A CONSTANT OF POROSITY FOR CONVEX BODIES

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Abstract. It was proved recently that a Banach space fails the Mazur intersection property if and only if the family of all closed, convex and bounded subsets which are intersections of balls is uniformly very porous. This paper deals with the geometrical implications of this result. It is shown that every equivalent norm on the space can be associated in a natural way with a constant of porosity, whose interplay with the geometry of the space is then investigated. Among other things, we prove that this constant is closely related to the set of ε-differentiability points of the space and the set of r-denting points of the dual. We also obtain estimates for this constant in several classical spaces.

1. Introduction

The present article is a companion piece to [8], in which the authors studied the question of whether the majority of closed, convex and bounded sets in a Banach space are intersections of closed balls. This question can be answered in the negative if there is at least one which is not. Indeed, it was shown in [8] that either every closed, convex and bounded set is an intersection of closed balls or the family $\mathcal{M}$ of all closed, convex and bounded sets with the above property is uniformly very porous. One of the most surprising features of this result is the existence (once the space and the norm have been fixed) of a positive constant which is everywhere a lower bound for the porosity of $\mathcal{M}$, in spite of the vast class of closed, convex and bounded sets involved.

This is the starting point of this paper in which we introduce a *constant of porosity* for each Banach space and each equivalent norm by considering the infimum of the porosity of $\mathcal{M}$ at each element. It is quite remarkable that this constant has many geometrical implications. For instance, it can be used as a measure of non-denseness (i) of the set of weak* denting points of the dual unit sphere, (ii) of the set of subdifferentials of the

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norm at the ε-differentiability points of the unit sphere of the space. The exact value of this constant is usually unknown. However, it is shown in Section 3 how to obtain reasonably good estimates by using the topological information provided by the set of points of ε-differentiability of the norm. With this tool, estimates for the constant are obtained in Section 4 for several classical spaces: $C(K)$ spaces, $L_1(\mu)$, $L_\infty(\mu)$, spaces with property (α, ε), spaces with property (β, ε) and others.

While porosity properties in hyperspaces of convex bodies have been widely investigated (see [6], [13], [17] and [18]), their connections with the geometrical features of the underlying spaces have been hardly explored. Moreover, relationships between topological questions in hyperspaces and geometry of Banach spaces are still far from being well known [1].

2. A constant of porosity for convex bodies

Throughout, X is a Banach space with norm $\|\cdot\|$, $B_{\|\cdot\|}$ is the unit ball and $S_{\|\cdot\|}$ its unit sphere. The dual of X is denoted by X^* with the dual norm $\|\cdot\|^*$. Let $\mathcal{M}$ be the collection of all intersections of balls, considered as a subset of the hyperspace $\mathcal{H}$ of all closed, convex and bounded sets of a Banach space, furnished with the Hausdorff metric. Consider a set $C \in \mathcal{M}$ and denote by $B(C, R)$ the closed ball centered at C with radius R and by $\gamma(C, R, \mathcal{M})$ the supremum of all r for which there exists $D \in \mathcal{H}$ such that $B(D, r) \subset B(C, R) \setminus \mathcal{M}$. The number

$$\rho(C, \mathcal{M}) = 2 \limsup_{R \to 0} \frac{\gamma(C, R, \mathcal{M})}{R}$$

is called the porosity of $\mathcal{M}$ at C. It was proved in [8] that $\mathcal{M} \neq \mathcal{H}$ if and only if there is a positive number α satisfying

$$\rho(C, \mathcal{M}) \geq \frac{\alpha}{1+\alpha} \tag{2.1}$$

for every $C \in \mathcal{M}$. The question of where α comes from needs a little explanation. As shown in [8, Proposition 1], if X fails the Mazur intersection property, then there is a norm one functional f such that $M_f = \{x \in B_{\|\cdot\|} : f(x) \leq 0\} \notin \mathcal{M}$. This means that there is $x_0 \in B_{\|\cdot\|} \setminus M_f$ such that every ball containing M_f also contains x_0. Then α is precisely $f(x_0)$. The lower-estimate for $\rho(C, \mathcal{M})$ stated in 2.1 suggests the possibility of considering a constant of porosity $\rho(X, \|\cdot\|)$ for the whole space X simply by setting

$$\rho(X, \|\cdot\|) \;=\; \inf\{\rho(C, \mathcal{M}) : \; C \in \mathcal{H}\} \,.$$

Though it is quite natural to define $\rho(X, \|\cdot\|)$ this way, there exist serious difficulties when trying to estimate it. On the other hand, denoting by $\widehat{C}$ the intersection of all balls containing $C \in \mathcal{H}$, the constant

$$\beta = \sup\{d(C, \widehat{C}) : \; C \in \mathcal{H}, \; C \subset B_{\|\cdot\|}\} \tag{2.2}$$

is in a sense quite the opposite of $\rho(X, \|\cdot\|)$. It can be estimated by using a technique described in Section 3 (relying on the study of points of ε-differentiability) but it has not such a natural definition. Fortunately, both constants are tightly related, as proved in the following proposition. As a consequence, a method for estimating $\rho(X, \|\cdot\|)$ will be at hand.

Proposition 2.1. *Every Banach space X with norm $\|\cdot\|$ satisfies*

$$\frac{\beta}{1+\beta} \;\leq\; \rho(X, \|\cdot\|) \;\leq\; 2\beta$$

Proof. To prove the right inequality, it is enough to consider the set $\{0\} \in \mathcal{H}$ consisting of a single point, the origin, in order to show that $\rho(\{0\}, \mathcal{M}) \leq 2\beta$. Notice that the ball $B(\{0\}, R)$ is just the family of all $C \in \mathcal{H}$ such that $C \subset RB_{\|\cdot\|}$. Pick $C \in \mathcal{H}$ with $\sup\{\|y\| : y \in C\} < R$ and suppose that there is $\gamma \in \mathbb{R}$, $0 < \gamma < R - \sup\{\|y\| : y \in C\}$, such that $D \in \mathcal{H} \setminus \mathcal{M}$ whenever $d(C, D) \leq \gamma$. Obviously $\gamma < d(C, \widehat{C})$ and, by the definition of β, we have $d(C, \widehat{C}) < R\beta$. Consequently $2\gamma/R \leq 2\beta$ thus implying that $\rho(\{0\}, \mathcal{M}) \leq 2\beta$, as desired.

In the case of the left inequality, it is convenient to express β by using only slices instead the collection of all convex sets. Recall that a slice S of $C \in \mathcal{H}$ is a set of the form $S = \{x \in C : f(x) \leq \lambda\}$ with $f \in X^*$ and $\lambda \in \mathbb{R}$. Denote by $\mathcal{S}$ the family of all slices of the unit ball. It can be shown, as a consequence of the Hahn-Banach separation theorem, that

$$\beta = \sup \; \left\{ \; d(S, \widehat{S}) : \; S \in \mathcal{S} \; \right\} . \tag{2.3}$$

Indeed, given $C \in \mathcal{H}$ with $C \subset B_{\|\cdot\|}$, $x \in \widehat{C} \setminus C$ and $r > 0$ such that $x + rB_{\|\cdot\|} \cap C = \emptyset$, there is a functional $f \in X^*$ separating both convex sets $x + rB_{\|\cdot\|}$ and C. Then f defines a slice S of $B_{\|\cdot\|}$ containing C. Since $x \in \widehat{S}$, we have $d(S, \widehat{S}) \geq d(C, \widehat{C})$.

Our plan now is to apply an argument similar to the one used in the proof of [8, Theorem 2.2]. It could be done easily if we knew the existence of a slice $S = \{x \in B_{\|\cdot\|} : f(x) \leq \lambda\}$ such that

$$\beta = \sup f(\widehat{S}) - \lambda$$

Notice that, by (2.3), for every $n \in \mathbb{N}$ there is $S = \{x \in B_{\|\cdot\|} : g(x) \leq \sigma\}$ such that $\beta \geq d(S, \widehat{S}) - 1/n$. However, we cannot work directly with S, even in the case $\beta = d(S, \widehat{S})$, since the only value which is relevant here is $\sup g(\widehat{S}) - \sigma$. In order to avoid this difficulty, observe that (2.3) implies that

$$\beta = \sup \Big\{ \sup_{\widehat{S}} f - \sup_{S} f : \ ||f||^* = 1, \ S \in \mathcal{S} \Big\} \tag{2.4}$$

and so, for every $n \in \mathbb{N}$ there is a slice $S = \{x \in B_{\|\cdot\|} : f(x) \leq \sigma\}$ such that $\beta \geq \sup f(\widehat{S}) - \sigma - 1/n$. Actually, there is no loss of generality if we assume that there is $x_0 \in \widehat{S}$ such that $\beta = f(x_0) - \sigma$. The proof now can be accomplished as in [8, Theorem 2.2] □

Obviously, the space X with norm $\|\cdot\|$ has the Mazur intersection property if and only if $\rho(X, \|\cdot\|) = \beta = 0$. Intuitively, $\rho(X, \|\cdot\|)$ says how far is $(X, \|\cdot\|)$ from satisfying this property. If we fix the space and consider $\rho(X, \|\cdot\|)$ as a real valued mapping from the metric space $\mathcal{N}$ of all equivalent norms on X, one could be tempted to think that $\rho(X, \|\cdot\|)$ is continuous. This is never true, since the set of norms satisfying the Mazur intersection property is always residual (and therefore dense) in $\mathcal{N}$ [4].

3. Applications to the geometry of the dual space

A basic connection between the porosity of $\mathcal{M}$ and the geometry of the dual unit ball, $S_{\|\cdot\|^*}$, is described in this section. Let us begin by recalling the definition of points of ε-differentiability [16]. Given $\varepsilon > 0$, define M_ε as the set of all points $x \in S_{\|\cdot\|}$ such that, for some $\delta(\varepsilon, x) > 0$,

$$\sup_{0<\lambda<\delta,\ \|y\|=1} \frac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda} \ < \ \varepsilon \ .$$

The set M_ε is open in $S_{\|\cdot\|}$ [5] and $\cap_{\varepsilon>0} M_\varepsilon$ is, precisely, the set of points in $S_{\|\cdot\|}$ where the norm is Fréchet differentiable. Given $x \in S_{\|\cdot\|}$, denote $D(x) = \{f \in S_{\|\cdot\|} : f(x) = 1\}$. It was proved in [5] that X has the Mazur Intersection Property if and only if, for every $\varepsilon > 0$, the set $D(M_\varepsilon) = \{D(x) : x \in M_\varepsilon\}$ is dense in $S_{\|\cdot\|^*}$. It means that, when X fails this property, there are $\varepsilon, \delta > 0$ so that $D(M_\varepsilon)$ is not a δ-net of $S_{\|\cdot\|^*}$ ($N \subset M$ is δ-net of M if for every $m \in M$ there is a $n \in N$ with $||m-n|| \leq \delta$). Is there any relationship between ε, δ and the constant of porosity β? Next we show that this is the case. Given $f \in S_{\|\cdot\|^*}$ and $|\lambda| \leq 1$, denote by $S_{f,\lambda}$ the slice $\{x \in B_{\|\cdot\|} : f(x) \leq \lambda\}$ and define

$$d_f = \sup\{\sup_{\widehat{S_{f,\lambda}}} f - \lambda : \ -1 \leq \lambda \leq 1\} \ .$$

Clearly, $\beta = \sup\{d_f : \ f \in S_{\|\cdot\|^*}\}$ as stated in (2.4). Finally, recall that $f \in X^*$ is a norm attaining functional if there is $x \in S_{\|\cdot\|}$ so that $f(x) = ||f||$. The role of d_f as a measure of non density of $D(M_r)$ when $0 < r < d_f$ and f is a norm attaining functional is illustrated in the following proposition.

Proposition 3.1. *Given two norm attaining functionals $f, g \in S_{\|\cdot\|^*}$,*

(i) $f \notin D(M_{d_f})$ *when* $d_f > 0$,
(ii) $g \notin D(M_r)$ *when* $0 < r < d_f$ *and* $\|g - f\| \leq \frac{d_f - r}{2}$.

Proof. (i) Without loss of generality we may assume that there are $-1 \leq \lambda \leq 1$ and $x_0 \in \widehat{S}_{f,\lambda}$ so that $d_f = f(x_0) - \lambda$. By symmetry, it is enough to see that $-f \notin D(M_{d_f})$. Toward this end, we take an arbitrary point $y \in S_{\|\cdot\|}$ satisfying $f(y) = 1$ and we consider the ball $B_n = B(-ny, n + \lambda + d_f - 1/n)$. It is clear that $x_0 \notin B_n$, so there is a norm one vector $x_n \in S_{f,\lambda} \setminus B_n$. Letting $y_n = x_n/n$, we have

$$\begin{aligned}\frac{\|-y+y_n\| + \|-y-y_n\| - 2}{\|y_n\|} &= \frac{\|-y+y_n\| - \|-y\|}{\|y_n\|} + \|-ny-x_n\| - n \\ &\geq -\frac{f(y_n)}{\|y_n\|} + (n+\lambda+d_f-\frac{1}{n}-n) \\ &= -f(x_n)+\lambda+d_f-\frac{1}{n} \\ &\geq d_f - \frac{1}{n} \quad .\end{aligned}$$

In order to prove (ii), note that if $||g-f|| \leq \frac{d_f-r}{2}$ then $S_{f,\lambda} \subseteq S_{g,\lambda+\frac{d_f-r}{2}} \subseteq S_{f,\lambda+d_f-r}$, implying that $\widehat{S}_{f,\lambda} \subseteq \widehat{S}_{g,\lambda+\frac{d_f-r}{2}}$ and hence $x_0 \in \widehat{S}_{g,\lambda+\frac{d_f-r}{2}}$. On the other hand, $g(x_0) = f(x_0)+(g(x_0)-f(x_0)) \geq \lambda + d_f - \frac{d_f-r}{2} = \lambda + \frac{d_f-r}{2} + r$ and, therefore, $d_g \geq r$. It follows by (i) that $g \notin D(M_r)$. As a consequence, $D(M_r)$ is not a $\frac{d_f-r}{2}$-net of $S_{\|\cdot\|^*}$. □

A functional $f \in S_{\|\cdot\|^*}$ is said to be a weak* r-denting point of $B_{\|\cdot\|^*}$ provided f is in a slice $S = \{g \in B_{\|\cdot\|^*} : \ g(x) \leq \lambda\}$ for some $x \in S_{\|\cdot\|}$, $-1 < \lambda < 1$, and diam $S < r$. Set D_r for the set of all weak* r-denting points of $S_{\|\cdot\|^*}$.

Corollary 3.2. *Let $(X, \|\cdot\|)$ be a Banach space with $\beta > 0$*

(i) *$D(M_r)$ is not a $\frac{s-r}{2}$-net of $S_{\|\cdot\|^*}$ when $r < s < \beta$.*
(ii) *D_r is not a $\frac{s-3r}{2}$-net of $S_{\|\cdot\|^*}$ when $r < s/3 < \beta/3$.*

Proof. (i) Since $\beta = \sup\{d_f : f \in S_{\|\cdot\|^*}\}$, there is $f \in S_{\|\cdot\|^*}$ satisfying $d_f > s > r$. Applying now (ii) of Proposition 3.1, we know that $D(M_r)$ is not a $\frac{d_f-r}{2}$-net of $S_{\|\cdot\|^*}$ and, consequently, is not a $\frac{s-r}{2}$-net either.

(ii) Notice that $D(M_r)$ is included in D_r [5, Lemma 2.1] and D_r is included in $D(M_r) + rB_{\|\cdot\|^*}$. Pick $f \in S_{\|\cdot\|^*}$ satisfying $d_f/3 > s/3 > r$. If we assume that D_r is a $\frac{d_f-3r}{2}$-net of $S_{\|\cdot\|^*}$, then $S_{\|\cdot\|^*} \subset D_r + \frac{d_f-3r}{2}B_{\|\cdot\|^*} \subseteq D(M_r) + \frac{d_f-r}{2}B_{\|\cdot\|^*}$, implying that $D(M_r)$ is a $\frac{d_f-r}{2}$-net of $S_{\|\cdot\|^*}$, a contradiction. □

The preceding Corollary enlightens the geometrical meaning of β (and thus of the constant of porosity). It yields an estimate of the holes of $D(M_\varepsilon)$ for every $\varepsilon < \beta$. Conversely, if we are trying to get estimates of β from the geometrical features of the space, how might we do so? It seems that there is no direct way to calculate β, but Proposition 3.3 gives insight to this question. It will be used as the main tool in next section to estimate β in many classical spaces.

Proposition 3.3. *For $f \in S_{\|\cdot\|^*}$ and $0 < \varepsilon < 1/2$, $\operatorname{dist}(f, D(M_{4\varepsilon})) \geq 2\varepsilon$ implies that $0 \in \widehat{S}_{f,-\varepsilon}$. Therefore, if $D(M_{4\varepsilon})$ is not a 2ε-net, then $\beta \geq \varepsilon$.*

Proof. We may assume that f is a norm attaining functional. Say that $f(x_0) = 1$ with $\|x_0\| = 1$. If $0 \notin \widehat{S}_{f,-\varepsilon}$, there is a ball $B = B(x_1, R)$ containing $S_{f,-\varepsilon'}$ in its interior and missing 0 for some $0 < \varepsilon' < \varepsilon$. Consider the following sets: $C = \operatorname{int}(\operatorname{conv}(S_{f,-\varepsilon'} \cup \{0\}))$ and $S = \{x \in C : \|x - x_1\| = R\}$. Let $z_0 \in S$ and $g \in D\big(\frac{z_0 - x_1}{R}\big)$. As a first step in the proof, we will see that $\|f - g\| \leq 2\varepsilon'$.

Indeed, the set $H = \{x \in X : g(x) = R + g(x_1)\}$ does not intersect $S_{f,-\varepsilon'}$. Also, note that $0 \notin \{x \in X : g(x) \leq g(x_1) + R\}$ so $H \cap S_{-f,-\varepsilon'} = \emptyset$. By symmetry, $-H$ does not intersect $S_{f,-\varepsilon'} \cup S_{-f,-\varepsilon'}$. Finally, this implies that $\ker g \cap B_{\|\cdot\|} \subseteq f^{-1}([-\varepsilon', \varepsilon'])$ and, by Phelps' Lemma [14], either $\|f - g\| \leq 2\varepsilon'$ or $||f + g|| \leq 2\varepsilon'$. Why is the second situation not possible? The reason is that $-x_0 \in S_{f,-\varepsilon'}$ so $g(-x_0) < g(z_0)$ and hence $0 = g(0) > g(z_0) > g(-x_0)$. Consequently $g(x_0) > 0$, implying that

$$\|f + g\| \geq f(x_0) + g(x_0) > 1 > 2\varepsilon > 2\varepsilon'$$

so therefore $\|f - g\| \leq 2\varepsilon'$, as claimed. Consequently, for every sequence $\{z_n\}$ with $\|z_n - x_1\| = R$ and $\lim \|z_n - z_0\| = 0$ and for every sequence $\{g_{z_n}\}$ with $g_{z_n} \in D\left(\frac{z_n - x_1}{R}\right)$ one has $\limsup \|g_{z_n} - g_{z_m}\| \leq 4\varepsilon'$. Applying now Lemma 2.1 of [5] we get that $\frac{z_0 - x_1}{R} \in M_{4\varepsilon}$ thus implying that $\operatorname{dist}(f, D(M_{4\varepsilon})) \leq 2\varepsilon' < 2\varepsilon$, a contradiction. □

Remark 3.4. Since the set $D(M_r) \subset D_r$ for every r, the above proposition tell us that $\beta \geq \varepsilon$ if the set $D_{4\varepsilon}$ of weak* 4ε-denting points of $B_{\|\cdot\|^*}$ is not a 2ε-net of $S_{\|\cdot\|^*}$.

4. Estimates for the constant of porosity in classical spaces

We have shown in Proposition 2.1 that $\rho\,(X, \|\cdot\|) \geq \frac{\beta}{1+\beta}$. In this section, we obtain estimates (for concrete spaces X) of the form $\beta \geq \varepsilon$ and then we can conclude that $\rho\,(X, \|\cdot\|) \geq \frac{\varepsilon}{1+\varepsilon}$ since the map $t \to \frac{t}{1+t}$ is strictly increasing.

1. Finite dimensional polyhedral spaces. *A finite dimensional space is said to be polyhedral provided its unit ball is the convex hull of a (symmetric) finite set. This is a particular case of* (not necessarily finite dimensional) *Banach spaces whose norms have property* (α,ε) *for some* $0<\varepsilon\le 1$*: there is a family* $\{x_i,x_i^*\}_{i\in I}\subset S_{\|\cdot\|}\times S_{\|\cdot\|^*}$ *so that (a)* $x_i^*(x_i)=1$*, (b)* $|x_i^*(x_j)|\le 1-\varepsilon$ *if* $i\neq j$ *and (c)* $B_{\|\cdot\|}=\overline{\text{conv}}\,(\{\pm x_i\}_{i\in I})$ [15]*. Spaces with property* (α,ε) *satisfy* $\beta\ge\varepsilon$ *and hence* $\rho\,(X,\|\cdot\|)\ge\frac{\varepsilon}{1+\varepsilon}$*.*

Let x_{i_0} be one of the points appearing in the definition of property (α,ε) . We claim that every ball containing the set $D=\overline{\text{conv}}\,(\{\pm x_i\}_{i\in I\setminus\{i_0\}})$ also contains x_{i_0}. As a consequence, we deduce that $x_{i_0}\in\widehat{S}_{1-\varepsilon,x_{i_0}^*}$ and so $\beta\ge\varepsilon$.

Indeed, suppose that B is a ball containing D and missing x_{i_0}. Set φ for the composition of the homothety and the translation mapping $B_{\|\cdot\|}$ onto B. We can separate x_{i_0} from B by a functional $g\in X^*$ supporting B at $\varphi(D)$. The existence of such a functional needs a little explanation. Since $B_{\|\cdot\|}=\overline{\text{conv}}\,(\{\pm x_i\}_{i\in I})=\text{conv}\,(\{\pm x_{i_0}\}\cup D)$, we know that every functional supporting $x\in S_{\|\cdot\|}$, $x\neq\pm x_{i_0}$, must also support D. Besides,

$$\|\cdot\| \;=\; \sup\{f_x(\cdot): x\in S_{\|\cdot\|}, x\neq\pm x_{i_0}\}$$

where $f_x(x)=1=\|f_x\|$. Now, finally, assume that g supports B at $f(d)$ with $d\in D$. Though it is clear that the segments $[d,x_{i_0}]$ and $[f(d),f(x_{i_0})]$ are parallel, x_{i_0} is separated from d by the functional g while this it not the case of $f(x_{i_0})$ and $f(d)$, a contradiction.

2. The $C(K)$-spaces. *If* K *is any compact Hausdorff space,* $C(K)$ *denotes the Banach space of all continuous real-valued functions* $f:K\to\mathbb{R}$ *endowed with the "sup norm"* $\|f\|_\infty=\max_{t\in K}|f(t)|$*. This space satisfies* $\beta\ge 1/2$ *and hence* $\rho(C(K),\|\cdot\|_\infty)\ge 1/3$*.*

We first consider $f\in S_{C(K)}$ which attains its norm at an accumulation point of K, so that there is a (not eventually constant) sequence $\{t_n\}\in K$ with $\lim_n|f(t_n)|=1$. We may assume, without loss of generality, that $\lim_n f(t_n)=1$. Denote by $\delta_t\in S_{C(K)^*}$ the evaluation at the point t. Note that $||\delta_t\pm\delta_{t'}||=2$ if $t\neq t'$. Then, for each $-1<\lambda<1$, every slice $S_{f,\lambda}$ of $B_{C(K)^*}$ contains two different elements of the sequence $\{\delta_{t_n}\}$ thus implying that $\text{diam}\,(S_{f,\lambda})\ge 2$ and, by [5, Lemma 2.1], that $f\notin M_2$.

Otherwise, f attains its norm at a point $t_0 \in K \setminus K'$ (where K denotes the set of all accumulation points of K). It is well known that, in this case, the sup norm $\|\cdot\|_\infty$ is Fréchet differentiable at f with differential δ_{t_0} so, therefore, $f \in M_r$ for every $r > 0$. Summarizing, $D(M_r) = \{\delta_t : \ t \in K \setminus K'\}$ for every $0 < r < 2$ and, consequently, $D(M_r)$ is not a $(r/2)$-net of $S_{\|\cdot\|_\infty^*}$. Proposition 3.3 ensures that $\beta \geq 1/2$.

3. The $L_\infty(\mu)$-spaces. *For a space of positive measure* (Ω, Σ, μ), *the Banach space* $L_\infty(\mu)$ *consists of all (essentially) bounded measurable functions* $f : \Omega \to \mathbb{R}$ *under the norm* $\|f\|_\infty = \inf\{C : |f(x)| \leq C \text{ a.e. in } \Omega\}$. *This space satisfies* $\beta \geq 1/2$ *and thus* $\rho\,(L_\infty(\mu), \|\cdot\|_\infty) \geq 1/3$.

In fact, we can prove a much more general result: given two Banach spaces Y and Z, the space $X = Y \oplus_\infty Z$ endowed with the norm $\|\cdot\|_\infty = \max\{\|\cdot\|_Y, \|\cdot\|_Z\}$ satisfies $\beta \geq 1/2$. Indeed, it is easily checked that a point $x = (x_1, x_2) \in X$ satisfying $\|x_1\|_X = 1$ and $\|x_2\|_Y = 1$ is not in M_2 for if you pick $x_1^* \in Y^*$, $x_2^* \in Z^*$ with $\|x_1^*\|_{Y^*} = x_1^*(x_1) = 1$ and $\|x_2^*\|_{Z^*} = x_2^*(x_2) = 1$ then the points $x^* = (x_1^*, 0)$, $y^* = (0, x_2^*)$ are in $D(x)$ and $\|x^* - y^*\| = 2$. As a consequence

$$D(M_2) \subset \{(x^*, 0) : x^* \in Y^*, \|x^*\|_{Y^*} = 1\} \cup \{(0, y^*) : y^* \in Z^*, \|y^*\|_{Z^*} = 1\}$$

so $D(M_2)$ is not an ε-net, for every $0 < \varepsilon < 1$, and $\beta \geq 1/2$.

4. The $L_1(\mu)$-spaces. *For a space of positive measure* (Ω, Σ, μ), *the Banach space* $L_1(\mu)$ *consists of all (equivalence classes of) measurable functions* $f : \Omega \to \mathbb{R}$ *under the norm* $\|f\|_1 = \int_\Omega |f|\, d\mu < \infty$. *This space satisfies* $\beta \geq 1$ *and consequently* $\rho(L_1(\mu), \|\cdot\|_1) \geq 1/2$.

As in the previous case, we can prove something stronger: given two Banach spaces Y, Z, the space $Y \oplus_1 Z$ endowed with the norm $\|\cdot\| = \|\cdot\|_Y + \|\cdot\|_Z$ satisfies $\beta \geq 1$. Letting $D = \{(y, 0) : \|y\|_Y = 1\}$, it suffices to see that every point $(0, z) \in Y \oplus_1 Z$ with $\|z\|_Z = 1$ is contained in $\widehat{D}$. To this end observe that, for every $x = (y, z) \in X$,

$$\|x\|_1 = \|y\|_Y + \|z\|_Z = \sup\{(y^*, z^*)(x) : \|y^*\|_{Y^*} = 1, \|z^*\|_{Z^*} = 1\}$$

Suppose now that B is a ball containing D and missing $x = (0, z)$ for some $z \in Z$, $\|z\|_Z = 1$. The above equation shows that x can be separated from B by a functional supporting $\varphi(D)$ (φ being the composition of the homothety and the translation mapping the unit ball onto B) and we derive a contradiction as in the proof of property (α, ε).

5. Spaces with property (β, ε). *A norm $\|\cdot\|$ on a Banach space X has property (β, ε) for some $0 < \varepsilon \leq 1$ if there is a family $\{x_i, f_i\}_{i \in I} \subset S_{\|\cdot\|} \times S_{\|\cdot\|^*}$ so that (a) $x_i^*(x_i) = 1$, (b) $|x_i^*(x_j)| \leq 1 - \varepsilon$, for $i \neq j$, and (c) $\|x\| = \sup\{\,|x_i^*(x)| : i \in I\}$ for every $x \in X$* [9]. *Spaces having norms with the above property satisfy $\beta \geq \frac{\varepsilon}{4-2\varepsilon}$ and then $\rho(X, \|\cdot\|) \geq \varepsilon/(4+\varepsilon)$.*

First, note that $\|x_i \pm x_j\| \leq 2 - \varepsilon$ for every $i, j \in I$, $i \neq j$. Then $\|x_i^* - x_j^*\|^* \geq (x_i^* - x_j^*)(\frac{x_i - x_j}{2-\varepsilon}) \geq \frac{2\varepsilon}{2-\varepsilon}$ and, similarily, $\|x_i^* + x_j^*\|^* \geq \frac{2\varepsilon}{2-\varepsilon}$. Now, consider $x \in S_{\|\cdot\|}$ for which there is a (not eventually constant) sequence $\{x_n^*\} \subset \{x_i^*\}_{i \in I}$ satisfying $\lim_n |x_n^*(x)| = 1$. We may assume, without loss of generality, that $\lim_n x_n^*(x) = 1$. Then every slice $S_{x,\lambda}$ of $B_{\|\cdot\|^*}$ with $-1 < \lambda < 1$ contains at least two different elements of the sequence. Thus $\operatorname{diam}(S_{x,\lambda}) \geq \frac{2\varepsilon}{2-\varepsilon}$ and $x \notin M_{\frac{2\varepsilon}{2-\varepsilon}}$. In the other case, $x \in S_{\|\cdot\|}$ lies in the relative interior (in $S_{\|\cdot\|}$) of some face $F_i = \{x \in S_{\|\cdot\|} : x_i^*(x) = 1\}$ (or $-F_i$) and then the norm is Fréchet differentiable at x with differential x_i^* (or $-x_i^*$, respectively) [11]. Therefore, $D(M_r) = \{\pm x_i^* : i \in I\}$ for every $0 < r < \frac{2\varepsilon}{2-\varepsilon}$ and, consequently, $D(M_r)$ is not a $(r/2)$-net. Finally, Proposition 3.3 yields the desired estimate $\beta \geq \frac{\varepsilon}{4-2\varepsilon}$.

5. Final remarks

A vertex point of a closed bounded convex body C is a point which is strongly exposed by an open set of functionals. Consequently, when a space X has a vertex point in its unit ball, there is an hyperplane in X^* whose intersection with $S_{\|\cdot\|^*}$ has non-empty (relative) interior and obviously, for some $r > 0$, weak* r-denting points cannot be dense in $S_{\|\cdot\|^*}$. This is the reason why $\rho(X, \|\cdot\|) > 0$. A particular class of vertex points are the strongly vertex points: the point x is a strongly vertex point of C if there is a closed, convex and bounded set D with $x \notin D$ satisfying $C = \operatorname{conv}(\{x\} \cup C)$. Information concerning vertex and strongly vertex points can be found in [11]. Porosity of spaces having a strongly vertex point in its unit ball can be estimated by using the same arguments as in the case of property (α, ε). This is the case, for instance, of Lorentz sequence spaces $d(w, 1)$.

The "natural" norms with property (β, ε) are finite dimensional polyhedral norms and the usual sup norm on c_0 and ℓ_∞. The first of these norms have property α and second are sup norms on $C(K)$ spaces. In both cases we may have better constants of porosity ($\frac{\varepsilon}{1+\varepsilon}$ and 1/3 respectively). However, norms with property (β, ε) are important in the theory of norm attaining operators. Partington proved that every Banach space can be equivalently renormed with property (β, ε) [12]. These norms are Fréchet differentiable on a dense open set of the space [11] but, as we have seen, they are far from having the Mazur intersection property.

The Mazur intersection property was introduced by Mazur [10] and later studied by Phelps [14], Sullivan [16], Giles, Gregory and Sims [5] and, after these pioneering works, by many other authors. Information concerning this property can be found in [2], [3], [7] and references therein. There are still a number of open problems concerning this subject, such as the existence of points of Fréchet differentiability in spaces with this property. While spaces with Fréchet diferentiable norms satisfy the Mazur intersection property, it is unknown if it is also the case of spaces with a (Fréchet) differentiable bump function.

Finally, the reader interested in knowing the state of the art in the field of hyperspaces and their topologies is referred to the authoritative book by G. Beer [1]. Connections between these topologies and geometrical features of the underlying spaces are still far from being well known.

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