Math. Nachr. (1998),

On the Nonseparable Subspaces of $J(\eta)$ and $C([1,\eta])$

By Antonio S. Granero, M. Mar Jiménez and José P. Moreno of Madrid

(Received Sept., 1998)

Abstract. Let η be a regular cardinal. It is proved, among other things, that: (i) if $J(\eta)$ is the corresponding *long James space*, then every closed subspace $Y \subseteq J(\eta)$, with $Dens(Y) = \eta$, has a copy of $\ell_2(\eta)$ complemented in $J(\eta)$; (ii) if Y is a closed subspace of the space of continuous functions $C([1,\eta])$, with $Dens(Y) = \eta$, then Y has a copy of $c_0(\eta)$ complemented in $C([1,\eta])$. In particular, every nonseparable closed subspace of $J(\omega_1)$ (resp. $C([1,\omega_1])$) contains a complemented copy of $\ell_2(\omega_1)$ (resp. $c_0(\omega_1)$). As a consequence, we give examples ($J(\omega_1)$, $C([1,\omega_1])$, $C(V)$, V being the "long segment") of Banach spaces X with the *hereditary density property* (HDP) (i.e., for every subspace $Y \subseteq X$ we have that $Dens(Y) = w^*\text{-}Dens(Y^*)$), even though these spaces are not *weakly countably determined* (WCD).

1. Notations and preliminaries

Throughout, $(X, \|\cdot\|)$ will be a real Banach space, $B(X)$ the closed unit ball of X, $S(X)$ the unit sphere and X^* its topological dual. Let O denote the ordinal numbers, LO the limit ordinals and C the cardinals. If $\eta \in C$, the *cofinality* $cf(\eta)$ of η is the smallest cardinal τ for which there exists a sequence of ordinals $\{\beta_i\}_{1\le i<\tau}$, $\beta_i \in O, \beta_i < \eta$ and $\eta = \sup\{\beta_i : 1 \le i < \tau\}$. A cardinal η is said to be *regular* if $cf(\eta) = \eta$. Denote by RC the family of regular cardinals.

If A, B are subsets of the ordinal η, we write $A < B$ iff, $\forall \alpha \in A$, $\forall \beta \in B$, we have $\alpha < \beta$. A transfinite sequence $\{A_\alpha\}_{1\le\alpha<\theta}$, $\theta \in O$, of subsets of η is said to be a *skipped* (transfinite) sequence iff $A_\alpha < A_\beta$, whenever $\alpha < \beta < \theta$, and for each $\alpha < \theta$, $\exists n_\alpha < \eta$ such that $A_\alpha < n_\alpha < A_{\alpha+1}$. A subset $A \subseteq \eta$ is said to be *nice* if $\min(A) \notin LO$. We say that S is a *segment* of η if $S \subseteq \eta$ and $[\kappa, \lambda] \subseteq S$ whenever $\kappa, \lambda \in S, \kappa \le \lambda$. Note that, if S is a nice segment of η, then $S = [\alpha, \beta)$, where $\alpha = \min(S) \notin LO$ and

1991 *Mathematics Subject Classification.* Primary 46B25; Secondary 46B26
Keywords and phrases. Long James spaces, nonseparable subspaces

$\beta = \min\{\gamma \in O : \gamma > S\}$. If $f : [1,\eta] \to \mathbb{R}$ is a map, define the support of f by $supp(f) = \{\alpha \leq \eta : f(\alpha) \neq 0\}$.

If X is a Banach space, in general, $\mathrm{Dens}(X^*, w^*) \leq \mathrm{Dens}(X, \|\cdot\|)$. So, X is said to have the *density property* if $\mathrm{Dens}(X^*, w^*) = \mathrm{Dens}(X, \|\cdot\|)$ and the *hereditary density property* (HDP) if every subspace $Y \subseteq X$ has the density property.

It is known that every *weakly countably determined* (WCD) Banach space X has the HDP (see [Va]). So, we can ask if there exists a non WCD Banach space with the HDP. The following easy Proposition gives a method to find Banach spaces with the HDP.

Proposition 1.1. *Let X be a Banach space such that every closed subspace $Y \subset X$ contains a WCD subspace $Z \subset Y$ with Dens(Z) =Dens(Y). Then X has the HDP.*

Proof. Let $i : Z \to Y$ be the canonical inclusion and suppose that $\mathrm{Dens}(Y, \|\cdot\|) > \mathrm{Dens}(Y^*, w^*)$. As $i^* : Y^* \to Z^*$ is a w^*-w^*-continuous quotient map, we would obtain that $\mathrm{Dens}(Z, \|\cdot\|) = \mathrm{Dens}(Y, \|\cdot\|) > \mathrm{Dens}(Y^*, w^*) \geq \mathrm{Dens}(Z^*, w^*)$, a contradiction because Z being WCD has the HDP. □

We show that the *"long James space"* $J(\omega_1)$ is a non WCD Banach space such that every closed subspace $Y \subset J(\omega_1)$ contains a complemented copy of $\ell_2(I)$ with $\mathrm{card}(I) = \mathrm{Dens}(Y)$. So, by Prop. 1.1, $J(\omega_1)$ has the HDP.

Recall ([DGZ], p.253) that a compact space K is: (i) Gulko iff $C(K)$ is WCD ; (ii) Corson iff K is homeomorphic to a compact subset of $\Sigma(I)$ for some set I, where $\Sigma(I) = \{x \in [0,1]^I : card(\{i \in I : x_i \neq 0\}) \leq \aleph_0\}$. It is known that Gulko $\Rightarrow$ Corson. If K is Gulko, clearly $C(K)$ has the HDP, but there exists a Corson compact such that $C(K)$ has not the HDP (see the Kunen-Haydon-Talagrand compact in [Ne], p. 1086). So, we can ask if there exists a non Gulko (even a non Corson) compact K such that $C(K)$ has the HDP. We give examples ($[1, \omega_1]$, the "long segment") of non Corson compacts K such that $C(K)$ has the HDP.

2. The subspaces of $J(\eta)$

If $\eta \in O$, consider in $[1, \eta]$ the order topology and observe that, if $f : [1,\eta] \to \mathbb{R}$ is continuous, then $supp(f)$ is a nice subset of $[1,\eta]$. Define the *square variation* of f to be:

$$\|f\| = \sup \left\{ \left(\sum_{i=1}^{n-1} |f(\alpha_{i+1}) - f(\alpha_i)|^2 \right)^{1/2} \right\}$$

where the supremum is taken over all possible choices $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq \eta$. Define the *long James space* $J(\eta)$ to be:

$$J(\eta) = \{ f : [1,\eta] \to \mathbb{R} \text{ continuous } : f(\eta) = 0, \|f\| < \infty \}.$$

As may be verified, $(J(\eta), \|\cdot\|)$ is a Banach space.

The space $J(\eta)$ was firstly considered (implicitly) by Hagler and Odell [HO]. They proved that every infinite dimensional closed subspace of $J(\eta)$ contains a copy of ℓ_2. It was Edgar who introduced and studied (explicitly) the space $J(\eta)$ (see [E1], [EW]). The norm $\|\cdot\|$ we have just defined is not Edgar's original norm [E1], but it is equivalent. In fact the space considered here is isometrically isomorphic to the space introduced by Edgar (see [B], p. 347). In [Z] Zhao studies the bidual of $J(\eta)$.

Remark. Although we will only consider the space $J(\eta)$, it is worthy to remark that the same proofs work if $J(\eta)$ is replaced by $J_p(\eta)$, $1 < p < \infty$, and the norm in $J_p(\eta)$ is defined in terms of the p-th variation instead of the square variation.

Concerning the subspaces, $J(\eta)$ works approximately like the James space J. Recall that in 1967 Herman and Whitley [HW] proved that every infinite dimensional closed subspace of J contains a copy of ℓ_2. In 1977 Casazza, Lin and Lohman [CLL] improved this result showing that every infinite dimensional closed subspace of J contains a complemented copy of ℓ_2.

If $S = [\alpha, \beta)$ is a nice segment of η, let $Y = \{f \in J(\eta) : supp(f) \subseteq S\}$ and define $Q_S : J(\eta) \to Y$ as follows:

$$\forall f \in J(\eta),\ \forall i \leq \eta, \qquad Q_S(f)(i) = \begin{cases} 0 & \text{,if } i \notin S, \\ f(i) - f(\beta) & \text{,if } i \in S \end{cases}$$

Clearly Q_S is a projection on Y such that $\|Q_S\| \leq \sqrt{2}$.

If $f, g \in J(\eta)$, we write $f < g$ iff $supp(f) < supp(g)$. A sequence $\{f_\alpha\}_{\alpha<\theta}$, $\theta \in O$, in $J(\eta)$ is said to be a *skipped sequence* iff $\{supp(f_\alpha)\}_{\alpha<\theta}$ is a skipped sequence of subsets of η. Observe that, if $\eta \in C$ satisfies $cf(\eta) \geq \aleph_1$ and $f \in J(\eta)$, then $\exists \alpha_f < \eta$ such that, $\forall \alpha > \alpha_f$, $f(\alpha) = 0$. If $\gamma \leq \eta$, let $P_\gamma : J(\eta) \to J(\eta)$ be such that, $\forall f \in J(\eta)$, $P(f) = f \cdot \mathbf{1}_{[1,\gamma]}$. Of course, P_γ is a projection with $\|P_\gamma\| = 1$.

Lemma 2.1. *Let $\eta \in O$. Then:*

1. *Let $\gamma \leq \eta$, $\gamma \in LO$, $f \in J(\eta)$ with $f(\gamma) = 0$ and $\{\gamma_\alpha\}_{\alpha<\tau} \subset [1, \eta)$ such that $\gamma_\alpha < \gamma_\beta < \gamma$, for $\alpha < \beta < \tau$, and $\lim_{\alpha\to\tau} \gamma_\alpha = \gamma$. Then $\|P_\gamma(f) - P_{\gamma_\alpha}(f)\| \to 0$ as $\alpha \to \tau$.*
2. *If $Y \subseteq J(\eta)$ is a closed subspace with $Dens(Y) \geq \aleph_0$ and $\gamma \leq \eta$, then $Y_0 = \{y \in Y : y(\gamma) = 0\}$ is a subspace of Y such that $dim(Y/Y_0) \leq 1$ and $Dens(Y_0) = Dens(Y)$.*

Proof. (1) This is true because, if there existed $\epsilon > 0$ such that $\|P_\gamma(f) - P_{\gamma_\alpha}(f)\| \geq \epsilon$, $\forall \alpha < \tau$, then we would obtain easily that $\|P_\gamma(f)\| = \infty$. Observe that in this argument it is essential that $f(\gamma) = 0$.

(2) This is trivial. □

Lemma 2.2. *If $\eta \in O$, every normalized skipped sequence $\{f_\alpha\}_{\alpha<\theta}$, $\theta \in O$, in $J(\eta)$ is a monotone basic sequence equivalent to the canonical basis of $\ell_2(\theta)$, and the closed subspace $Y := [\{f_\alpha\}_{\alpha<\theta}]$ generated is complemented in $J(\eta)$.*

Proof. First of all, note that, if $\{f_i\}_{i=1}^p$ is a finite skipped normalized sequence in $S(J(\eta))$ such that $supp(f_1) < n_1 < supp(f_2) < \cdots < n_{p-1} < supp(f_p) < \eta$ and $a_i \in \mathbb{R},\ i = 1, 2, .., p$, then:

$$\sum_{i=1}^{p} |a_i|^2 \leq \| \sum_{i=1}^{p} a_i f_i \|^2 \leq 2 \sum_{i=1}^{p} |a_i|^2.$$

Indeed, if $g := \sum_{i=1}^p a_i f_i$, for the first inequality (i.e., $\sum_{i=1}^p |a_i|^2 \leq \| \sum_{i=1}^p a_i f_i\|^2$) take finite subsets $J_i = \{n_{i-1} = k_1^i < k_2^i < \cdots < k_{j_i}^i = n_i\},\ i = 1, 2, \cdots, p,\ (n_0 = 1, n_p = \eta)$, and $J = \{k_1 < k_2 < \cdots < k_r\} \subset [1, \eta]$ and observe that:

$$\sum_{i=1}^{p} |a_i|^2 = \sum_{i=1}^{p} \sup_{J_i} \sum_{s=1}^{j_i - 1} |g(k_{s+1}^i) - g(k_s^i)|^2 \leq \sup_{J} \sum_{i=1}^{r-1} |g(k_{i+1}) - g(k_i)|^2 = \|g\|^2,$$

where the suprema are taken over all possible choices of J_i and J.

For the second inequality (i.e., $\| \sum_{i=1}^p a_i f_i\|^2 \leq 2 \sum_{i=1}^p |a_i|^2$) observe that $(a-b)^2 \leq 2(a^2 + b^2)$. So, if $J = \{j_1 < j_2 < \cdots j_r\} \subset [1, \eta]$ is an arbitrary finite subset and $J \cup \{n_0, n_1, .., n_p\} = \{k_1 < k_2 < \cdots < k_s\} =: K_J$, then:

$$\|g\|^2 = \sup_{J} \sum_{i=1}^{r-1} |g(j_{i+1}) - g(j_i)|^2 \leq 2 \sup_{K_J} \sum_{i=1}^{s-1} |g(k_{i+1}) - g(k_i)|^2 \leq 2 \sum_{i=1}^{p} |a_i|^2,$$

where the suprema are taken over all possible choices of J.

So, if $\{f_\alpha\}_{\alpha<\theta},\ \theta \in O$, is a skipped normalized sequence in $J(\eta)$, it is equivalent to the canonical basis of $\ell_2(\theta)$. Consider the closed subspace $Y := [\{f_\alpha\}_{\alpha<\theta}]$ and prove that Y is complemented in $J(\eta)$:

(A) Let $S_\alpha = [p_\alpha, q_\alpha)$, where $p_\alpha = \min(supp(f_\alpha)),\ q_\alpha = \min\{\gamma \in O : \gamma > supp(f_\alpha)\}$, and $Y_\alpha = \{f \in J(\eta) : supp(f) \subseteq S_\alpha\}$. Then $\{S_\alpha\}_{\alpha<\theta}$ is a skipped sequence of nice segments of η such that $f_\alpha \in Y_\alpha$. Let $Q_\alpha := Q_{S_\alpha}$ and pick $g_\alpha \in S(Y_\alpha^*)$ such that $g_\alpha(Q_\alpha(f_\alpha)) = \|Q_\alpha(f_\alpha)\| = \|f_\alpha\| = 1$.

(B) If $f \in J(\eta)$, define $Q(f) : [1, \eta] \to \mathbb{R}$ so that $Q(f) = \sum_{\alpha<\theta} g_\alpha(Q_\alpha(f)) \cdot f_\alpha$. Clearly $Q(f)(\eta) = 0$ and we claim that $\sum_{\alpha<\theta} |g_\alpha(Q_\alpha(f))|^2 \leq 2\|f\|^2$. Indeed:

$$\begin{gathered}
\textstyle\sum_{\alpha<\theta} |g_\alpha(Q_\alpha(f))|^2 \leq \sum_{\alpha<\theta} \|Q_\alpha(f)\|^2 = \sum_{\alpha<\theta} \sup \Big\{ |f(\alpha_n) - f(q_\alpha)|^2 + \\
+ \textstyle\sum_{i=2}^n |f(\alpha_i) - f(q_\alpha) - f(\alpha_{i-1}) + f(q_\alpha)|^2 + |f(\alpha_1) - f(q_\alpha)|^2 : \\
: p_\alpha \leq \alpha_1 < \alpha_2 < .. < \alpha_n < q_\alpha,\ n \in \mathbb{N} \Big\} = \\
= \textstyle\sum_{\alpha<\theta} \sup \Big\{ |f(\alpha_n) - f(q_\alpha)|^2 + \sum_{i=2}^n |f(\alpha_i) - f(\alpha_{i-1})|^2 + |f(\alpha_1) - f(q_\alpha)|^2 : \\
: p_\alpha \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n < q_\alpha,\ n \in \mathbb{N} \Big\} = \\
= \sup \Big[\textstyle\sum_{\alpha \in A} \sup \Big\{ |f(\alpha_n) - f(q_\alpha)|^2 + \sum_{i=2}^n |f(\alpha_i) - f(\alpha_{i-1})|^2 + |f(\alpha_1) - f(q_\alpha)|^2 : \\
: p_\alpha \leq \alpha_1 < \alpha_2 < .. < \alpha_n < q_\alpha, n \in \mathbb{N} \Big\} : A \subset \theta \text{ finite} \Big] \leq \\
\leq 2 \sup \Big\{ \textstyle\sum_{i=1}^{m-1} |f(\beta_{i+1}) - f(\beta_i)|^2 : 1 \leq \beta_1 < \beta_2 < .. < \beta_m \leq \eta, m \in \mathbb{N} \Big\} = 2\|f\|^2.
\end{gathered}$$

In consequence, $Q(f) \in J(\eta)$ and Q is a projection on Y such that $\|Q\| \leq 2$. □

Proposition 2.3. *Let $\eta \in O$ and $Y \subset J(\eta)$ be a closed infinite dimensional subspace. Then Y contains a copy of ℓ_2 complemented in $J(\eta)$.*

Proof. We apply transfinite induction. If $\eta = \omega$, the statement was proved by Casazza, Lin and Lohman [CLL]. Assume that the result is true for every $\alpha < \eta$:

(1) If $\eta = \alpha + 1$, by Lemma 2.1, $Y_0 = \{f \in Y : f(\alpha) = 0\}$ is a closed infinite dimensional subspace of $V := \{f \in J(\eta) : f(\alpha) = f(\eta) = 0\}$. Since V can be canonically identified with $J(\alpha)$, by induction hypothesis, there exists in Y_0 a copy Z of ℓ_2 and a projection $P : V \to Z$. Thus, PP_α is a projection from $J(\eta)$ onto Z.

(2) Suppose that $\eta \in LO$ and that there exists $\alpha < \eta$ such that P_α is an isomorphism between Y and $P_\alpha(Y)$. Since $P_\alpha(J(\eta))$ can be canonically identified with $J(\alpha + 1)$, by the induction hypothesis there exists in $P_\alpha(Y_0)$ a copy W of ℓ_2 and a projection $P : P_\alpha(J(\eta)) \to W$. Consider $Z = (P_{\alpha|Y})^{-1}(W)$. Then $Z \subset Y$ is a copy of ℓ_2 and $(P_{\alpha|Y})^{-1} \circ P \circ P_\alpha$ is a projection from $J(\eta)$ onto Z.

In the contrary case (i.e., if P_α, $\forall \alpha < \eta$, doesn't work as an isomorphism over Y), we have that, $\forall \alpha < \eta$, $\forall \epsilon > 0$, there exists $f \in S(Y)$ such that $\|P_\alpha(f)\| \leq \epsilon$. Fix $\epsilon > 0$ and pick $y_1 \in S(Y)$. By Lemma 2.1, there exists $\alpha_1 < \eta$ such that $\|y_1 - P_{\alpha_1}(y_1)\| \leq \epsilon 4^{-1}$. Choose $y_2 \in S(Y)$ and $\alpha_1 < \alpha_2 < \eta$ such that $\|P_{\alpha_1}(y_2)\| \leq \frac{\epsilon}{2} 4^{-2}$ and $\|y_2 - P_{\alpha_2}(y_2)\| \leq \frac{\epsilon}{2} 4^{-2}$. Thus $\|y_2 - (P_{\alpha_2} - P_{\alpha_1})y_2\| \leq \epsilon 4^{-2}$. By reiteration we get sequences $\{y_n\}_{n\geq 1} \subset S(Y)$ and $\{\alpha_n\}_{n\geq 1} \subset [1, \eta)$ such that: (i) $\alpha_n < \alpha_{n+1} < \eta$; (ii) $\|y_n - (P_{\alpha_n} - P_{\alpha_{n-1}})y_n\| \leq \epsilon 4^{-n}$, $(P_{\alpha_0} = 0)$. Let $x_n = \frac{(P_{\alpha_n} - P_{\alpha_{n-1}})y_n}{\|(P_{\alpha_n} - P_{\alpha_{n-1}})y_n\|}$. Then $\{x_{2n}\}_{n\geq 1}$ is a skipped normalized sequence of $J(\eta)$ and, by Lemma 2.2, it is equivalent to the canonical basis of ℓ_2 and there exists a projection $Q : J(\eta) \to [\{x_{2n}\}_{n\geq 1}]$ with $\|Q\| \leq 2$. Moreover, we have:

$$\sum_{n\geq 1} \|y_{2n} - x_{2n}\| \leq \sum_{n\geq 1} \|y_n - \frac{(P_{\alpha_n} - P_{\alpha_{n-1}})y_n}{\|(P_{\alpha_n} - P_{\alpha_{n-1}})y_n\|}\| \leq$$
$$\sum_{n\geq 1} \|y_n - (P_{\alpha_n} - P_{\alpha_{n-1}})y_n\| + \sum_{n\geq 1} |1 - \|(P_{\alpha_n} - P_{\alpha_{n-1}})y_n\|| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So, by [LT], Prop. 1.a.9, if $\epsilon \leq \frac{1}{16}$, the subspace $[\{y_{2n}\}_{n\geq 1}]$ is complemented in $J(\eta)$. □

Proposition 2.4. *Let $\eta \in C$, $cf(\eta) = \tau \geq \omega_1$, and $Y \subseteq J(\eta)$ a closed subspace with $Dens(Y) = \eta$. Consider the following statements:*

1. *w^*-$Dens(Y^*) = Dens(Y) = \eta$.*
2. *There exists in Y a normalized transfinite sequence $\{f_\alpha\}_{\alpha<\tau}$ with $supp(f_\alpha) < supp(f_\beta)$, whenever $\alpha < \beta < \tau$.*
3. *There exists in Y a normalized skipped transfinite sequence $\{g_\alpha\}_{\alpha<\tau}$.*
4. *There exists in Y an isomorphic copy of $\ell_2(\tau)$ complemented in $J(\eta)$.*
5. *There exists in Y an isomorphic copy of $\ell_2(\tau)$.*

Then (1)⇒(2)⇒(3)⇒(4)⇒(5) and, if $\eta \in RC$, they are all equivalent.

Proof. (1) ⇒ (2). Let $1 \leq \alpha < \eta$. We claim that $\exists f \in S(Y)$ such that $[1, \alpha] \cap supp(f) = \emptyset$. Indeed, in the contrary case, the family of functionals $d_\beta \in J(\eta)^*$, $1 \leq \beta \leq \alpha$, such that $d_\beta(g) = g(\beta)$, $\forall g \in J(\eta)$, would be total over Y, which contradicts (1). So, to find in $S(Y)$ a sequence $\{f_\alpha\}_{\alpha<\tau}$ with $supp(f_\alpha) < supp(f_\beta)$, whenever $\alpha < \beta < \tau$, pick $f_1 \in S(Y)$ arbitrarily. Since $cf(\eta) = \tau \geq \omega_1$ and $f_1(\eta) = 0$, there exists $\gamma_1 < \eta$ such that $supp(f_1) < \gamma_1 < \eta$. Now, by the above, choose $f_2 \in S(Y)$ and $\gamma_2 < \eta$ such that $\gamma_1 < supp(f_2) < \gamma_2$, and so on. Finally, by a standard transfinite argument, the choice can be completed.

(2) ⇒ (3). Let $\alpha < \tau$. If $\alpha \in LO$, take $g_\alpha = f_\alpha$. If $\alpha = \gamma + n$, $\gamma \in LO$, $1 \leq n < \omega$, take $g_\alpha = f_{\alpha+n} = f_{\gamma+2n}$.

(3) ⇒ (4) follows from Lemma 2.2; (4)⇒(5) is obvious.

Finally suppose that $\eta \in RC$ and prove that (5) ⇒ (1). From the embedding $i : \ell_2(\eta) \to Y$ we get the quotient map $i^* : Y^* \to \ell_2(\eta)$, which is $w^* - w^*$ continuous. So, if w^*- Dens$(Y^*) < \eta$, we would have w^*-Dens$(\ell_2(\eta)) < \eta$, a contradiction. □

Proposition 2.5. *Let $\omega_1 \leq \eta \in RC$, and $Y \subseteq J(\eta)$ a closed subspace with $Dens(Y) = \eta$. Then there exists in Y a normalized sequence $\{f_\alpha\}_{\alpha<\eta}$ with $supp(f_\alpha) < supp(f_\beta)$, whenever $\alpha < \beta < \eta$.*

Proof. First of all, we prove that, $\forall \rho < \eta$, $\exists \gamma \in LO$, $\rho < \gamma < \eta$, and a closed subspace $Y_0 \subseteq Y$ such that $dim(Y/Y_0) \leq 1$ and $\{0\} \neq P_\gamma(Y_0) \subseteq Y_0$. So, fix $\rho < \eta$ and assume, without loss of generality, that $P_\rho(Y) \neq \{0\}$. Choose a countable sequence of ordinals $\{\gamma_n\}_{n\geq 0}$ and a countable sequence $\{Y_n\}_{n\geq 1}$ of closed subspaces of Y such that:

1. $\rho = \gamma_0 < \gamma_1 < \gamma_2 < .. < \eta$.
2. $Dens(P_{\gamma_n}(Y)) \leq Dens(Y_{n+1}) < \eta$.
3. $P_{\gamma_n}(B(Y_{n+1}))$ is dense in $P_{\gamma_n}(B(Y))$.
4. $supp(Y_{n+1}) \subseteq [1, \gamma_{n+1}]$.

Let $\gamma = \sup_{n\geq 0} \gamma_n$ and $Y_0 = \{y \in Y : y(\gamma) = 0\}$. By Lemma 2.1, $dim(Y/Y_0) \leq 1$. Pick $y \in S(Y_0)$ and select x_n in $B(Y_{n+1})$ so that $\|P_{\gamma_n}(y) - P_{\gamma_n}(x_n)\| \to 0$ as $n \to \infty$. Since $\|P_\gamma(y) - P_{\gamma_n}(y)\| \to 0$ as $n \to \infty$, we get $\|P_\gamma(y) - P_{\gamma_n}(x_n)\| \to 0$ as $n \to \infty$.

Consider $u_n = x_n - P_{\gamma_n}(x_n)$, $n \geq 1$, which satisfies $u_n < u_{n+1}$. So $\{u_{2n}\}_{n\geq 1}$ is a bounded skipped sequence of $J(\eta)$. By Lemma 2.2, $u_{2n} \to 0$ weakly. Hence $P_\gamma(y) - x_{2n} \to 0$ weakly as $n \to \infty$. Therefore $P_\gamma(y) \in Y_0$.

Since $\gamma < \eta$, we have that $Dens(P_\gamma(Y_0)) \leq Card(\gamma) < \eta = Dens(Y_0)$. Hence, there exists $f_1 \in S(Y_0) \subset S(Y)$ such that $\gamma < supp(f_1)$. Choose $\gamma_1 < \eta$ such that $\gamma < supp(f_1) < \gamma_1$. By a new application of the above argument, find $f_2 \in S(Y)$ and $\gamma_2 < \eta$ such that $\gamma_1 < supp(f_2) < \gamma_2$, and so on. Thus, by a standard transfinite induction we obtain in Y a normalized sequence $\{f_\alpha\}_{\alpha<\eta}$ with $supp(f_\alpha) < supp(f_\beta)$, whenever $\alpha < \beta < \eta$. □

Corollary 2.6. *(1) If $\eta \in RC$, every closed subspace $Y \subseteq J(\eta)$ with $Dens(Y) = \eta$, contains a copy of $\ell_2(\eta)$, complemented in $J(\eta)$.*

(2) Every closed subspace $Y \subset J(\omega_1)$ contains a copy of $\ell_2(I)$ complemented in $J(\omega_1)$, with $Card(I) = Dens(Y)$.

(3) $J(\omega_1)$ is a non WCD Banach space which has the HDP.

Proof. (1) If $\eta = \omega$, the statement is the result of Cassaza, Lin and Lohman [CLL]. If $\eta \geq \omega_1$, we apply Prop. 2.5 and Lemma 2.2.

(2) This follows from (1) and Prop. 2.3.

(3) It is well known that $J(\omega_1)$ is not WCD ([B], [E1]). If $Y \subset J(\omega_1)$ is an infinite dimensional closed subspace, there exists in Y an isomorphic copy of $\ell_2(I)$ with card(I) =Dens(Y). So, applying Prop. 1.1 we obtain that $J(\omega_1)$ has the HDP. □

3. Subspaces of $C([1,\eta])$

In this §3 we apply the Prop.1.1 like in §2, but with $c_0(I)$ instead of $\ell_2(I)$. If K is a compact Hausdorff space and $f : K \to \mathbb{R}$ is continuous, denote by $supp(f) = \{k \in K : f(k) \neq 0\}$. If $\eta \in O$ is an infinite ordinal and $K = [1,\eta]$, with the order topology, then $C([1,\eta])$ is clearly isomorphic to $C_0([1,\eta]) := \{f \in C([1,\eta]) : f(\eta) = 0\}$. If $\alpha \leq \eta$, let $P_\alpha : C([1,\eta]) \to C([1,\eta])$ be the projection such that, $\forall f \in C([1,\eta])$, $P(f) = f \cdot \mathbf{1}_{[1,\alpha]}$. Of course, P_α is a projection with $\|P_\alpha\| = 1$ and $P_\alpha(C([1,\eta]))$ is isometric to $C([1,\alpha])$.

Lemma 3.1. *If $\eta \in O$, then every normalized skipped sequence $\{f_\alpha\}_{\alpha<\theta}$, $\theta \in O$, in $C_0([1,\eta])$ is a monotone basic sequence equivalent to the canonical basis of $c_0(\theta)$ and the closed subspace $Z := [\{f_\alpha\}_{\alpha<\theta}]$ generated is 2-complemented in $C_0([1,\eta])$.*

Proof. Let $\{f_\alpha\}_{\alpha<\theta}$ be a normalized skipped sequence in $C_0([1,\eta])$. Clearly, $\{f_\alpha\}_{\alpha<\theta}$ is a monotone basic sequence equivalent to the canonical basis of $c_0(\theta)$. Prove that the closed subspace $Z := [\{f_\alpha\}_{\alpha<\theta}]$ is complemented in $C_0([1,\eta])$. This proof is similar to the one of Lemma 2.2, namely:

(A) Let $S_\alpha = [p_\alpha, q_\alpha)$, where $p_\alpha = \min(supp(f_\alpha))$, $q_\alpha = \min\{\gamma \in O : \gamma > supp(f_\alpha)\}$, and $Z_\alpha = \{f \in C_0([1,\eta]) : supp(f_\alpha) \subseteq S_\alpha\}$. Then $\{S_\alpha\}_{\alpha<\theta}$ is a skipped sequence of nice segments of η such that $f_\alpha \in Z_\alpha$. Let $Q_\alpha : C_0([1,\eta]) \to Z_\alpha$ be defined as follows:

$$\forall f \in C_0([1,\eta]),\ \forall i \leq \eta, \qquad Q_\alpha(f)(i) = \begin{cases} 0 & ,\text{if } i \notin S_\alpha \\ f(i) - f(q_\alpha) & ,\text{if } i \in S_\alpha \end{cases}$$

It can be easily seen that Q_α is a projection on Z_α such that $\|Q_\alpha\| \leq 2$.

Pick $g_\alpha \in S(Z_\alpha^*)$ such that $g_\alpha(Q_\alpha(f_\alpha)) = \|Q_\alpha(f_\alpha)\| = \|f_\alpha\| = 1$.

(B) If $f \in C_0([1,\eta])$, define $Q(f) : [1,\eta] \to \mathbb{R}$ so that $Q(f) = \sum_{\alpha<\theta} g_\alpha(Q_\alpha(f)) \cdot f_\alpha$. Clearly $Q(f)(\eta) = 0$ and we claim that, $\forall f \in C_0([1,\eta])$, $Q(f) \in Z$. To prove this it is enough to show that, $\forall f \in C_0([1,\eta])$, $F(f) := \{\|Q_\alpha(f)\|\}_{\alpha<\theta} \in c_0(\theta)$. Use transfinite induction over $\alpha \leq \eta$ such that $supp(f) \subseteq [1,\alpha]$. It is clear that, if $supp(f) \subseteq [1,n], 1 \leq n < \omega$, then $F(f) \in c_0(\theta)$. Let $\gamma \leq \eta$ be such that, $\forall \beta < \gamma$ and $\forall g \in C_0([1,\eta])$, with $supp(g) \subseteq [1,\beta]$, we have that $F(g) \in c_0(\theta)$, and assume that $supp(f) \subseteq [1,\gamma]$. There are two possibilities:

(I) Suppose that $\gamma = \beta + 1$. Since $F(f \cdot \mathbf{1}_{[1,\beta]})$ and $F(f \cdot \mathbf{1}_{\{\beta+1\}})$ are in $c_0(\theta)$, clearly $F(f) \in c_0(\theta)$.

(II) Assume that $\gamma \in LO$.

(a) If $f = \mathbf{1}_{[1,\gamma]}$, then $Q_\alpha(f) \neq 0$ for, at most, a unique $\alpha < \eta$. Hence $F(f) \in c_0(\theta)$.

(b) If $f(\gamma) = 0$, then $\|f \cdot \mathbf{1}_{[1,\gamma_i]} - f\| \to 0$, where $\gamma_i \uparrow \gamma,\ i < \tau,\ \tau = cf(\gamma)$. Hence $\|Q_\alpha(f \cdot \mathbf{1}_{[1,\gamma_i]}) - Q_\alpha(f)\| \to 0$ (uniformly for $\alpha < \theta$) as $i \uparrow \tau$. Therefore, as $F(f \cdot \mathbf{1}_{[1,\gamma_i]}) \in c_0(\theta),\ \forall i < \theta$, we get that $F(f) \in c_0(\theta)$.

(c) Finally, if $f(\gamma) \neq 0$, taking into account $f = (f - f(\gamma) \cdot \mathbf{1}_{[1,\gamma]}) + f(\gamma) \cdot \mathbf{1}_{[1,\gamma]}$ and (1), (2), we obtain that $F(f) \in c_0(\theta)$.

So Q is a projection on Z such that $\|Q\| \leq 2$. □

Proposition 3.2. *Let $\eta \in O$ be an infinite ordinal and $Y \subset C([1,\eta])$ a closed infinite dimensional subspace. Then:*

1. *If Y is isomorphic to c_0, Y is complemented in $C([1,\eta])$.*
2. *Y contains a copy of c_0 complemented in $C([1,\eta])$.*
3. *If $\omega_1 \leq \eta \in RC$, $Y \subset C_0([1,\eta])$ and $Dens(Y) = \eta$, there exists a normalized sequence $\{f_\alpha\}_{\alpha<\eta} \subset Y$ with $supp(f_\alpha) < supp(f_\beta)$, whenever $\alpha < \beta < \eta$.*
4. *If $\eta \in RC$ and $Dens(Y) = \eta$, Y contains a copy of $c_0(\eta)$, complemented in $C([1,\eta])$.*

P r o o f. (1) Choose $A \subset [1,\eta]$ such that $\mathrm{card}(A) = \aleph_0$ and, $\forall f \in Y, \|f\| = \sup\{|f(\alpha)| : \alpha \in A\}$. We can assume, without loss of generality, that A is closed (if not, take $\overline{A}$ instead of A and note that $\overline{A}$ is also countable (by standard transfinite induction)). Observe that A is order and topologically homeomorphic to some ordinal interval $[1,\beta]$ with $\beta < \omega_1$. Consider the projection $P_1 : C([1,\eta]) \to C([1,\eta])$ such that $P_1(f) = f \cdot \mathbf{1}_A,\ \forall f \in C([1,\eta])$. Then $P_1(C([1,\eta])) = C(A) \cong C([1,\beta])$ (isometry) and $P_{1|Y}$ is an order isometric isomorphism between Y and $P_1(Y)$. As $C(A)$ is separable, Sobczyk's Theorem gives us a projection $P_2 : C(A) \to C(A)$ such that $P_2(C(A)) = P_1(Y)$. So $P = (P_{1|Y})^{-1} \circ P_2 \circ P_1$ is a projection of $C([1,\eta])$ onto Y.

(2) We apply transfinite induction. As $C([1,\eta])$ is isomorphic to c_0, for $\eta < \omega^\omega$, the statement is true if $\eta < \omega^\omega$ (see [LT], Prop. 2.a.2). Assume that the result holds for every $\alpha < \eta$ and proceed as in Prop. 2.3. If $\eta = \alpha + 1$, $Y_0 = \{f \in Y : f(\eta) = 0\}$ is a closed infinite dimensional subspace of $C_0([1,\eta]) \cong C([1,\alpha])$ and the statement follows by induction hypothesis.

Assume that $\eta \in LO$. As $C([1,\eta]) \simeq C_0([1,\eta])$ (isomorphism), we suppose that we work in $C_0([1,\eta])$. If there exists $\alpha < \eta$ such that P_α is an isomorphism between Y and $P_\alpha(Y)$, the result follows by induction hypothesis (because $P_\alpha(C_0([1,\eta])) \cong C([1,\alpha])$.

In the contrary case, like in Prop. 2.3, we get two normalized sequences $\{y_n\}_{n\geq 1} \subset S(Y)$ and $\{x_n\}_{n\geq 1}$ such that $\{x_{2n}\}_{n\geq 1}$ is a skipped normalized sequence of $C_0([1,\eta])$ and $\sum_{n\geq 1} \|y_{2n} - x_{2n}\| \leq \epsilon$, where $\epsilon > 0$ is arbitrary. So, applying Lemma 3.1 and [LT], Prop. 1.a.9, we complete the proof.

(3) To prove this statement use the argument of Prop. 2.5, taking into account that every normalized sequence $\{u_\alpha\}_{\alpha<\theta}$, $\theta \in O$, in $C_0([1,\eta])$, with $u_\alpha < u_\beta$, $\alpha < \beta < \theta$, is equivalent to the canonical basis of $c_0(\theta)$. So, it is shrinking and $u_{\alpha_n} \to 0$ weakly for every subsequence $\{u_{\alpha_n}\}_{n<\omega}$.

(4) Work in $C_0([1,\eta])$, which is isomorphic to $C([1,\eta])$. Consider a closed subspace $Y \subseteq C_0([1,\eta])$ with $Dens(Y) = \eta$. If $\eta = \omega$, apply (2). If $\omega_1 \leq \eta$, by (3) there exists in Y a normalized sequence $\{f_\alpha\}_{\alpha<\eta}$, $f_\alpha < f_\beta$ whenever $\alpha < \beta < \eta$, equivalent to the canonical basis of $c_0(\eta)$. Without loss of generality, we can suppose that $\{f_\alpha\}_{\alpha<\eta}$ is skipped. So, by Lemma 3.1, we have that the closed subspace $[\{f_\alpha\}_{\alpha<\eta}]$ is complemented in $C_0([1,\eta])$. □

Corollary 3.3. *(1) Every closed subspace $Y \subset C([1,\omega_1])$ contains a copy of $c_0(I)$ complemented in $C([1,\omega_1])$, with Card(I) =Dens(Y).*

(2) $[1,\omega_1]$ is a non Gulko (even non Corson) compact K such that $C(K)$ has the HDP.

Proof. (1) This follows from (2) and (4) of Prop. 3.2.

(2) It is well known that $[1,\omega_1]$ is non Gulko (see [DGZ], p. 259). Also, $[1,\omega_1]$ is non Corson because a Corson Radon-Nikodym compact is Eberlein compact ([OSV]). Finally, applying Prop. 1.1 we obtain that $C([1,\omega_1])$ has the HDP. □

The "long segment". ([E], p. 297) Consider in $V_0 = [1,\omega_1) \times [0,1)$ the linear order $<$ defined by letting $(\alpha,t) < (\beta,s)$ whenever $\alpha < \beta$ or $\alpha = \beta$ but $t < s$. Adjoining the point ω_1 to V_0 and assuming that $x < \omega_1$ for every $x \in V_0$, we obtain a linearly ordered set V. The set V with the topology induced by the linear order $<$ is called the *"long segment"*. Clearly, Dens$(C(V)) = \aleph_1$.

To prove that $C(V)$ has the HDP, it is enough to prove that $C_0(V) = \{f \in C(V) : f(\omega_1) = 0\}$ has. Consider a closed subspace $Y \subset C_0(V)$ such that Dens$(Y) = \aleph_1$. We claim that for every $\alpha < \omega_1$ there exist $\alpha < \beta < \omega_1$ and $f \in Y \setminus \{0\}$ such that supp$(f) \subset ((\beta,0),\omega_1)$. Indeed:

A) Consider the restriction $R_1(Y)$ of the functions of Y to $[(1,0),(\alpha,0)]$ which satisfies $R_1(Y) \subset C([(1,0),(\alpha,0)]) \simeq C([0,1])$ (isomorphism). So, there exists a separable subspace $Y_1 \subset Y$ such that $R_1(B(Y_1))$ is dense in $R_1(B(Y))$.

B) Let $\alpha_1 < \alpha_2 < \omega_1$ be such that supp$(Y_1) \subset [(1,0),(\alpha_2,0)]$. Let R_2 be the restriction on $[(1,0),(\alpha_2,0)]$ and choose a separable subspace $Y_1 \subset Y_2 \subset Y$ such that $R_2(B(Y_2))$ is dense in $R_2(B(Y))$.

By reiteration, we get a sequence (α_n) of ordinals. Let $\beta = \lim_{n\to\infty} \alpha_n$, $Y_0 = \{f \in$

$Y : f((\beta,0)) = 0\}$ and choose $f \in B(Y_0)$ such that $f_{|[(\beta,0),\omega_1]} \neq 0$. Let us see that $g := f \cdot \mathbf{1}_{[(1,0),(\beta,0)]} \in Y$. Indeed, pick $y_n \in B(Y_n)$ such that $\|R_n(y_n) - R_n(g)\|_\infty \leq 1/n$. We want to prove that $y_n \to g$ weakly, i.e., that $\mu(y_n) \to \mu(g)$ for every Radon measure μ on V. Without loss of generality, we can suppose that $|\mu|([(\beta,0),\omega_1]) = 0$. Let $\epsilon > 0$ and pick $n_0 \geq 1$ such that $|\mu|((\alpha_{n_0},0),(\beta,0)) < \epsilon$. Then for $n \geq n_0$ we have:

$$|\mu(y_n - g)| \leq |\mu| \left(|y_n - g|_{|[(1,0),(\alpha_{n_0},0)]} \right) + \|y_n - g\| \cdot |\mu|((\alpha_{n_0},0),(\beta,0)) \leq \frac{\|\mu\|}{n} + 2\epsilon,$$

which implies that $y_n \to g$ weakly. As $y_n \in Y$, we conclude that $g \in Y$, $f \cdot \mathbf{1}_{((\beta,0),\omega_1]} \in Y$ and this proves the claim. Applying the claim we obtain a sequence $\{f_\alpha\}_{\alpha<\omega_1} \subset Y$ of normalized functions with skipped supports. Clearly, $[\{f_\alpha\}_{\alpha<\omega_1}] \cong c_0(\omega_1)$ and, by Prop. 1.1, we get that $C(V)$ has the HDP.

Acknowledgements

Supported in part by DGICYT grants PB 97-0240 and PB 96-0607

References

[B] R. D. Bourgin, Geometric Aspects of Convex Sets with the Radon-Nikodým Property, Springer-Verlag, Lect. Notes in Math., Vol. **993** (1983).

[CLL] P. G. Casazza, B. L. Lin and R. H. Lohman, On James'quasi-reflexive Banach space, Proc. Amer. Math. Soc., **67** (1977), p. 265-271.

[DGZ] R. Deville, G. Godefroy and V. Zizler, Smoothness and Renormings in Banach Spaces, Pitman Monographs and Surveys in Pure and Applied Mathematics, **64** (1993).

[E] R. Engelking, General Topology, PWN-Polish Scientific Publishers (1977).

[E1] G. A. Edgar, A long James space, in: Measure Theory, Oberwolfach 1979, Lectures Notes in Math. **794**, Springer-Verlag (1980) p. 31-37.

[E2] G. A. Edgar, Measurability in Banach spaces, Indiana Univer. Math. J.,Vol.**26**, n. 4 (1977), p. 663-677.

[EW] G. A. Edgar and R. F. Wheeler, Topological properties of Banach spaces, Pac. J. Math., **115** (1984), p. 317-350.

[HO] J. Hagler and E. Odell, A Banach space not containing ℓ_1 whose dual ball is not weak* sequentially compact, Illinois J. Math., **22** (1978), p. 290-294.

[HW] R. Herman and R. Whitley, An example concerning reflexivity, Studia Math.,**28** (1967), p. 289-294.

[LT] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I, Springer-Verlag (1977).

[Ne] S. Negrepontis, Banach spaces and topology, in:K. Kunen and J. E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland (1984). p. 1045-1142.

[OSV] J. Orihuela, W. Schachermayer and M. Valdivia, Every Radon-Nikodym Corson compact is Eberlein compact, Studia Math., **98** (1991), p. 157-174.

[PS] A. Pełczynski and Z. Semadeni, Spaces of continuous functions (III), Studia Math., **18** (1959), p211-222.

[Va] L. Vašák, On one generalization of weakly compactly generated Banach spaces, Studia Math., **70** (1981), p. 11-19.

[Z] J. F. Zhao, The transfinite basis of the bidual space of the long James space, Acta Math. Sci. (English Ed.), **5** (1985), p.295-301.

Departamento de Análisis Matemático,
Facultad de Matemáticas
Universidad Complutense de Madrid
28040-MADRID, Spain

Departamento de Matemáticas
Universidad Autónoma de Madrid, Cantoblanco
28049-MADRID, Spain