach spaces with an uncountable minimal system*, C. R. Acad. Bulg. Sci. **39** No. 12 (1986), 13–14.

[13] H. E. Lacey, *The Isometric Theory of Classical Banach spaces*, Springer-Verlag 1974.

[14] A. J. Lazar, *Points of support for closed convex sets*, Illinois J. Math. **25** (1981), 302–305.

[15] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II: Function spaces*, Springer–Verlag, Berlin 1979.

[16] S. Mazur, *Über schwache Konvergentz in den Raumen L^p*, Studia Math. **4** (1933), 128–133.

[17] V. Montesinos, *Solution to a problem of S. Rolewicz*, Studia Math. **81** (1985), 65–69.

[18] S. Negrepontis, *Banach spaces and Topology*, Handbook of set theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, 1984, 1045–1142

[19] A. N. Plichko, *A Banach space without a fundamental biorthogonal system*, Soviet. Math. Dokl. **22** (1980) No. 2, 450–453.

[20] S. Rolewicz, *On convex sets containing only points of support*, Comment. Math., Tomus specialis in honorem Ladislai Orlicz, I, Warszawa, 1978, 279–281.

[21] W. Schachermayer, *Norm attaining operators and renormings of Banach spaces*, Isr. J. Math. **44** (1983), 201–212.

[22] A. Sersouri, *w-independence in non-separable Banach spaces*, Contemp. Math. **85** (1989), 509–512.

[23] S. Shelah, *Uncountable constructions for B. A., e.c. and Banach spaces*, Isr. J. Math. **51** (1985), No. 4, 273–297.

Dpto. de Análisis Matemático. Facultad de Ciencias Matemáticas. Universidad Complutense de Madrid. Madrid, 28040. SPAIN

E–mail Address: granero@eucmax.sim.ucm.es, marjim@sunam1.mat.ucm.es, moreno@sunam1.mat.ucm.es.

current address of the third named author:

Dpto. Matemáticas. Universidad Autonoma de Madrid. Madrid, 28049.

E–mail Address: josepedro.moreno@uam.es