A NOTE ON POROSITY AND THE MAZUR INTERSECTION PROPERTY

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ABSTRACT. Let $\mathcal{M}$ be the collection of all intersections of balls, considered as a subset of the hyperspace $\mathcal{H}$ of all closed, convex and bounded sets of a Banach space, furnished with the Hausdorff metric. We prove that $\mathcal{M}$ is uniformly very porous if and only if the space fails the Mazur intersection property.

1. INTRODUCTION

Motivated by problems in Real Analysis and, especially, in differentiation theory, several authors considered what came to be known as *porosity*, a notion which concerns the size of holes of a set near a point. Topologically speaking, porous sets are smaller than merely being a countable union of nowhere dense closed sets [15]. Consequently, porosity has been usually used to describe smallness in a topological sense. Precisely, let M be a metric space, P a subset of M, $B(x,R)$ the closed ball centered at x with radius R and $\gamma(x,R,P)$ the supremum of all r for which there exists $y \in M$ such that $B(y,r) \subset B(x,R) \setminus P$. The number

$$\rho(x,P) = 2 \limsup_{R\to 0} \frac{\gamma(x,R,P)}{R}$$

is called the porosity of P at x. We say that P is porous at x whenever $\rho(x,P) > 0$ and, when P is porous at every point of M, we simply say that P is a porous set. If there is $\varepsilon > 0$ satisfying $\rho(x,P) > \varepsilon$ for every $x \in M$, then P is said to be uniformly porous. Finally, replacing "$\limsup$" by "$\liminf$" in the above definition, we encounter the notions of *very porosity* and *very porous set*, respectively. The unit sphere of a normed linear space is an easy example of an uniformly very porous set.

In convex geometry, the use of porosity received in recent years a great deal of attention. Several topics as smoothness, strict convexity, diameters, nearest points and others have

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been investigated by using porosity. We refer to the works of Zamfirescu [16], [17] and Gruber [8], [9] for more information about this rich line of research.

In Banach space theory, porosity has been used to describe topological properties of the set of points of Frechet nondifferentiability [13], [14] and also in relation with questions of best approximation [3] and variational principles [5]. For these and other applications of porosity, we refer to Zajicek's survey [15] and Phelps' book [13].

The purpose of this paper is obtaining a new characterization for the Mazur Intersection property (every closed, convex and bounded subset is an intersection of closed balls) [12] in terms of porosity. More precisely, let $\mathcal{M}$ be the collection of all intersections of balls, considered as a subset of the hyperspace $\mathcal{H}$ of all closed, convex and bounded sets of a Banach space, furnished with the Hausdorff metric. We prove that $\mathcal{M}$ is uniformly very porous if and only if the space fails the Mazur intersection property, thus improving a result obtained in [6]. Actually, in this cases, $\mathcal{M}$ turns out to be porous in a much strong sense, somehow close to the notion of cone-meager introduced by Preiss and Zajicek [13], [14] .

Throughout, X is a Banach space with norm $\|\cdot\|$, $B_{\|\cdot\|}$ is the unit ball and $S_{\|\cdot\|}$ its unit sphere. We denote by X^* its dual space with dual norm $\|\cdot\|^*$. The Hausdorff distance between $C_1, C_2 \in \mathcal{H}$ is

$$d(C_1, C_2) = \inf \{\varepsilon > 0 : C_1 \subset C_2 + \varepsilon B_{\|\cdot\|},\ C_2 \subset C_1 + \varepsilon B_{\|\cdot\|}\} \ .$$

Given $C \in \mathcal{H}$, the ball centered at C with radius R will be denoted by $B_d(C, R)$ and the distance from C to a subset $\mathcal{A} \subset \mathcal{H}$ by $d(C, \mathcal{A}) = \inf\{d(C, A) : A \in \mathcal{A}\}$. It is well known that $(\mathcal{H}, d)$ is a complete metric space [11] and, hence, a Baire space. If $C \in \mathcal{H}$, we denote the boundary and the interior of C by ∂C and $\operatorname{int} C$ respectively.

2. Porosity and Mazur intersection property.

Consider the set $\mathcal{M} = \{C \in \mathcal{H} : C = \cap_{i\in I} B_i,\ B_i \text{ closed ball},\ i \in I\}$. The space X has the Mazur intersection property if and only if $\mathcal{H} = \mathcal{M}$. Among the Banach spaces failing the above property are separable spaces with non-separable dual [7]. In $\mathbb{R}^2$, a norm is Fréchet differentiable if and only if it has the Mazur intersection property. The aim of this note is to show that $\mathcal{M}$ is uniformly very porous whenever $\mathcal{M} \neq \mathcal{H}$. To this end, we need a handy description of the elements of $\mathcal{H}\backslash\mathcal{M}$, obtained as a consequence of Proposition 2.1. In what follows, given $f \in X^*$, we denote $K_f = \ker f \cap B_{\|\cdot\|}$, $L_f = \{x \in B_{\|\cdot\|} : f(x) \geq 0\}$ and $M_f = \{x \in B_{\|\cdot\|} : f(x) \leq 0\}$.

Proposition 2.1. *Given a Banach space, the following conditions are equivalent:*

(i) *The space has the Mazur intersection property.*
(ii) *There is a dense set* $F \subset S_{\|\cdot\|^*}$ *satisfying* $M_f \in \mathcal{M}$ *(*$L_f \in \mathcal{M}$*) for each* $f \in F$.
(iii) *There is a dense set* $F \subset S_{\|\cdot\|^*}$ *satisfying* $K_f \in \mathcal{M}$ *for each* $f \in F$.

Proof. To see (ii) $\Rightarrow$ (iii) just observe that $M_f \in \mathcal{M}$ if and only if $-M_f \in \mathcal{M}$ so the difficulty of the proof lies only in (iii) $\Rightarrow$ (i). Before proving this implication, we shall prove the next assertion: *consider* $f \in X^*$, $y \in X$, $\lambda > 0$ *and a ball* B *containing* $y+\lambda K_f$*; then,* B *contains either* $y+\lambda L_f$ *or* $y+\lambda M_f$.

We may assume that B is the unit ball and, consequently, that $\lambda \leq 1$. Indeed, if $B \supset y+\lambda K_f$, then $B \supset -y-\lambda K_f = -y+\lambda K_f$ and so $B \supset \lambda K_f$. Therefore $K_f \supset \lambda K_f$ and $\lambda \leq 1$. The proof of the above assertion will be accomplished by studying two cases:

Case 1. $f(y) = 0$. Observe that, in this case, $\|y+\lambda(y/\|y\|)\| \leq 1$ and thus $\lambda \leq 1-\|y\|$. As a consequence, $y+\lambda B \subset B$ and, hence, B contains both $y+\lambda L_f$ and $y+\lambda M_f$.

Case 2. $f(y) \neq 0$. We claim that $y+\lambda M_f \subset B$ provided $f(y) > 0$. To prove this fact, we will use a two-dimensional argument showing that

$$y+\lambda M_f \ \subset \overline{\mathrm{conv}}\,(\,(y+\lambda K_f) \cup M_f) \ \subset \ B \tag{2.1}$$

and, analogously, if $f(y) < 0$, a similar argument can be used to prove that $y+\lambda L_f \ \subset \overline{\mathrm{conv}}\,(\,(y+\lambda K_f) \cup L_f) \ \subset \ B$.

Consider an arbitrary point $z \in M_f$ and the two-dimensional subspace Y spanned by y and z. Let us call φ to the afin function which maps $K_f \cap Y$ onto $y+\lambda(K_f \cap Y)$. We will treat first the special case where $\lambda = 1$.

The function φ is a translation by the vector y. To prove 2.1, it is sufficient to show that the ray $\{z+ty : t \geq 0\}$ has nonempty intersection with $(y+K_f)$ or, equivalently, that $z \in \{K_f+ty : t \in \mathbb{R}\}$. We argue geometrically: if $z \notin \{K_f+ty : t \in \mathbb{R}\}$ then $\overline{\mathrm{conv}}\,(\{z\} \cup (y+K_f)) \cap \mathrm{Ker}\, f \not\subset K_f$ which is a contradiction.

In the general case, where $\lambda < 1$, φ is just a dilation relative to the point $y_0 = \frac{1}{1-\lambda}\,y$. If R is the line passing through z and y_0, then clearly $\varphi(z) \in R$. As in the previous case, we must verify that R has nonempty intersection with $(y+\lambda K_f)$. If it is not so then z does not belong to the cone $\{y_0+t(z-y_0) : z \in K_f, t \geq 0\}$ and, again, it implies that $\overline{\mathrm{conv}}\,(\{z\} \cup (y+\lambda K_f)) \cap \mathrm{Ker}\, f \not\subset K_f$. Thus $R \cap (y+\lambda K_f) \neq \emptyset$, say $w = R \cap (y+\lambda K_f)$. We only need to prove that $\varphi(z) = y+\lambda z$ lies into the segment $[w,z] \subset R$. To this end, notice that $\lambda \leq 1$ and, hence,

$$f(z) \leq f(y+\lambda z) \leq f(y) = f(w)\ .$$

Once the claim has been proved, let C be a closed, bounded, convex set and let $z \in X \setminus C$. By the Hahn-Banach separation theorem, there is $f \in F$ and $\sigma \in \mathbb{R}$ such that $f(z) > \sigma$ and $\sup f(C) < \sigma$. Take $y \in X$, $f(y) = \sigma$ and $n \in \mathbb{N}$ large enough so that $z \in y + nL_f$ and $C \subset y + nM_f$. Let $B \subset X \setminus \{z\}$ be a ball containing $y + nK_f$. By the above assertion, B contains $y + nM_f$ and hence it contains also C. $\square$

Norm one functionals $f \in X^*$ satisfying that for every $\varepsilon > 0$ there exists a weak* slice $S = \{x^* \in B_{\|\cdot\|^*} : x^*(x) \geq 1 - \delta\}$ (where $x \in S_{\|\cdot\|}$ and $\delta > 0$) such that $\operatorname{diam}(f \cup S) < \varepsilon$ were introduced in [2] under the name of *semi-denting* points. In Proposition 5 of [2], semi-denting points are characterized as those points satisfying $M_f \in \mathcal{M}$. This result combined with a similar argument to the one used at the end of Proposition 2.1 show that f is semi-denting if and only if $K_f \in \mathcal{M}$.

Clearly, the set of semi-denting points is closed. Indeed, if $f \in S_{\|\cdot\|^*}$ is not semi-denting, there is $\varepsilon > 0$ such that the set $B(f, \varepsilon) = \{x^* \in S_{\|\cdot\|^*} : \|x^* - f\|^* < \varepsilon\}$ contains no weak* slices and thus no point g of $B(f, \varepsilon)$ is semi-denting, either. As a consequence, condition (iii) of Proposition 2.1 easily implies that $K_f \in \mathcal{M}$ for every $f \in X^*$. We come now to the main result of this paper.

Theorem 2.2. *The set $\mathcal{M}$ is uniformly very porous if and only if the space fails the Mazur Intersection Property.*

Proof. We find it convenient to isolate from the argument the following observation: *consider $C \in \mathcal{H}$ and $\lambda > 0$ so that $D = \{x \in C : d(x, \partial C) \geq \lambda\} \neq \emptyset$; every set $E \in \mathcal{H}$ with $d(C, E) < \lambda$ contains also D.* The proof is fairly easy: if $x \in D \setminus E$, there is a norm one functional f separating x and E. Say, for instance, that $f(x) > \sup f(E)$. Clearly, $\sup f(C) \geq f(x) + \lambda > \sup f(E) + \lambda$, so $d(C, E) > \lambda$, a contradiction.

By Proposition 2.1, if X fails the Mazur Intersection Property there is a norm one functional f such that $M_f \notin \mathcal{M}$. It means that there is also $x_0 \in B_{\|\cdot\|} \setminus M_f$ such that every ball containing M_f contains also x_0. Denote by $\alpha = f(x_0) > 0$ and consider an arbitrary subset $C \in \mathcal{B}$. We will prove that

$$\rho(C, \mathcal{M}) = 2 \liminf_{R \to 0} \frac{\gamma(C, R, \mathcal{M})}{R} \geq \frac{\alpha}{1 + \alpha} \quad .$$

and the proof will be accomplished by looking at two cases.

Case 1. The functional f attains its maximum over C, say at $y_0 \in C$. Define the sets $C_R = \overline{C + RB_{\|\cdot\|}}$ and $D_R = \{x \in C_R : f(x) \leq \sup f(C)\}$. Notice that $D_R \notin \mathcal{M}$ since D_R contains $y_0 + RM_f$ and misses the point $y_0 + Rx_0$. However, we do not know the existence of $r > 0$ such that $B_d(D_R, r) \subset \mathcal{H} \setminus \mathcal{M}$, which is necessary to compute the porosity of C. It is then convenient to select a suitable modification of D_R, namely the set $E_R = \overline{D_R + \frac{\alpha R}{2} B_{\|\cdot\|}}$. We claim that the ball $B_d(E_R, \alpha R/2 - 1/n)$ satisfies

$$B_d(E_R, \alpha R/2 - 1/n) \cap \mathcal{M} = \emptyset$$

for $n \in \mathbb{N}$ large enough so that $\alpha R/2 - 1/n > 0$. Indeed, if $G \in \mathcal{H}$ and $d(G, E_R) \leq \alpha R/2 - 1/n$ then $y_0 + Rx_0 \notin G$ but, due to the first remark, $y_0 + RM_f \subset G$ so every ball containing G should contain also $y_0 + Rx_0$.

Now, since $d(E_R, C) \leq R + R\alpha/2$, then $B_d(E_R, \alpha R/2 - 1/n) \subset B(C, R + R\alpha)$. It means that $\gamma(C, R + R\alpha, \mathcal{M}) \geq \alpha R/2 - 1/n$, for n large enough, so $\gamma(C, R + R\alpha, \mathcal{M}) \geq \alpha R/2$, thus implying that

$$2 \liminf_{R\to 0} \frac{\gamma(C, R + R\alpha, \mathcal{M})}{R + R\alpha} \geq \liminf_{R\to 0} \frac{\alpha R}{R + R\alpha} = \frac{\alpha}{1+\alpha} .$$

Case 2. The functional f does not attain its maximum over C. Given $R > 0$, we take y_m so that $f(y_m) = \sup f(C)$ and $d(y_m, C) < R/m$. Consider now $C_m = \overline{\mathrm{conv}}\,(\{y_m \cup C\})$. Since C_m satisfies the condition of Case 1, $\gamma(C_m, R + R\alpha, \mathcal{M}) \geq \alpha R/2$ and, consequently, $\gamma(C, R + R\alpha + R/m, \mathcal{M}) \geq \alpha R/2$. Therefore

$$2 \liminf_{R\to 0} \frac{\gamma(C, R + R\alpha + R/m, \mathcal{M})}{R + R\alpha + R/m} \geq \liminf_{R\to 0} \frac{\alpha R}{R + R\alpha + R/m} = \frac{\alpha}{1 + \alpha + 1/m}$$

for every $m \in \mathbb{N}$ and the theorem is proved. □

Notice that, if $C \notin \mathcal{M}$, then $x + \lambda C \notin \mathcal{M}$ for every $x \in X$ and $\lambda \in \mathbb{R}$. It means that $\mathcal{M}$ is porous in a much stronger sense than stated in Theorem 2.2, and close to the notions of cone meager and angle-smallness introduced by Preiss and Zajicek (see [14] and [13]).

The Mazur intersection property was introduced by Mazur [12] and later studied by many other authors. Information concerning this property can be found in [1], [4], [10] and references therein. There are still a number of open problems concerning this subject, as the existence of points of Fréchet differentiability in spaces with this property. While spaces with Fréchet differentiable norm satisfy the Mazur intersection property, it is unknown if it is also the case of spaces with a (Fréchet) differentiable bump function.

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