ON DENSENESS OF CERTAIN NORMS IN BANACH SPACES

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Abstract. We give several results dealing with denseness of certain classes of norms with many vertex points. We prove that, in Banach spaces with the Mazur or the weak* Mazur intersection properties, every ball (convex body) can be uniformly approximated by balls (convex bodies) being the closed convex hull of their strongly vertex points. We also prove that given a countable set F, every norm can be uniformly approximated by norms which are locally linear at each point of F.

1. Introduction

The problem of approximation of plane convex compacta by polygones has been investigated by many authors and there exist also some results in n-dimensional spaces for approximation of convex convex bodies by polytopes. In the infinite dimensional case, spaces where the unit ball of each of its finite dimensional subspaces is a polytope, called *polyhedral* spaces, have been largely studied (see, for instance [3] and references therein). However, the unit ball of a polyhedral space is not the only possible generalization of polytopes for infinite dimensional spaces. Our approach to these questions finds its motivation in the following simple fact: it is well known that every ball in a finite dimensional space can be uniformly approximated by balls being the convex hull of its *vertex* points. The concept of *vertex* point in a polygon can be easily generalized to infinite dimensional Banach spaces. Thus, it is a natural question to ask whether every convex body in these spaces can be approximated by convex bodies being the closed convex hull of

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their vertex points. Assuming continuum hypothesis, Kunen [10] constructed a non separable Banach space for which it has been showed in [9] that, for every equivalent norm, the set of denting points of the unit ball lies in a separable subspace. Therefore, no ball of this space can be expressed as the closed convex hull of its *denting* points. This property is shared by Shelah space [15], constructed assuming the diamond principle for $\aleph_1$.

Nevertheless, we shall prove that there exists a wide class of infinite dimensional Banach spaces satisfying this mentioned property of approximation. This problem is closely related with a geometric property, first studied by Mazur, [11] and later called the Mazur intersection property: *every bounded closed convex set can be represented as an intersection of closed balls*, and with its dual property, introduced in the remarkable paper of Giles, Gregory and Sims [6], called the weak* Mazur intersection property: *every bounded weak* closed convex set can be represented as an intersection of closed dual balls*.

During the last years, some authors drived their attention to the study of these properties and its connections with differentiability among others topics in the geometry of Banach spaces. It is shown in this paper that every ball (convex body) in a Banach space with the Mazur intersection property can be uniformly approximated by balls (convex bodies) being the closed convex hull of their strongly vertex points. Also, we prove the same property of approximation in Banach spaces having a dual with the weak* Mazur intersection property. We give also some other results of approximation by norms with many vertex points in their unit ball. For instance, every ball in a separable Banach space can be uniformly approximated by norms having a *dense* set of vertex points in their unit sphere.

The second part of this paper concerns approximation by locally smooth norms. Recently, Georgiev [4] and Vanderwerff [16] proved that, given a Banach space X and a countable set $F \subset X \setminus \{0\}$, almost all (in the Baire sense) equivalent norms in X are Fréchet differentiable at each point of F. Then, it is quite natural to consider the validity of this result if we replace Fréchet by a higher order of smoothness. The answer is negative due to the fact that the set of Lipschitz smooth norms at a fixed point is always of first Baire category. This implies, for instance, that the set of twice Gâteaux smooth norms in a Banach space is always of first Baire category (and maybe empty). On the other hand, the set of equivalent norms which are locally linear on an open set containing F is dense. To prove this result, a different argument than this provided in [13] to construct locally linear norms on open dense sets is required. Some results of this paper have been announced in [8].

2. Vertex points

We consider only infinite dimensional real Banach spaces. Given a Banach space $(X, \|\cdot\|)$ we denote by $B_{\|\cdot\|}$ its closed unit ball and by $(X^*, \|\cdot\|^*)$ its dual space. We denote by $(N(X), \rho)$ the complete metric space of all equivalent norms on X endowed with the metric induced by the uniform convergence on bounded sets. Similarly, $(N^*(X^*), \rho)$ denotes the complete metric space of all equivalent *dual* norms on $(X^*, \|\cdot\|^*)$ endowed with the same metric. Given a functional $f \in X^* \setminus \{0\}$ and $0 < \rho \leq 1$, the set

$$K(f, \rho) = \{x \in X : f(x) \geq \rho \|f\| \|x\|\}$$

is the cone generated by f. Let C be a subset of X and $x \in C$. The point x is a *vertex point* of C if there exist $f \in X^* \setminus \{0\}$ and $0 < \rho \leq 1$ such that $C \subseteq x - K(f, \rho)$, that is,

$$f(x - y) \geq \rho \|f\|^* \|x - y\| \tag{2.1}$$

for every $y \in C$. We also say that x is a vertex point of C with respect to f.

Let C be now a closed, bounded convex set and $x \in C$. The point x is said to be a *strongly vertex point* of C if there exist a closed bounded convex subset $D \subset C$ with $x \notin D$ verifying $C = \operatorname{conv}(\{x\} \cup D)$. The set of (strongly) vertex points of C will be denoted as $(\operatorname{strver} C)$ $\operatorname{ver} C$.

Given $f \in X^* \setminus \{0\}$, and $\delta < \sup_C f$, the set $S(C, f, \delta) = \{x \in C : f(x) > \delta\}$ is a *slice* of C. A point $x \in C$ is said to be a *denting point* of C if for every $\varepsilon > 0$ there exists $f \in X^*$ and $0 < \delta < f(x)$ such that $\operatorname{diam} S(C, f, \delta) < \varepsilon$. It is obvious that strongly vertex points of C are vertex point. Also, it can be easily deduced from (2.1) that vertex points are denting points. Let $f \in X^* \setminus \{0\}$ be a functional attaining its maximum on C. The set

$$C_f = \{x \in C : f(x) = \sup_C f\}$$

is called a *face* (with respect to f) whenever it has nonempty interior in their relative topology. We denote this interior by $\operatorname{int} C_f$. Suposse $C \subset X^*$ a weak* closed and bounded set. A point $f \in C$ is said to be a *weak* denting point* of C if for every $\varepsilon > 0$ there exists $x \in X$ and $0 < \delta < f(x)$ such that $\operatorname{diam} S(C, x, \delta) < \varepsilon$. The following lemma establishes the duality between vertex and faces.

Lemma 2.1. [12] *Let X be a Banach space, C a bounded, closed, convex set of X with $0 \in \operatorname{int} C$, C^o its polar set, $f \in C^o$ and $x \in C$.*

(i) *The set $C_f = \{y \in C : f(y) = 1\}$ is a face of C with $x \in \operatorname{int} C_f$ if and only if f is a vertex point of C^o with respect to x.*
(ii) *The set $C^o_x = \{g \in C^o : g(x) = 1\}$ is a face of C^o with $f \in \operatorname{int} C_x$ if and only if x is a vertex point of C with respect to f.*

Our first result of approximation shows that, given a Banach space $(X, \|\cdot\|)$ and a countable set $\{x_n\} \subset X \setminus \{0\}$, we can approximate $\|\cdot\|$ by a norm $|\cdot|$ with $\{x_n/|x_n|\} \subset \operatorname{ver} B_{|\cdot|}$. To prove this result, we need the following simple but useful lemma.

Lemma 2.2. *Let X be a Banach space and C a bounded closed convex subset of X with $0 \in \operatorname{int} C$ and boundary ∂C. Let $f \in X^* \setminus \{0\}$ such that $F = \{x \in C : f(x) = \sup_C f\}$ is a face of C. Then, if $g \in X^* \setminus \{0\}$ attains its maximum on C in the interior of F, there exists $\alpha \in \mathbb{R}^+$ with $f = \alpha g$.*

Proof. Let us consider $\ker f = \{x \in X : f(x) = 0\}$ and $\ker g$. If $H = \ker f \cap \ker g$ is an hyperplane, obviously $f = \alpha g$ for some $\alpha \in \mathbb{R}$.

Assume now that H is a subspace of codimension two. If we consider $y \in \ker f \setminus H$, clearly $|g(y)| > 0$. Take $x \in \operatorname{int} F$ with $g(x) = \sup_C g$ and $\varepsilon > 0$ with $x + \varepsilon B_{\|\cdot\|} \cap \{x \in X : f(x) = \sup_C f\} \subset C$. Defining

$$z = x + \varepsilon \operatorname{sign}\Big(g(y)\Big)\frac{y}{\|y\|} \quad ,$$

we have that $z \in F \subset C$ and $g(z) > g(x) = \sup_C g$, a contradiction. $\square$

Recall that, given $\lambda > 1$, the norm $|\cdot|$ is said to be λ-isometric to the norm $\|\cdot\|$ provided $B_{\|\cdot\|} \subset B_{|\cdot|} \subset \lambda B_{\|\cdot\|}$.

Proposition 2.3. *Let $(X, \|\cdot\|)$ be a Banach space and $\{x_n\} \subset X \setminus \{0\}$. Then, for each $\lambda > 1$, there exist a λ-isometric norm $|\cdot|$ verifying that $\{x_n/|x_n|\}_{n=1}^{\infty} \subset \operatorname{ver} B_{|\cdot|}$.*

Proof. We may assume that $x_n\|x_n\|^{-1} \neq x_m\|x_m\|^{-1}$ for every $n \neq m$. Consider $B_0 = \lambda B_{\|\cdot\|^*}$ and define

$$B_1 = \{f \in B_0 : |f(x_1)| \leq \delta_1\}$$

where $\|x_1\| < \delta_1 < \lambda\|x_1\|$. Clearly, $B_{\|\cdot\|^*} \subset \operatorname{int} B_1$ and B_1 is the unit ball of a dual norm containing the face $\{f \in B_0 : f(x_1) = \delta_1\}$ in its unit sphere. Take $0 < \mu_1 < 1$ so that $\{f \in \mu_1 B_0 : |f(x_1)| > \delta_1\} \neq \emptyset$ and consider the set $F_1 = \mu_1 B_0 \cap \partial B_1 \subset \{f \in B_1 : |f(x_1)| = \delta_1\}$. Suppose that for every $n \in \mathbb{N}$ we have

$$\{f \in B_1 : |f(x_2)| \geq \sup_{B_1} x_2 - \frac{1}{n}\} \cap F_1 \neq \emptyset \ .$$

Then F_1 is weak* compact and there exists $f_0 \in F_1$ such that $|f_0(x_2)| = \sup_{B_1} x_2$ which is impossible by Lemma 2.2. Analogously, it is not possible that

$$\{f \in B_1 : |f(x_2)| > \sup_{B_1} x_2 - (1/n)\} \cap B_{\|\cdot\|^*} \neq \emptyset$$

for each $n \in \mathbb{N}$ since $B_{\|\cdot\|^*} \subset \operatorname{int} B_1$. Hence, there is $0 < \delta_2 < \sup_{B_1} x_2$ such that, defining

$$B_2 = \{f \in B_1 : |f(x_2)| \le \delta_2\},$$

then $B_{\|\cdot\|^*} \subset \operatorname{int} B_2$ and $F_1 \subset \partial B_2$. We procede now inductively. Assume that B_n is already defined satisfying

$$B_{\|\cdot\|^*} \subset \operatorname{int} B_n \tag{2.2}$$

$$F_k = \mu_k B_{k-1} \cap \partial B_k \subset \operatorname{int}\{f \in \partial B_n : |f(x_k)| = \delta_k\} \qquad k = 1, ..., n-1\,. \tag{2.3}$$

Take $0 < \mu_n < 1$ such that $\{f \in \mu_n B_{n-1} : |f(x_n)| > \delta_n\} \neq \emptyset$ and define

$$F_{k+1} = \{f \in \mu_n B_{n-1} : |f(x_n)| > \delta_n\} \cap \partial B_n \ .$$

Using again that F_{k+1} is weak* compact and Lemma 2.2, we can find, for each $k = 1, ..., n$, $0 < \delta_{n+1}^k < \sup_{B_n} x_{n+1}$ such that

$$\{f \in B_n : |f(x_{n+1}| \ge \delta_{n+1}^k\} \cap \ F_k = \emptyset$$

since, by (2.3), F_k lies in the interior of a face in B_n. Also, as $B_{\|\cdot\|^*} \subset \operatorname{int} B_n$, there is δ_{n+1}^0 verifying

$$B_{\|\cdot\|^*} \subset \{f \in B_n : |f(x_{n+1})| > \delta_{n+1}^0\} \ .$$

Let us consider now $\delta_{n+1} = \max\{\delta_{n+1}^0, \delta_{n+1}^1, ..., \delta_{n+1}^n\}$ and

$$B_{n+1} = \{f \in B_n : |f(x_{n+1})| \le \delta_{n+1}\} \ .$$

The set B_{n+1} also satisfies (2.2) and (2.3). If we define $B = \bigcap_{n=1}^\infty B_n$, then the set B is a unit ball of a dual norm $|\cdot|^*$ (B is weak* closed) and $B_{\|\cdot\|^*} \subset B \subset \lambda B_{\|\cdot\|^*}$. Also, for each $n \in \mathbb{N}$, $F_n \subset S_{|\cdot|^*}$, and therefore every point of the sequence $\{x_n/|x_n|\}$ induces a face in $B_{|\cdot|^*}$. This implies that $x_n/|x_n| \in \operatorname{ver} B_{|\cdot|}$, as we wanted to prove. $\square$

Note that $x_n/|x_n|$ is a strongly vertex point of the dual (and predual) unit ball of B_m, for each $m \ge n$. Nevertheless, in general, it is not true that $x_n/|x_n|$ is a strongly vertex point of $B_{|\cdot|}$, as the following corollary points out.

Corollary 2.4. *Let $(X, \|\cdot\|)$ be a separable Banach space and let $\{x_n\}$ be a dense set of points in the unit sphere. Then, for every $\lambda > 1$, there is a λ-isometric norm $|\cdot|$ such that $\{x_n/|x_n|\} \subseteq \operatorname{ver} B_{|\cdot|}$.*

On the other hand, it can be easily proved that the set of vertex points of a ball is always of first Baire category.

Corollary 2.5. *Let X be a separable Banach space and F a countable dense set in X. Then, norms which are not Gâteaux differentiable in each point of F are dense in $(N(X), \rho)$.*

There exist some classical Banach spaces whose unit ball is the closed convex hull of its strongly vertex points. For instance, ℓ_1 and the Lorentz sequence space $d(\omega, 1)$ with their usual norms have this property. Moreover, in both cases, every extreme point of the unit ball is a strongly vertex point. The class of Banach spaces admitting an equivalent norm with the unit ball being the closed convex hull of its strongly vertex points is considerably wide. Actually, this class includes every Banach space with property α [12] and, therefore, every Banach space admitting a biorthogonal system with cardinal equals to the density of the space [7]. Unfortunately, norms with the property α are only known to be dense in superreflexive Banach spaces [14]. However, it is possible to give some positive results in this direction provided the space has the Mazur intersection property or the dual has the weak* Mazur intersection property.

Theorem 2.6. *Let X be a Banach space with the Mazur intersection property. Then, every equivalent norm in X can be uniformly approximated by norms with their unit balls being the closed convex hull of their strongly vertex points.*

Proof. In [6] Banach spaces with the Mazur intersection property are characterized as those having in the unit sphere of its dual ball a dense set of weak* denting points. It is showed in [5] that the set of all equivalent norms with the Mazur intersection property is residual in $(N(X),\rho)$. Then, we only need to consider an equivalent norm $\|\cdot\|$ with a dense set of weak* denting points in its dual unit sphere.

Let $0<\delta<1$ and find a family $(f_\alpha,x_\alpha,\rho_\alpha)_{\alpha\in I}$, with $f_\alpha\in X^*$, $\|f_\alpha\|^*=1$, $x_\alpha\in X$, $\|x_\alpha\|=1$, $1>\rho_\alpha\geq 1-\delta$ and $f_\alpha\in S(B_{\|\cdot\|^*},\ x_\alpha,\ \rho_\alpha)$, which is maximal with respect to the following condition

$$S(B_{\|\cdot\|^*},\ x_\alpha,\ \rho_\alpha)\cap S(B_{\|\cdot\|^*},\ \varepsilon x_\beta,\ \rho_\beta)=\emptyset,\qquad \alpha\neq\beta,\ \ \varepsilon=\pm 1.$$

This family induces a dual equivalent norm $|\cdot|^*$ in X^* with unit ball

$$B_{|\cdot|^*}=B_{\|\cdot\|^*}\setminus\bigcup_{\substack{\alpha\in I\\ \varepsilon=\pm 1}}S(B_{\|\cdot\|^*},\ \varepsilon x_\alpha,\ \rho_\alpha). \tag{2.4}$$

Obviously, $(1-\delta)\|\cdot\|\leq|\cdot|\leq\|\cdot\|$. The maximality and the density of the weak* denting points in $S_{\|\cdot\|^*}$ imply that the subset

$$\bigcup_{\alpha\in I}\{f\in B_{\|\cdot\|^*}:\ |f(x_\alpha)|=\rho_\alpha\}$$

is dense in the unit sphere of $B_{|\cdot|^*}$. From this fact it can be deduced in the usual way that $B_{|\cdot|}=\overline{\mathrm{conv}}(\{\pm y_\alpha\}_{\alpha\in I})$, where $y_\alpha=(1/\rho_\alpha)x_\alpha$. Finally, we will show that the points $\{\pm y_\alpha\}_{\alpha\in I}$ are strongly vertex points. Consider $\alpha\in I$; it follows from (2.4) that $f_\alpha(y_\alpha)>1$ and $|f_\alpha(y_\beta)|\leq 1$ for every $\beta\in I$, $\beta\neq\alpha$. This means that

$$y_\alpha\notin B_\alpha=\overline{\mathrm{conv}}(\{\pm y_\beta\}_{\beta\in I\setminus\{\alpha\}}\cup\{-y_\alpha\}),$$

and obviously, $B_{|\cdot|}=\mathrm{conv}(B_\alpha\cup\{y_\alpha\})$. The point $-y_\alpha$ is also a strongly vertex point by symmetry. $\square$

Let us denote by C_D, C_V and C_S the classes of Banach spaces where every equivalent norm can be uniformly approximated by norms whose unit balls are the closed convex hull of their denting, vertex and strongly vertex points, respectively. Obviously, $C_S\subseteq C_V\subseteq C_D$ and next proposition shows that, actually, these classes are the same.

Proposition 2.7. *Let* $(X, \|\cdot\|)$ *be a Banach space such that its unit ball* $B_{\|\cdot\|}$ *is the closed convex hull of its denting points. Then the norm* $\|\cdot\|$ *can be uniformly approximated by norms with their unit balls being the closed convex hull of their strongly vertex points.*

Proof. Let $0 < \delta < 1$ and find a family of denting points $(x_\alpha)_{\alpha \in I}$ in X, $\|x_\alpha\| = 1$, which is maximal with respect to the condition that for $\alpha \neq \beta$

$$\|x_\alpha - x_\beta\| \geq \delta \qquad \text{and} \qquad \|x_\alpha + x_\beta\| \geq \delta. \tag{2.5}$$

Let B the closed convex hull of $(\pm x_\alpha)_{\alpha \in I}$ and $|\cdot|$ the Minkowski functional of B. Clearly $\|\cdot\| \leq |\cdot|$. On the other hand, if we denote $|\cdot|^*$ the dual Minkowski functional of $|\cdot|$ on X^*, for every denting point x of $B_{\|\cdot\|}$ there is an $\alpha \in I$ such that $\|x - x_\alpha\| < \delta$ or $\|x + x_\alpha\| < \delta$ and then,

$$\begin{aligned} \|f\|^* &= \sup\{|f(x)|,\ \|x\| = 1\} \\ &= \sup\{|f(x)|,\ \|x\| = 1, \quad x \in \operatorname{dent} B_{\|\cdot\|}\} \\ &\leq \sup\{|f(x_\alpha)| + |f(y)| : \quad \alpha \in I,\ \|y\| < \delta\} \\ &\leq |f|^* + \delta \|f\|^*, \end{aligned}$$

and this implies that, $\|\cdot\| \leq |\cdot| \leq (1-\delta)^{-1}\|\cdot\|$. From the fact that the points $(\pm x_\alpha)_{\alpha \in I}$ are denting in $B_{\|\cdot\|}$ and condition (2.5), we get that

$$x_\alpha \notin B_\alpha = \overline{\operatorname{conv}}\left(\{\pm x_\beta\}_{\beta \in I \setminus \{\alpha\}} \cup \{-x_\alpha\}\right)$$

and obviously, $B_{|\cdot|} = \operatorname{conv}(B_\alpha \cup \{x_\alpha\})$. The point $-x_\alpha$ is also a strongly vertex point by symmetry. □

Corollary 2.8. *Let* X^* *be a dual Banach space with the weak* Mazur intersection property. Then, every equivalent norm in* X *can be uniformly approximated by norms with their unit balls being the closed convex hull of their strongly vertex points.*

Proof. Banach spaces with the *weak** Mazur intersection property are characterized in [6] as those having in the unit sphere of their predual ball a dense set of denting points and it is showed in [5] that the set of all equivalent norms with the *weak** Mazur intersection property is residual in $(N^*(X^*),\rho)$. Therefore, we only need to apply Proposition 2.7. $\square$

Corollary 2.9. *Let X be a Banach space with a fundamental biorthogonal system. Then every norm can be uniformly approximated by norms with their unit balls being the closed convex hull of their strongly vertex points.*

Proof. It is proved in [13] that for a Banach space X with a fundamental biorthogonal system, X^* can be equivalent renormed to have the *weak** Mazur intersection property. Now, the assertion follows from Corollary 2.8. $\square$

Recall that the Banach space $(X,\|\cdot\|)$ has property α if there is a λ with
$0\le\lambda<1$ and a family $\{x_\alpha,x^*_\alpha\}_{\alpha\in I}\subset X\times X^*$ with $\|x_\alpha\|=\|x^*_\alpha\|=x^*_\alpha(x_\alpha)=1$ such that,

(i) for $\beta\neq\alpha$, $|x^*_\alpha(x_\beta)|\le\lambda$,

(ii) $B_{\|\cdot\|}=\overline{\mathrm{conv}}\,(\{\pm x_\alpha\}_{\alpha\in I})$.

This property was introduced by Schachermayer [14] in the study of norm attaining operators. It could be interesting to know the relation between the class of Banach spaces with the Mazur intersection property or with dual having the weak* Mazur intersection property and the class of Banach spaces admitting an equivalent norm with the property α. Next result is a partial answer to this question.

If η is a cardinal, recall that $\mathrm{cf}\,(\eta)$ is the smallest cardinal β such that there exists a sequence of cardinals $\{\beta_i\}_{i<\beta}$ strictly less than η such that $\eta=\sup\{\beta_i:\ i<\beta\}$.

Corollary 2.10. *Let X be a Banach space such that $\mathrm{dens}\,X=\eta$ with $cf(\eta)>\chi_0$ and X does not contain ℓ_1. Suposse that X has the Mazur intersection property or X^* has the weak* Mazur intersection property. Then, X can be equivalently renormed with the property α.*

Proof. By Proposition 2.6 and Corollary 2.8 there is a bounded family $\{x_i\}_{i\in I}$ in X with $card\,(I) = \eta$ such that $x_i \notin \overline{\text{conv}}\,(\{x_j\}_{j\in I\setminus\{i\}})$. That means that there is a bounded family $\{x_i, f_i\}_{i\in I} \subset X \times X^*$ and, for every $i \in I$, there is a $\delta_i > 0$ satisfying $f_i(x_i) = 1$ and $|f_i(x_j)| \leq 1 - \delta_i$ for every $j \neq i$. Consider for each $n \in \mathbb{N}$ the set $I_n = \{i \in I : \ \delta_i \geq 1/n\}$. Clearly, $I = \cup_{n\in\mathbb{N}} I_n$ and the hypothesis that $\text{cf}\,(\eta) > \chi_0$ implies that there exists $n_0 \in \mathbb{N}$ with $\text{card}(I_{n_0}) = \eta = \text{dens}\,X$. Finally, we relabel the family $\{x_i, f_i\}_{i\in I_{n_0}}$ as $\{x_i^n, f_i^n\}_{i\in I, n\in\mathbb{N}}$ and apply [14, Thm. 4.1] to construct a norm with the property α. $\square$

A convex body is a bounded closed convex set having nonempty interior. Most of the previous results of approximation for balls can be expressed in terms of convex bodies as it is proved in the following results. We denote the boundary of C by ∂C and the interior of C by $\text{int}\,C$. Let $\mathcal{V}(\mathcal{X})$ be the set of all convex, closed, bounded and nonempty subsets of X, $\mathcal{V}'(\mathcal{X}) = \{\mathcal{C} \in \mathcal{V}(\mathcal{X}) : \ \prime \in \text{int}\,\mathcal{C}\}$, $\mathcal{V}(\mathcal{X}^*)$ the set of all convex, weak*-compact and nonempty subsets of the dual space X^*, and $\mathcal{V}'(\mathcal{X}^*) = \{\mathcal{C} \in \mathcal{V}(\mathcal{X}^*) : \ \prime \in \text{int}\,\mathcal{C}\}$. Recall that $C^o = \{f \in X^* : \ f(x) \leq 1 \text{ for all } x \in C\}$ is the polar set of $C \in \mathcal{V}'(\mathcal{X})$ and, for $C \in \mathcal{V}'(\mathcal{X}^*)$, the set C_o is $C^o \cap X$. The Hausdorff metric between two subsets of $\mathcal{V}(\mathcal{X})$ is defined as follows:

$$h(C_1, C_2) = \inf\{\varepsilon > 0 : C_1 \subset C_2 + \varepsilon B_{\|\cdot\|},\ C_2 \subset C_1 + \varepsilon B_{\|\cdot\|}\}\ .$$

It is well known that $(\mathcal{V}(\mathcal{X}), \langle)$ is a complete metric space and that $\mathcal{V}'(\mathcal{X})$ is an open set of $\mathcal{V}(\mathcal{X})$. Given $C \in \mathcal{V}'(\mathcal{X})$, we say that X has the Mazur intersection property with respect to C if for every closed, convex, bounded set D and every point $x \notin D$ there exists $y \in X$ and $\lambda > 0$ such that $D \subset y + \lambda C$ and $x \notin y + \lambda C$. For a dual Banach space and $C \in \mathcal{V}'(\mathcal{X}^*)$ we consider the corresponding weak* Mazur intersection property with respect to C. These two properties can be characterized in the same manner that Mazur and weak* Mazur intersection properties in Theorem 2.1 and 3.1 of [6], with a similar proof.

Proposition 2.11. *(i) A Banach space X have the Mazur intersection property with respect to $C \in \mathcal{V}'(\mathcal{X})$ if and only if weak* denting points of C^o are dense in ∂C^o.*

(ii) A dual Banach space X^* has the weak Mazur intersection property with respect to $C \in \mathcal{V}'(\mathcal{X}^*)$ if and only if denting points of C_o are dense in ∂C_o.*

Proposition 2.12. *(i) A Banach space X admits a norm with the Mazur intersection property if and only if there exists $C \in \mathcal{V}'(\mathcal{X})$ such that the set of weak* denting points of C^o is dense in ∂C^o.*

(ii) A dual Banach space X^* admits a dual norm with the weak Mazur intersection property if and only if there exists $C \in \mathcal{V}'(\mathcal{X}^*)$ such that the set of denting points of C_o is dense in ∂C_o.*

Proof. We prove only (i). Let us consider in X^* the functional $\sigma_C(f) = \sup_{x \in C} f(x)$ and the dual norm

$$\|f\|^{*2} = \sigma_C^2(f) + \sigma_C^2(-f) \quad .$$

It can be easily verified that $f\|f\|^{*-1}$ is a weak* denting point of $B_{\|\cdot\|^*}(X^*)$ whenever $f\sigma_C(f)^{-1}$ or $-f\sigma_C(-f)^{-1}$ is a weak* denting point of C^o. □

The following result is similar to a characterization given in [5]. The proof can be carried out in one sense directly from Theorem 2.12 and, in the other sense, using Proposition 2.11 and the techniques used in [5].

Proposition 2.13. *(i) A Banach space X admits a norm with the Mazur intersection property if and only if there exists a dense G_δ subset $\mathcal{V}_I \subset \mathcal{V}'(\mathcal{X})$ such that for every $C \in \mathcal{V}_I$ the set of weak* denting points of C^o is dense in ∂C^o.*

(ii) A dual Banach space X^* admits a norm with the weak Mazur intersection property if and only if there exists a dense G_δ subset $\mathcal{V}_I \subset \mathcal{V}'(\mathcal{X}^*)$ such that for every $C \in \mathcal{V}_I$ the set of denting points of C_o is dense in ∂C_o.*

As a consequence of the previous results and using analogous arguments than in Theorem 2.6 and Proposition 2.7 for approximation of balls, it turns out the following corollary.

Corollary 2.14. *Let X be a Banach space with the Mazur intersection property or such that X^* has the weak* Mazur intersection property. Then, every convex body in X can be approximated (in the Hausdorff metric) by convex bodies being the closed convex hull of its strongly vertex points.*

Given a Banach space $(X, \|\cdot\|)$, the set $\operatorname{strver} B_{\|\cdot\|}$ is discrete and this implies that $\operatorname{card}(\operatorname{strver} B_{\|\cdot\|}) \leq \operatorname{dens} X$. Surprisingly, this is not the case with the set of vertex points, as we can see in the following example.

Example 2.15. *The usual norm of ℓ_1 can be uniformly approximated by equivalent norms with uncountable many vertex points.*

Proof. For each $\varepsilon = (\varepsilon_j)_{j=1}^{\infty} \in \{-1,1\}^{\mathbb{N}}$, set $x_\varepsilon = (\varepsilon_j/2^j)_{j=1}^{\infty}$. Then $x_\varepsilon \in \ell_1$, $\varepsilon \in \ell_\infty$ and $\|x_\varepsilon\|_1 = 1 = \|\varepsilon\|_\infty = \varepsilon(x_\varepsilon)$. For every δ, $0 < \delta < 1/4$, we consider the family of cones $K_\varepsilon = x_\varepsilon - K(\varepsilon, \delta)$ for $\varepsilon \in \{-1,1\}^{\mathbb{N}}$. We will show that for every $\varepsilon, \varepsilon' \in \{-1,1\}^{\mathbb{N}}$,

(1) the point $x_{\varepsilon'} \in K_\varepsilon$,
(2) if B_1 is usual unit ball of ℓ_1, then $(1-4\delta)B_1 \subseteq K_\varepsilon$.

The first assertion follows from the fact that

$$\begin{aligned}\varepsilon(x_\varepsilon - x_{\varepsilon'}) &= \sum_{j\in\mathbb{N}} \frac{\varepsilon_j(\varepsilon_j - \varepsilon'_j)}{2^j} = \sum_{j\in\mathbb{N}} \frac{|\varepsilon_j - \varepsilon'_j|}{2^j} = ||x_\varepsilon - x_{\varepsilon'}||_1 \\ &\geq \delta ||x_\varepsilon - x_{\varepsilon'}||_1,\end{aligned}$$

and we obtain (1). Consider now $x \in (1-4\delta)B_1$, then

$$\varepsilon(x_\varepsilon - x) = 1 - \varepsilon(x) \geq 1 - (1-4\delta) > 2\delta \geq \delta ||x_\varepsilon - x||_1,$$

and this implies (2). The set $B = \bigcap_{\varepsilon\in\{-1,1\}^{\mathbb{N}}} K_\varepsilon$ is closed, convex, symmetric, bounded and contains a neighbourhood of 0. More precisely

$$(1-4\delta)B_1 \subseteq B \subseteq B_1. \tag{2.6}$$

The first inclusion is obvious from (2). For the second, let us consider $x \in B$ and $\varepsilon \in \{-1,1\}^{\mathbb{N}}$ such that $\|x\|_1 = \varepsilon(x)$. The point $x \in K_\varepsilon$, so we have

$$1 - \|x\|_1 = 1 - \varepsilon(x) = \varepsilon(x_\varepsilon - x) \geq \delta \|x_\varepsilon - x\|_1,$$

and we obtain $1 \geq 1 - \delta\|x_\varepsilon - x\|_1 \geq \|x\|_1$. It follows from (2.6) that B is the unit ball on an equivalent norm $\|\cdot\|$ in ℓ_1. Using (1) we have that $x_\varepsilon \in B$. The fact that $B \subseteq K_\varepsilon$ implies that x_ε is a vertex point of the unit ball of $\|\cdot\|$. $\square$

Observe that, according with Lemma 2.1, $\operatorname{card}(\operatorname{ver} B_{\|\cdot\|}) \leq \operatorname{dens} X^*$ since different vertex points produce different faces with disjoint interiors in $B_{\|\cdot\|^*}$. On the other hand, if we consider the Kunen space $\mathcal{K}$ mentioned above and an equivalent norm $\|\cdot\|$ in $\mathcal{K}$, there is a separable subspace $H \subseteq \mathcal{K}$ such that $\operatorname{ver} B_{\|\cdot\|} \subseteq H \cap B_{\|\cdot\|}$ [9]. Therefore, $\operatorname{ver} B_{\|\cdot\|} \subseteq \operatorname{ver}(H \cap B_{\|\cdot\|})$. Also, $\mathcal{K}$ is a nonseparable Asplund space and thus

$$\operatorname{card}(\operatorname{ver} B_{\|\cdot\|}) \leq \operatorname{dens} H^* = \operatorname{dens} H = \chi_0 < \operatorname{dens} \mathcal{K}^*.$$

It could be interesting then to determine the class of Banach spaces X for which there exists an equivalent norm $\|\cdot\|$ with $\operatorname{card}(\operatorname{ver} B_{\|\cdot\|}) = \operatorname{dens} X^*$.

3. Approximation by locally linear norms. Differentiability properties

We say that a norm $\|\cdot\|$ on a Banach space X is locally linear at a point $x \in X \setminus \{0\}$ provided there exists a neighbourhood U of x and a functional $f \in X^* \setminus \{0\}$ such that $\|y\| = f(y)$ for all $y \in U$. Every Banach space admits a locally linear norm on a dense open set [12]. Moreover, the set of norms with this property is dense in Banach spaces with dual having the weak* Mazur intersection property [13]. The following theorem, in the same direction of these results, is the key to prove the validity of Georgiev and Vanderwerff result [5, 16] concerning denseness of smooth norms at a fixed sequence for higher orders of smoothness.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space and $\{x_n\}$ a sequence of points in $X\setminus\{0\}$. For each $\lambda > 1$ there exist a λ-isometric norm $|\cdot|$ which is locally linear in a neighbourhood of $\{x_n\}$.*

Proof. We construct in X^* a sequence of dual norms $\|\cdot\|_n^*$ in the following way. Choose a strictly decreasing sequence $\{\gamma_n\}$ converging to $1/2$ such that $1/2 < \gamma_n < 1$, $n \in \mathbb{N}$. Suposse that $\|x_1\| = 1$ and take $f_1 \in X^*$, with $\|f_1\|^* = 1 = f_1(x_1)$. Then, if we select $1 < \lambda_1 < 1 + (\log\lambda)/2$, the set

$$B_{\|\cdot\|_1^*} = \operatorname{conv}(\{\pm\lambda_1 f_1\} \cup B_{\|\cdot\|^*})$$

which is obviously weak* closed is the unit ball of $\|\cdot\|_1^*$. The point $g_1 = \lambda_1 f_1$ is vertex with respect to x_1. Then, there exist $0 < \rho_1 \leq 1$ such that

$$B_{\|\cdot\|_1^*} \subseteq g_1 - K(x_1, \rho_1)\ .$$

Suppose we have constructed a dual norm $\|\cdot\|_n^*$, such that there are different points h_j^n, $j = 1, \dots, k(n)$, in $S_{\|\cdot\|_n^*}$ such that for every x_i, $i = 1, \dots, n$, there exists $j(i) \in \{1, \dots, k(n)\}$ verifying

$$B_{\|\cdot\|_n^*} \subseteq h_{j(i)}^n - K(x_i,\ \rho_i\gamma_n)\ . \tag{3.1}$$

Assume that $\|x_{n+1}\|_n = 1$ and take f_{n+1} such that $\|f_{n+1}\|_n^* = 1 = f_{n+1}(x_{n+1})$. Then:

(a) if $f_{n+1} \notin \{\pm h_j^n,\ j = 1, \dots, k(n)\}$, we take $1 < \lambda_{n+1} < 1 + (\log\lambda)/2^{n+1}$ such that, if $g_{n+1} = \lambda_{n+1} f_{n+1}$,

$$\pm\, g_{n+1} \in h_{j(i)}^n - K(x_i,\ \rho_i\gamma_{n+1}) \qquad\qquad i = 1, ..., n \tag{3.2}$$

(b) if $f_{n+1} \in \{\pm h_j^n,\ j = 1, \dots, k(n)\}$, say $f_{n+1} = h_1^n = g_1$, we can find $1 < \lambda_{n+1} < 1 + (\log\lambda)/2^{n+1}$ such that $g_{n+1} = \lambda_{n+1} f_{n+1}$ verifies (3.2) whenever $j(i) \neq 1$. If $j(i) = 1$ then

$$g_1 - K(x_1,\ \rho_1\gamma_{n+1}) \subseteq g_{n+1} - K(x_1,\ \rho_1\gamma_{n+1})\ . \tag{3.3}$$

Define now the unit ball of $\|\cdot\|_{n+1}^*$ as

$$B_{\|\cdot\|_{n+1}^*} = \mathrm{conv}\,(\{\pm g_{n+1}\} \cup B_n^*).$$

In the case (a), by (3.1) and (3.2) we have that there exist h_j^{n+1}, $j = 1, \ldots, k(n+1)$, vertex points in the unit sphere of $B_{\|\cdot\|_{n+1}^*}$ such that for every $i = 1, \ldots, n+1$, there exists $j(i) \in \{1, \ldots, k(n+1)\}$ verifying

$$B_{\|\cdot\|_{n+1}^*} \subseteq h_{j(i)}^{n+1} - K(x_i,\ \rho_i \gamma_{n+1})\ .$$

In the second case we can deduce it from (3.2) for $j(i) \neq 1$ and (3.3) for $j(i) = 1$.

Now, it can be easily proved that the sequence of dual norms $\|\cdot\|_n^*$ converges in $(N^*(X^*), \rho)$ to a dual norm $|\cdot|^*$ which is $\lambda-$isometric to $\|\cdot\|^*$. The norm $|\cdot|^*$ satisfies that, for every $n \in \mathbb{N}$, there exists a vertex point h_n in its unit sphere such that

$$B_{|\cdot|^*} \subseteq h_n - K(x_n,\ \rho_n/2)\ .$$

By Lemma 2.1 (i), the predual norm $|\cdot|$ in X is locally linear in a neighbourhood of $\{x_n\}$. $\square$

Corollary 3.2. *Let $(X, \|\cdot\|)$ be a separable Banach space and F a countable dense subset of $X \setminus \{0\}$. For every $\lambda > 1$ there exists a λ-isometric norm which is locally linear in an open dense set containing F.*

Once we have solved the problem of denseness, we are concerned with a Baire category question. Let φ be a real valued function defined on an open subset D of a Banach space $(X, \|\cdot\|)$ and $x \in D$. The function φ is Lipschitz smooth at x if there exist $M > 0$, $\delta > 0$, and $f \in X^*$ such that

$$|\varphi(x+h) - \varphi(x) - f(h)| \leq M||h||^2$$

whenever $h \in X$, $||h|| \leq \delta$. It can be deduced [2] that the norm $\|\cdot\|$ in X is Lipschitz smooth at $x \in X \setminus \{0\}$ if and only if there exists $M > 0$

such that

$$||x+h||^2 + ||x-h||^2 - 2||x||^2 \leq M||h||^2 \ , \qquad \text{for every } h \in X.$$

Proposition 3.3. *Let X be a Banach space and F a countable subset of $X \setminus \{0\}$. The set of norms in $(N(X), \rho)$ Lipschitz smooth in a neighbourhood of F is dense and first Baire category.*

Proof. The denseness follows from Theorem 3.1. The second assertion follows from the fact that the set $\mathcal{L}_{\S,}$ of Lipschitz smooth norms at a fixed point x_0 is always of first Baire category. Indeed, let us consider, for every $n \in \mathbb{N}$, the set

$$\mathcal{F}_\setminus = \{\| \cdot \| \in (\mathcal{N}(\mathcal{X}), \rho) : \|\S, + \langle\|^\in + \|\S, - \langle\|^\in - \in\|\S,\|^\in \leq \setminus\|\langle\|^\in, \text{ for } \langle \in \mathcal{X}\}$$

Obviously, $\mathcal{L}_{\S,} = \bigcup_{\setminus=\infty}^{\infty} \mathcal{F}_\setminus$. The sets $\mathcal{F}_\setminus$ are closed with empty interior since the set of norms which are not Gâteaux smooth at x_0 is dense in $(N(X), \rho)$. □

It seems to be still an open question to know if the set of Fréchet differentiable norms is of secondary category. This is the case, for instance, in Banach spaces with dual locally uniformly rotund.

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