ON ω-INDEPENDENCE AND THE KUNEN-SHELAH PROPERTY

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Abstract. We prove that spaces with an uncountable ω-independent family fail the Kunen-Shelah property. Actually, if $\{x_i\}_{i\in I}$ is an uncountable ω-independent family, there exists an uncountable subset $J \subset I$ such that $x_j \notin \overline{\mathrm{conv}}\,(\{x_i\}_{i\in J\setminus\{j\}})$ for every $j \in J$. This improves a previous result due to Sersouri, namely that every uncountable ω-independent family contains a convex right-separated subfamily.

1. Introduction.

In recent years, two remarkable Banach spaces constructed by Kunen and Shelah respectively served as a source of counterexamples for a number of different problems. Shelah [9] was the first to construct a nonseparable space satisfying what we called in [3] the *Kunen-Shelah* property: for every uncountable family $\{x_i\}_{i\in I}$, there is $i_0 \in I$ such that

$$x_{i_0} \in \overline{\mathrm{conv}}\,(\{x_i\}_{i\in I\setminus\{i_0\}})\,.$$

Spaces with the above property share in a rather striking way some of the features of separable spaces (see [3], [4] and [8]). The Shelah space was constructed assuming the diamond principle for ω_1 and solved a problem by Davis and Johnson [10]. Later, assuming only the continuum hypothesis, Kunen [6] constructed a Banach space enjoying a stronger property: for every family $\{x_\alpha : \alpha < \omega_1\}$ there is $\alpha < \omega_1$ satisfying $x_\alpha \in \overline{\{x_\beta :\ \alpha < \beta\}}^{\text{weak}}$ (recall that a family $\{x_\alpha : \alpha < \omega_1\}$ in a topological space is *rigth-separated* if $x_\alpha \notin \overline{\{x_\beta :\ \alpha < \beta\}}$ for all $\alpha < \omega_1$). The Kunen space is an Asplund $C(K)$ space with no Fréchet differentiable norms [4] and has many other interesting properties (see, for instance, [1] and [3]).

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Clearly, spaces with an uncountable biorthogonal system cannot satisfy the Kunen-Shelah property. We encounter the notion of ω-independence when looking for a weaker condition that ensures the failure the above property. A family $\{x_i\}_{i\in I}$ is said to be ω-independent if for every sequence $(i_n)_{n\geq 1}\subset I$ of distinct indices, and every sequence $(\lambda_n)_{n\geq 1}\subset\mathbb{R}$, the series $\sum_{n=1}^{\infty}\lambda_n x_{i_n}$ converges (in norm) to zero if and only if $\lambda_n=0$ for every $n\geq 1$ (see [2] and [5]). There are ω-independent families which are not biorthogonal systems. Here is one example: $X=C([0,1]^{\omega_1})$ and $\{f_\alpha^n\}_{\alpha<\omega_1,n\in\mathbb{N}}$ defined as

$$f_\alpha^n\left((t_\gamma)_{\gamma<\omega_1}\right) = t_\alpha^n$$

for every $x=(t_\gamma)_{\gamma<\omega_1}\in[0,1]^{\omega_1}$. The purpose of this note is proving that spaces with the Kunen-Shelah property contain no uncountable ω-independent families.

Unless otherwise stated, by a *family* we understand always an uncountable set and then we often omit this adjective. Let us say that the family $F=\{x_\alpha:\alpha<\omega_1\}$ has the Kunen-Shelah property if among any ω_1 elements of F there is one that belongs to the closed convex hull of the rest (i.e., the space has the Kunen-Shelah property if and only if every uncountable family so has). The family F is said to be *convex right separated* if $x_\alpha\notin\overline{\mathrm{conv}}\left(\{x_\beta\}_{\beta>\alpha}\right)$ for every $\alpha<\omega_1$. Finally, the family $\{x_i\}_{i\in I}$ is *polyhedral* provided

$$x_j\notin\overline{\mathrm{conv}}\left(\{x_i\}_{i\in I\setminus\{j\}}\right)$$

for every $j\in I$. Sersouri [8] proved that an ω-independent family contains always a convex right-separated family. We will improve his result by showing that an ω-independent family contains always a polyhedral subfamily. As a consequence, ω-independent families have never the Kunen-Shelah property.

2. Uncountable ω-independent families fail the Kunen-Shelah property.

The Sersouri proof, as our, is based in the following result due to Kalton [5]. Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_n a_n=0$ and $\sum_{n\geq 1}a_n=\infty$. Let $b_n=\sup_{m\geq n}a_m$. If $\{x_\alpha:\alpha<\omega_1\}$ is an uncountable family of norm one elements then for every $x\in\overline{\mathrm{span}}\{x_\alpha:\alpha<\omega_1\}$, every $\delta>0$ and every $n\in\mathbb{N}$ there exist $m>n$, a sequence of signs $\{\varepsilon_i\}_{i=n+1}^m$ and a sequence of (not necessarily distinct) ordinals $\{\alpha_i\}_{i=n+1}^m$

such that

$$\|x + \sum_{i=n+1}^{m} a_i \varepsilon_i x_{\alpha_i}\| \; < \; \delta$$

and

$$\sup_{n<k<m} \|x + \sum_{i=n+1}^{k} a_i \varepsilon_i x_{\alpha_i}\| \; < \; \|x\| + \delta + b_n\,.$$

Using these two conditions and that (taking a subfamily, if necessary) every x_α is an accumulation point of $\{x_\alpha : \alpha < \omega_1\}$ when the space is separable, Kalton constructs inductively a series $\sum a_j \varepsilon_j x_{\alpha_j}$ convergent to zero with $\{\alpha_j\}$ different ordinals. In the Kunen space, Sersouri first showed that it can be assumed $\overline{\mathrm{conv}}\,\{x_\alpha : \alpha < \omega_1\} = \overline{\mathrm{conv}}\,\{x_\alpha : \beta < \alpha < \omega_1\}$ for every $\beta < \omega_1$ and, consequently, that every x_γ is an accumulation point of $\mathrm{conv}\,\{x_\alpha : \beta < \alpha < \omega_1\}$ to apply then the Kalton technique. In our case, we use a preparatory Lemma that, together with the Kalton construction, are the key tools of the proof.

Lemma 1. *If a convex right separated family $\{y_\alpha : \alpha < \omega_1\}$ satisfies the Kunen-Shelah property, there exists a subfamily $\{x_\alpha : \alpha < \omega_1\} \subset \{y_\alpha : \alpha < \omega_1\}$, an ordinal $\beta_0 < \omega_1$ and a positive number ε_0 so that every x_i, with $\beta_0 \leq i < \omega_1$, is an accumulation point of $[0, 1 + 1/\varepsilon_0]D_\gamma - [0, 1/\varepsilon_0]D_\gamma$, for every $\gamma < \omega_1$ and $D_\gamma = \mathrm{conv}\,(\{x_\alpha\}_{\gamma<\alpha})$.*

Proof. Assume that $\|y_\alpha\| \leq 1$ for every $\alpha < \omega_1$. Since the family $\{y_\alpha : \alpha < \omega_1\}$ is convex right separated, we can choose a subfamily $\{x_\alpha : \alpha < \omega_1\} \subset \{y_\alpha : \alpha < \omega_1\}$ and a positive number $\varepsilon_0 > 0$ satisfying

$$\mathrm{dist}\,(x_\alpha, \overline{\mathrm{conv}}\,(\{x_\beta\}_{\beta>\alpha})) \; \geq \; 3\varepsilon_0$$

for every $\alpha < \omega_1$. Let $n_0 \in \mathbb{N}$ be so that $\frac{1}{n_0} < \varepsilon_0$. Define A to be the subset of all ordinals $\alpha < \omega_1$ for which there are $\alpha < \rho < \gamma < \omega_1$ such that

$$x_\rho \; \notin \; \overline{\mathrm{conv}}\,(\{x_\beta\}_{\beta\leq\alpha} \cup \{x_\beta\}_{\gamma\leq\beta<\omega_1})\,.$$

Since $\{x_\alpha : \alpha < \omega_1\}$ satisfies the Kunen-Shelah property, A must be countable. Let $\tau = \sup\{\alpha : \alpha \in A\}$ and $B = \overline{\mathrm{conv}}\,(\{x_\beta\}_{\beta\leq\tau})$. It is clear that

$$x_\rho \; \in \; \overline{\mathrm{conv}}\,(B \cup D_\gamma)$$

whenever $\tau < \rho < \gamma < \omega_1$. Define now $B(\rho,\gamma,n)$ to be the subset of B of all u with the property that there is $v \in D_\gamma$ and $\lambda \in (0,1]$ with

$$\|\lambda u + (1-\lambda)v - x_\rho\| < \frac{1}{2n} \tag{2.1}$$

Note first that $B(\rho,\gamma,n) \neq \emptyset$ if $n > n_0$. By definition, $B(\rho,\gamma,n+1) \subset B(\rho,\gamma,n)$ and $B(\rho,\gamma',n) \subset B(\rho,\gamma,n)$ for every $\gamma' > \gamma$. If u, v, λ satisfy the conditions of $B(\rho,\gamma,n)$ with $n > n_0$, then 2.1 implies that

$$2\lambda \geq \|\lambda(u-v)\| > 3\varepsilon_0 - \frac{1}{2n} > 2\varepsilon_0$$

so, in the definition of $B(\rho,\gamma,n)$ we can write $\lambda \in (\varepsilon_0, 1]$ instead of $\lambda \in (0,1]$. Given $\tau \leq \beta < \omega_1$ and $n \geq n_0$, define $B(\beta,n)$ as the closure of the union of all $B(\rho,\gamma,n)$ with $\beta \leq \rho < \gamma < \omega_1$. Again, $B(\beta,n+1) \subset B(\beta,n)$ and $B(\beta',n) \subset B(\beta,n)$ if $\tau \leq \beta < \beta'$, $B(\beta,n) \neq \emptyset$ if $n \geq n_0$.

Since B is hereditarily Lindelöff (it is separable metric complete), for each $n \geq n_0$ there exists $\tau \leq \beta_n < \omega_1$ such that for every $\beta_n \leq \beta < \omega_1$ we have $B(\beta,n) = B(\beta_n,n)$. Let $\beta_0 = \sup_n \beta_n$ and fix $\beta_0 \leq \rho < \gamma < \omega_1$, $n \geq n_0$. Pick $u \in B(\rho,\gamma,n)$, $\mu \in (\varepsilon_0,1]$ and $\omega \in D_\gamma$ such that $\|(\mu u + (1-\mu)w) - x_\rho\| < 1/(2n)$. Since $u \in B(\gamma,n)$ also, there exist $\gamma \leq \sigma < \theta < \omega_1$, $\lambda \in (\varepsilon_0,1]$ and $v \in D_\theta$ such that $\|(\lambda u + (1-\lambda)v) - x_\sigma\| < 1/(2n)$. Letting $y = x_\sigma - (\lambda u + (1-\lambda)v)$ we have $u = (x_\sigma - y - (1-\lambda)v)/\lambda$ and

$$\left\| \mu \, \frac{x_\sigma - y - (1-\lambda)v}{\lambda} + (1-\mu)w - x_\rho \right\| < \frac{1}{2n}\,.$$

Taking now into account that $0 < \mu/\lambda < 1/\varepsilon_0$ and $\|y\| < 1/2n$, we obtain

$$\begin{aligned}\left\| \mu \, \frac{x_\sigma - (1-\lambda)v}{\lambda} + (1-\mu)w - x_\rho \right\| &< \frac{1}{2n} + \frac{\mu}{\lambda}\,\|y\| \\ &< \frac{1}{2n} + \frac{1}{2n}\cdot\frac{1}{\varepsilon_0} = \frac{1}{2n}\left(1 + \frac{1}{\varepsilon_0}\right).\end{aligned}$$

which implies that $x_\rho \in E_\gamma := \overline{[\{x_i\}_{\gamma \leq i < \omega_1}]}$ since $x_\sigma, v, w \in E_\gamma$ and $n > n_0$ is arbitrary. In particular, it means that $E_{\beta_0} = E_\beta$ for every $\beta_0 \leq \beta < \omega_1$. Finally, if we denote

$z = x_\rho - [(\mu/\lambda)\,(x_\sigma - y - (1-\lambda)v) + (1-\mu)w]$, then

$$x_\rho = \left(\mu v + (1-\mu)w + \frac{\mu}{\lambda}\,x_\sigma\right) - \frac{\mu}{\lambda}\,v \,+\, z$$

thus implying that x_ρ is an accumulation point of $[0, 1+\frac{1}{\varepsilon_0}]D_\gamma - [0, \frac{1}{\varepsilon_0}]D_\gamma = F_\gamma$. Indeed, $0 < \mu/\lambda < 1/\varepsilon_0$, $\mu v + (1-\mu)w, x_\sigma, v \in D_\gamma$ and $\|z\| < 1/(2n)(1+\frac{1}{\varepsilon_0})$. □

Theorem 2. *Every uncountable ω-independent family fails the Kunen-Shelah property. Consequently, spaces with the Kunen-Shelah property have no ω-independent families.*

Proof. By the Sersouri result, we know that an uncountable ω-independent family $F = \{x_\alpha : \alpha < \omega_1\}$ always contains a convex right separated family so we may assume that F is convex right separated. By Lemma 1 (choosing a subfamily, if neccesary) we can suppose also the existence of $\beta_0 < \omega_1$ and $\varepsilon_0 > 0$ so that every x_i, with $\beta_0 \le i < \omega_1$, is an accumulation point of $[0, 1+1/\varepsilon_0]D_\gamma - [0, 1/\varepsilon_0]D_\gamma$, for every $\gamma < \omega_1$. Pick $x_j \in F$ with $\beta_0 < j$. As in [5] we can construct inductively a sequence of signs $(\varepsilon_n)_{n\ge1}$, a sequence of positive real numbers $\{\lambda_p^n, \mu_p^n\}_{n\ge1, 1\le p\le k(n)}$ and a sequence of ordinals $(\gamma_p^n)_{n\ge1,1\le p\le k(n)}$ such that,

$$(a) \qquad \sum_{p=1}^{k(n)} \lambda_p^n \in [0, 1+\tfrac{1}{\varepsilon_0}], \quad \sum_{p=1}^{k(n)} \mu_p^n \in [0, \tfrac{1}{\varepsilon_0}] \qquad \text{for every} \quad n \ge 1$$

$$(b) \qquad \beta_0 < j < \gamma_1^n < \gamma_2^n < \cdots < \gamma_{k(n)}^n < \gamma_1^{n+1} < \omega_1 \qquad \text{for every} \quad n \ge 1$$

and

$$x_j \,+ \sum_{n\ge1} a_n \varepsilon_n y_n = 0 \tag{2.2}$$

where $y_n = \sum_{p=1}^{k(n)} (\lambda_p^n - \mu_p^n) x_{\gamma_p^n}$. Now it is easy to see that the series

$$x_j \,+ \sum_{n\ge1} a_n \varepsilon_n \left(\sum_{p=1}^{k(n)} (\lambda_p^n - \mu_p^n) x_{\gamma_p^n} \right)$$

also converges to zero, thus proving that $\{x_i\}_{i<\omega_1}$ is not ω-independent, a contradiction. □

3. Remarks

An Asplund space X with the Kunen-Shelah property has no convex right separated families. Indeed, every weak* compact convex subset of X^* is the weak* closed convex hull of its weak* strongly exposed points [7]. As shown in [4], the set of weak* denting points of the dual unit ball of a Banach space with the Kunen-Shelah property lies in a separable subspace. Consequently, X^* is weak* separable and then X cannot contain a convex right separated family $\{x_\alpha : \alpha < \omega_1\}$. Otherwise, the nested sequence $\{C_\beta = \overline{\text{conv}}\,(\{x_\alpha\}_{\beta<\alpha}),\, \beta < \omega_1\}$ would produce, by duality, an (uncountable) nested sequence of different weak* closed convex sets in X^*.

In general, the Kunen-Shelah property seems to be a weaker condition than the absence of convex right separated families. However, there is no example of a Banach space with the Kunen-Shelah property admitting such a family. The Kunen-Shelah property suffices for most of the purposes and it can be connected with many usual geometrical features of Banach spaces.

Let us finish this note with an open problem. It is natural to ask if the condition $\overline{\text{conv}}\,\{x_\alpha : \beta_0 \le \alpha < \omega_1\} = \overline{\text{conv}}\,\{x_\alpha : \beta \le \alpha < \omega_1\}$ for every $\beta_0 \le \beta < \omega_1$, used by Sersouri, and the condition $x_i \in \overline{[0, 1 + 1/\varepsilon_0]D_\gamma - [0, 1/\varepsilon_0]D_\gamma}$ for every $\beta_0 < i < \omega_1$ and every $\gamma < \omega_1$, used in our proof, can be replaced by other *stationary* conditions. For instance, is it true that a family $\{x_\alpha : \alpha < \omega_1\}$ satisfying

$$\overline{\text{span}}\,\{x_\alpha : \beta_0 \le \alpha < \omega_1\} = \overline{\text{span}}\,\{x_\alpha : \beta \le \alpha < \omega_1\}$$

for every $\beta_0 \le \beta < \omega_1$ cannot be ω-independent?

References

[1] C. Finet and G. Godefroy, *Biorthogonal systems and big quotient spaces*, Contemporary Math. 85 (1989), 87-110.

[2] D.H. Fremlin and A. Sersouri, *On ω- independence in separable Banach spaces*, Quart. J. Math., 39 (1988), 323- 331.

[3] A. S. Granero, M. Jiménez Sevilla and J. P. Moreno, *Convex sets in Banach spaces and a problem of Rolewicz*, Studia Math. 129 (1) (1998), 19-29.

[4] M. Jiménez Sevilla and J. P. Moreno, *Renorming Banach spaces with the Mazur intersection property*, J. Funct. Anal. 144 (1997), 486-504.

[5] N. J. Kalton, *Independence in separable Banach spaces*, Contemporary Math., vol. 85 (1989), 319-323.

[6] S. Negrepontis, *Banach Spaces and Topology* , Handbook of Set-Theoretic Topology, North-Holland, 1984, p.1045-1142.

[7] R. R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Math. 1364, Springer-Verlag.

[8] A. Sersouri, *ω-independence in nonseparable Banach spaces*, Contemporary Math., vol. 85 (1989), 509-512

[9] S. Shelah, *Uncountable constructions for B.A., e.c. groups and Banach spaces*, Israel J. Math., 51(1985), 273-297.

[10] S. Shelah, *A Banach space with few operators*, Israel J. Math., 30(1978), 181-191.

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