RENORMING BANACH SPACES WITH THE MAZUR INTERSECTION PROPERTY

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Abstract. In this paper we give new sufficient and necessary conditions for a Banach space to be equivalently renormed with the Mazur intersection property. As a consequence, several examples and applications of these results are obtained. Among them, it is proved that every Banach space embeds isometrically into a Banach space with the Mazur intersection property, answering a question asked by Giles, Gregory and Sims. We prove that, for every tree T, the space $C_0(T)$ admits a norm with the Mazur intersection property, solving a problem posed by Deville, Godefroy and Zizler. Finally, assuming continuum hypothesis, we find an example of an Asplund space admitting neither equivalent norm with the above property nor nicely smooth norm.

1. Introduction

It was Mazur [24] who began the investigation to determine those normed linear spaces which possess the later called **Mazur intersection property**: every bounded closed convex set can be represented as an intersection of closed balls.

Further results of sufficiency conditions for a Banach space to have this property were obtained by Phelps [29] and it was characterized later by Sullivan [31] in the case of smooth spaces. Giles, Gregory and Sims gave complete characterizations in a remarkable paper [11]. They also characterized the associated property for a dual space which has been called **weak* Mazur intersection property**: every bounded weak* closed convex set can be represented as an intersection of closed dual balls.

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After these pioneering works, many authors directed their attention to the systematic study of both properties (see, for instance, the book of Deville, Godefroy and Zizler [4] and references therein).

The nice structural properties of Banach spaces with the Mazur intersection property suggested the question of whether this property has any implications for the differentiability of convex functions. Actually, it has been asked in [11] and in most of the papers on this subject whether every Banach space with the Mazur intersection property is an Asplund space. In this direction, significant results were obtained in [9], [10] and [21] by exploiting the rich structure of Banach spaces which have the Mazur intersection property and whose duals have the weak* Mazur intersection property. For instance, it is proved in [9] that, in Banach spaces with both properties, the set of norms which are Fréchet differentiable on a G_δ dense set of the space is residual.

We prove in this paper a renorming theorem for the Mazur intersection property similar to the theorem obtained in [26] for the weak* Mazur intersection property. To be more precise, we prove that every Banach space X whose dual admits a fundamental biorthogonal system $\{x_i, f_i\}_{i\in I} \subset X^* \times X$ can be equivalently renormed to have the Mazur intersection property. We use in the proof of this result a renorming technique due to Troyanski [32] in a similar manner as it was done by Zizler in [33]. We obtain several applications of this result. Among them, we prove that every Banach space can be *almost isometrically* and *complementably* embedded into a Banach space with the above property. On the other hand, using a result of Partington, it can be proved that every Banach space embeds *isometrically* into a Banach space with the Mazur intersection property. The above assertion answers by the negative the question raised by Giles, Gregory and Sims [11] cited before.

Spaces of continuous functions on scattered compacta are an important class of Asplund spaces. We give a sufficient condition to renorm these spaces with the Mazur intersection property. This condition is satisfied, in particular, by spaces of continuous functions on tree spaces answering positively a question posed by Deville, Godefroy and Zizler in [4, Ch. VII]. The relevance of renormings in spaces of continuous functions on a tree is due to the deep results obtained in the pioneering works of Haydon [17], [18] in this area. To obtain these results, we discuss some three-space problems related with the Mazur and weak* Mazur intersection properties.

On the other hand, we give a new necessary condition for a Banach space to admit an equivalent norm with the Mazur intersection property. As a consequence, we prove that the Asplund space constructed by Kunen [27] using the continuum hypothesis admits no equivalent norm with this property and its dual space admits no equivalent norm with the weak* Mazur intersection property. Moreover, this space not even has equivalent nicely smooth norm. In particular, this space has neither a norm with the ball intersection property [13] nor a strongly subdifferentiable norm ([2], [8], [14]). The techniques used to obtain these results are tightly related with those used in [6] and [20]. Some results of this paper have been announced in [19].

2. A renorming theorem with the Mazur intersection property

We consider only Banach spaces over the reals. Given a Banach space X with norm $\|\cdot\|$, we denote by $B_{\|\cdot\|}$ the unit ball of X, by $S_{\|\cdot\|}$ the unit sphere and by X^* the dual space with the dual norm $\|\cdot\|^*$. A slice of the ball $B_{\|\cdot\|}$ is a set of the form $S(B_{\|\cdot\|}, f, \delta) = \{x \in B_{\|\cdot\|} : \; f(x) > \delta\}$ with $f \in X^* \setminus \{0\}$, and $\delta < ||f||^*$. A point $x \in S_{\|\cdot\|}$ is said to be a *denting point* of $B_{\|\cdot\|}$ if for every $\varepsilon > 0$ there exists $f \in X^*$ and $0 < \delta < f(x)$ such that $\operatorname{diam} S(B_{\|\cdot\|}, f, \delta) < \varepsilon$. A point $f \in S_{\|\cdot\|^*}$ is said to be a *weak* denting point* of $B_{\|\cdot\|^*}$ if for every $\varepsilon > 0$ there exists $x \in X$ and $0 < \delta < f(x)$ such that $\operatorname{diam} S(B_{\|\cdot\|^*}, x, \delta) < \varepsilon$. The norm $\|\cdot\|$ is *locally uniformly rotund* at the point $x \in X$ if whenever $\{x_n\}_{n\in\mathbb{N}} \subset X$ is such that $\lim_n 2(||x||^2 + ||x_n||^2) - ||x + x_n||^2 = 0$, then $\lim_n ||x - x_n|| = 0$. A *biorthogonal system* in X is a subset $\{x_i, f_i\}_{i\in I} \subset X \times X^*$ which satisfies $f_i(x_j) = \delta_{ij}$ for $i, j \in I$. The biorthogonal system $\{x_i, f_i\}_{i\in I} \subset X \times X^*$ is *fundamental* if $X = \overline{\text{span}}\,(\{x_i\}_{i\in I}\,)$.

We begin with the fundamental characterization given in [11] of the Mazur intersection property and with some lemmas.

Theorem 2.1 (Giles, Gregory and Sims)**.** *Given a Banach space $(X, \|\cdot\|)$, the norm $\|\cdot\|$ has the Mazur intersection property if and only if the set of weak* denting points of $B_{\|\cdot\|^*}$ is dense in $S_{\|\cdot\|^*}$.*

Lemma 2.2. *Given a dual Banach space $(X^*, \|\cdot\|^*)$ and a weak*-closed subspace H of X^*, the distance function $f(z) = \text{dist}(z, H) = \inf_{h\in H} ||z-h||^*$, $z \in X^*$, is weak*–lower semicontinuous (weak*–l.s.c.).*

Proof. Note that for every $r \geq 0$ we have

$$\{z \in X^* :\ f(z) \leq r\} = \bigcap_{\varepsilon>0}\Big(H + (r+\varepsilon)B_{\|\cdot\|^*}\Big),$$

where $B_{\|\cdot\|^*}$ is the dual unit ball. Using the weak* compactness of $B_{\|\cdot\|^*}$ it is straightforward to verify that $H + (r+\varepsilon)B_{\|\cdot\|^*}$ is weak* closed and therefore the set $\{z \in X^* :\ f(z) \leq r\}$ is also weak* closed. □

Lemma 2.3. *Let $(X^*, \|\cdot\|^*)$ be a dual Banach space with a biorthogonal system $\{x_i, f_i\}_{i\in I} \subset X^* \times X$ and $X_0 = \text{span}\,(\{x_i\}_{i\in I})$. Then, X^* admits an equivalent dual norm $|\cdot|^*$ which is locally uniformly rotund at the points of X_0.*

Proof. We may assume that $\|f_i\| = 1$, for every $i \in I$ and let us consider
$\Delta = \{0\} \cup \mathbb{N} \cup \mathbb{I}$. Define the map T from X_0 into $\ell_\infty(\Delta)$ as follows:

$$T(x)(\delta) \;=\; \begin{cases} \|x\|^* & \text{if} \quad \delta = 0 \\ 2^{-n}G_n(x) & \text{if} \quad \delta = n \in \mathbb{N} \\ f_i(x) & \text{if} \quad i \in I \end{cases}$$

for every $x \in X^*$ and $\delta \in \Delta$, where

$$F_A(x) = \sum_{i\in A} |f_i(x)|$$

$$E_A(x) = \operatorname{dist}\Big(x, \operatorname{span}(\{x_i\}_{i\in A})\Big) \quad A \subset I, \quad \operatorname{card} A < \infty$$

$$G_n(x) = \sup_{\operatorname{card} A \leq n} \{E_A(x) + nF_A(x)\}.$$

Clearly $T(X^*) \subseteq \ell_\infty(\Delta)$ and $T(X_0) \subseteq c_0(\Delta)$. On the other hand, since
$2^{-n}(1+n^2) \leq 2$ for every $n \in \mathbb{N}$, we have $\|x\|^* \leq \|T(x)\|_\infty \leq 2\|x\|^*$.

For every $\delta \in \Delta$, consider the map $T_\delta(x) = T(x)(\delta)$, $x \in X^*$. Obviously, if $\delta \in I \cup \{0\}$ the map T_δ is weak*-l.s.c. . Note also that the maps F_A and, by Lemma 2.2, the maps E_A are weak*-l.s.c., so T_δ is weak*-l.s.c. for every $\delta \in \Delta$.

Let p be the Day norm [4, p.69] in $\ell_\infty(\Delta)$, and consider in X^* the map $n(x) = p(T(x))$, $x \in X^*$. It can be easily proved that $n(\cdot)$ is an equivalent norm in X^*. The norm $n(\cdot)$ has the following expression:

$$n(x)^2 = \sup_{\substack{n\in\mathbb{N} \\ (\delta_1,\delta_2,\ldots,\delta_n)\subset\Delta \\ \delta_i\neq\delta_j}} \sum_{i=1}^{n} \frac{|T_{\delta_i}(x)|^2}{4^i},$$

so $n(\cdot)$ is weak*-l.s.c. , that is, n is a dual norm $|\cdot|^*$. The norm p defined in $\ell_\infty(\Delta)$ is locally uniformly rotund at the points of $c_0(\Delta)$. It can be proved as in [26] that the norm $|\cdot|^*$ is locally uniformly rotund at the points of X_0. We give this proof here for a sake of completness. Take $x \in X_0$, $\{x_m\}_{m=1}^\infty \subset X^*$ so that $\lim_m |x_m|^* = |x|^* = 1$ and $\lim_m |x_m + x|^* = 2$. This implies that $\lim_m p(T(x_m)) = p(T(x)) = 1$ and $\lim_m p(T(x_m) + T(x)) = 2$. The norm p is locally uniformly rotund at each point of $c_0(\Delta)$ so we have that $\lim_m p\Big(T(x_m) - T(x)\Big) = 0$ and, then,

$$\lim_m \|T(x_m) - T(x)\|_\infty = 0\ . \tag{2.1}$$

Define $A = \{i \in I : f_i(x) \neq 0\}$, $M = \max\{\|x_i\| : i \in A\}$ and take $N \geq \max\{M, \operatorname{card} A\}$. For every $B \subset I$ with $\operatorname{card} B < \infty$ we have

$$\begin{aligned} E_B(x) + NF_B(x) &\leq M \sum_{i \in A \setminus B} |f_i(x)| + N \sum_{i \in B \cap A} |f_i(x)| \\ &\leq N \sum_{i \in A} |f_i(x)| \\ &= NF_A(x) \end{aligned}$$

so $G_N(x) = NF_A(x)$. By (2.1) there is m_0 such that, for every $m \geq m_0$,
$|G_N(x_m) - G_N(x)| < \varepsilon$ and $|NF_A(x_m) - NF_A(x)| < \varepsilon$, thus

$$G_N(x_m) - NF_A(x_m) \leq 2\varepsilon + G_N(x) - NF_A(x) = 2\varepsilon$$

and, hence,

$$E_A(x_m) \leq G_N(x_m) - NF_A(x_m) \leq 2\varepsilon \ .$$

This implies that $\lim_m \operatorname{dist}\Big(x_m, \operatorname{span}(\{x_i\}_{i \in A})\Big) = 0$ and the existence of $\{z_m\}_{m=1}^{\infty} \subset \operatorname{span}(\{x_i\}_{i \in A})$ such that $\lim_m \|x_m - z_m\|^* = 0$. Using again (2.1),

$$\lim_m \max_{i \in A} |f_i(z_m) - f_i(x)| = \lim_m \max_{i \in A} |f_i(x_m) - f_i(x)| = 0,$$

and this implies that

$$\lim_m \|x_m - x\|^* = \lim_m \|z_m - x\|^* = 0$$

as we wanted to prove. □

Theorem 2.4. *Let X^* be a dual Banach space with a fundamental biorthogonal system $\{x_i, f_i\}_{i \in I} \subset X^* \times X$. Then, X admits an equivalent norm with the Mazur intersection property.*

Proof. Using Lemma 2.3, we obtain an equivalent dual norm $|\cdot|^*$ in X^* which is locally uniformly rotund at every point of $X_0 = \operatorname{span}(\{x_i\}_{i \in I})$. Now, it is straightforward to verify that the points of $X_0 \cap S_{|\cdot|^*}$ are weak* denting points of $B_{|\cdot|^*}$. The subspace X_0 is dense in X^* and then, by the characterization given in Theorem 2.1, we obtain that the space X endowed with the predual norm of $|\cdot|^*$ has the Mazur intersection property. $\square$

3. Some examples and consequences

Let us recall that the density of a Banach space X, denoted by $\operatorname{dens} X$, is the minimal cardinality of a dense set in X.

Theorem 3.1. *Let X, Y be Banach spaces such that $\operatorname{dens} X^* \leq \operatorname{dens} Y$. Suppose that Y^* has a fundamental biorthogonal system $\{y_i, f_i\}_{i\in I} \subset Y^* \times Y$. Then, the Banach space $X \oplus Y$ admits an equivalent norm with the Mazur intersection property.*

Proof. Let us consider $Z = X \oplus Y$ with the norm

$$||(x, y)||_Z = ||x||_X + ||y||_Y.$$

By Theorem 2.4 we need only to show that $Z^* \approx X^* \oplus Y^*$ has also a fundamental biorthogonal system in $Z^* \times Z$. An element $x^* + y^*$ of $X^* \oplus Y^*$ is considered an element of Z^* in the usual way: $(x^* + y^*)(x + y) = x^*(x) + y^*(y)$ for every $x \in X$, $y \in Y$. Let us take a dense set $\{x_i\}_{i\in I}$ of X^* and relabel the fundamental biorthogonal system given in Y^* as $\{y_i^n, f_i^n\}_{i\in I, n\in\mathbb{N}}$. We may assume that $\|y_i^n\|_Y \leq 1/n$ for every $i \in I$, $n \in \mathbb{N}$. Then, the system

$$S = \{x_i + y_i^n,\ f_i^n\}_{i\in I, n\in\mathbb{N}} \subset Z^* \times Z$$

is a fundamental biorthogonal system in Z^*. We use now Theorem 2.4 to conclude the proof. $\square$

Corollary 3.2. *Every Banach space X can be almost isometrically complementably embedded into a Banach space with the Mazur intersection property.*

Proof. The set of all equivalent norms on a Banach space endowed with the topology of the uniform convergence on bounded sets is a complete metric space. A useful result of Georgiev [10] ensures that the set of norms with the Mazur intersection property is either empty or residual. Let us consider the Banach space $Z = X \oplus \ell_2(\Gamma)$ with $\operatorname{card}\Gamma = \operatorname{dens} X^*$. By Theorem 3.1, Z can be renormed with the Mazur intersection property and this finishes the proof. $\square$

We observe that, from the proof of last corollary, it follows that every Banach space X embeds into a Banach space Z with the Mazur intersection property such that $\operatorname{dens} Z = \operatorname{dens} X^*$. Clearly, this is sharp in the sense that, necessarily, $\operatorname{dens} Z = \operatorname{dens} Z^* \geq \operatorname{dens} X^*$ as it can be deduced from the following remark.

Remark 3.3. [11] A Banach space with the Mazur intersection property has the same density character than its dual.

It arises now a quite natural question: is it also possible also to embed *isometrically* every Banach space X into a Banach space Z with the above property? The answer is affirmative although, in this case, we cannot assure neither it is embedded in a complemented way nor $\operatorname{dens} Z = \operatorname{dens} X^*$.

Corollary 3.4. *Every Banach space X may be isometrically embedded into a Banach space Z with the Mazur intersection property.*

Proof. We denote by $\alpha = \operatorname{dens} X$, $\beta = \min\{\gamma \text{ ordinal number} : \operatorname{card}\gamma > \alpha\}$, and the Banach space

$$m_\alpha(\beta) = \{\, x \in \ell_\infty(\beta) : \ \operatorname{supp} x \ \text{ has cardinality at most } \alpha \,\},$$

with the supremum norm $\|x\| = \sup_{\gamma<\beta} |x_\gamma|$. Obviously, X may be isometrically embedded into $(m_\alpha(\beta), \|\cdot\|)$. On the other hand, by Corollary 3.2, $m_\alpha(\beta)$ embeds into a Banach space $(Z, |\cdot|)$ with the Mazur intersection property and, by a result of Partington [28], $(m_\alpha(\beta), \|\cdot\|)$ embeds *isometrically* into $(m_\alpha(\beta), |\cdot|)$. Therefore, X embeds isometrically into $(Z, |\cdot|)$. Note that, with this argument, we have $\operatorname{dens} X^* < \operatorname{dens} Z^*$ $\square$

Proposition 3.5. *Every Banach space can be isometrically embedded into a Banach space whose dual has the weak* Mazur intersection property. Consequently, every dual Banach space is isometric to a quotient of a dual space with the above property.*

Proof. It has been proved in [26] that, for every Γ, the dual space of $\ell_\infty(\Gamma)$ can be equivalently renormed to have the weak* Mazur intersection property. The proof can be obtained using again the result of Partington [28], as in previous corollary. $\square$

Corollaries 3.2 and 3.4 show the existence of many non Asplund spaces with the Mazur intersection property, answering by the negative the question asked by Giles, Gregory and Sims [11]. The following space is an interesting example of the last assertion.

Example 3.6. *In Theorem 3.1 we can consider $X = \ell_1$ and $Y = \ell_2(\mathbb{R})$. Then, the Banach space $\ell_1 \oplus \ell_2(\mathbb{R})$ admits an equivalent norm with the Mazur intersection property.*

It is proved in [1] that for an Asplund space $(X, \|\cdot\|)$ with the Mazur intersection property and for every subspace $Y \subseteq X$, there exists a subspace $Z \subseteq X$ such that $\operatorname{dens} Y = \operatorname{dens} Z$ with $Y \subseteq Z \subseteq X$ and $(Z, \|\cdot\|)$ has the Mazur intersection property. This result is sharp in two senses. First, it is not true if the condition of being an Asplund space is removed. Indeed, the space of the Example 3.6 has the Mazur intersection property but there is no separable subspace $Z \subseteq X$ containing ℓ_1 with the Mazur intersection property, even for any equivalent norm, since, from Remark 3.3, a separable Banach space with the Mazur intersection property is an Asplund space. Secondly, we will see later that there exist Asplund spaces with the Mazur intersection property having subspaces admitting no equivalent norm with the above property. These spaces are obtained from Kunen's space [27] which is constructed using the continuum hypothesis.

Corollary 3.7. *Banach spaces with the Mazur intersection property and whose dual have the weak* Mazur intersection property are not necessarily Asplund spaces.*

Proof. As a counterexample we can take the Banach space $Z = \ell_1 \oplus \ell_2(\mathbb{R})$ of the Example 3.6, which has a norm with the Mazur intersection property and whose dual space Z^* has a dual norm with the weak* Mazur intersection, since Z can be renormed to have a locally uniformly rotund norm. By [10], the set of equivalent norms on Z having the Mazur intersection property and whose dual norms have the weak* Mazur intersection property is residual. □

We also obtain as an application of Theorem 3.1 the following result of Deville [3]. Denote by η an ordinal number and by $J(\eta)$ the set of all continuous functions f on $[0, \eta]$ such that $f(0) = 0$ and

$$\|f\| = \sup \left(\sum_{i=1}^{n} \Big(f(\alpha_{i+1}) - f(\alpha_i) \Big)^2 \right)^{1/2} < \infty$$

where the supremum is taken over all choices of $(\alpha_1, ..., \alpha_{n+1}) \subset [0, \eta]$ such that $\alpha_1 < \alpha_2 < \cdots < \alpha_{n+1}$, $n \in \mathbb{N}$. This space is usually called the long James space. Some of its properties can be found in [5].

Corollary 3.8. *For every ordinal η, the long James space $J(\eta)$, its predual $M(\eta)$ and every finite dual of $J(\eta)$ admit an equivalent norm with the Mazur intersection property.*

Proof. First, we need to observe that $\ell_2(\eta)$ can be complementably embedded into $J(\eta)$. Indeed, consider the subset

$$A = \{\alpha \in [0, \eta] : \alpha = 2n \text{ or } \alpha = \gamma + 2n \text{ , with } \gamma \text{ ordinal limit and } n \in \mathbb{N} \ \}$$

and the subspace

$$H(\eta) = \{f \in J(\eta) : \ f(\alpha) = 0 \quad \text{if} \quad \alpha \notin A\}.$$

The subspace $H(\eta)$ is isomorphic to $\ell_2(A)$ and $\operatorname{card} A = \operatorname{card}\eta$. On the other hand, the projection $f \in J(\eta) \longrightarrow p(f) \in H(\eta)$ defined as

$$p(f)(\alpha) = \begin{cases} f(\alpha) - f(\alpha - 1) & \text{if} \quad \alpha \in A \\ 0 & \text{if} \quad \alpha \notin A \end{cases}$$

is continuous and, therefore, $H(\eta)$ is complemented in $J(\eta)$. Thus, we have that $J(\eta) \approx \ell_2(\eta) \oplus Y$ for a Banach space Y (which can be easily identified with $J(\eta)$) and $J(\eta)^* \approx \ell_2(\eta) \oplus Y^*$.

On the other hand, $M(\eta)$, $J(\eta)$ and every finite dual of $J(\eta)$ are Asplund spaces [5]. Consequently $\operatorname{dens}\ell_2(\eta) = \operatorname{card}\eta \geq \operatorname{dens} Y = \operatorname{dens} Y^* = \operatorname{dens} Y^{**}$ and, applying Theorem 3.1, we obtain that $J(\eta)$ and $J(\eta)^*$ admit a norm with the Mazur intersection property. The assertion for $M(\eta)$ and the dual spaces of $J(\eta)^*$ follows from the fact that $M(\eta)$ is isometric to $J(\eta)^*$ (cf. [5]). $\square$

Example 3.9. *Consider the James'tree space JT. It is shown in* [23] *that JT^{**} is isomorphic to $JT \oplus \ell_2(\mathbb{R})$. Then, from Theorem 3.1, we have that JT^{**} and finite even duals of JT^{**} admit an equivalent norm with the Mazur intersection property. Note also that the space JT^* and finite odd duals of JT admit a Fréchet differentiable norm since their duals are WCG.*

In the following, we study the three space property for renormings in Banach spaces with the Mazur (weak* Mazur) intersection property.

Proposition 3.10. *Let X be a Banach space and Y be a closed subspace of X such that Y admits a norm whose dual norm has a dense set of locally uniformly rotund points and X/Y admits a norm with the Mazur intersection property. Then X admits a norm with the Mazur intersection property.*

Proof. It is enterely similar to the proof for the three space problem for locally uniform rotund renormings given in [15]. We consider, under the standard identifications, $(X/Y)^*$ to be the annihilator subspace $Y^\perp$ with the weak* topology in $(X/Y)^*$ being the same as the induced weak* topology which $Y^\perp$ inherites as a subspace of X^*. By hypothesis, there is a norm on $Y^\perp$ which is $\sigma(Y^\perp, X)$-l.s.c. and has a G_δ dense set of weak* denting points (note that the set of weak* denting (denting, locally uniformly rotund) points of a norm is always a G_δ set). The subspace $Y^\perp$ is weak* closed so this norm can be extended to an equivalent dual norm $\|\cdot\|^*$ on X^*. Let $|\cdot|^*$ be an equivalent dual norm on Y^* which is locally uniformly rotund at a G_δ dense set. Consider the restriction map $Q : X^* \to Y^*$, which is weak*-weak* continuous and the Bartle-Graves continuous selection mapping $B : Y^* \to X^*$, which is bounded on bounded sets, $B(y^*) = |y^*|^* B(y^*/|y^*|^*)$ and $B(0) = 0$. For every $y^* \in S_{|\cdot|^*} = \{y^* \in Y^* : \ |y^*|^* = 1\}$, take $y \in Y$ such that $y^*(y) = 1$ and $|y| \leq 2$. Define $P_{y^*}(x^*) = x^*(y)B(y^*)$, for $x^* \in X^*$, which is weak*-weak* continuous. The following family of weak* l.s.c. convex functions defined on X^*

$$\varphi_{y^*}(x^*) = |Q(x^*) + y^*|^*,$$
$$\psi_{y^*}(x^*) = ||x^* - P_{y^*}(x^*)||^*, \qquad y^* \in S_{|\cdot|^*},$$

is uniformly bounded on bounded sets. Therefore, if we consider

$$\phi_k(x^*) = \sup\{\varphi_{y^*}(x^*)^2 + \tfrac{1}{k}\psi_{y^*}(x^*)^2 : \ y^* \in S_{|\cdot|^*}\},$$

$$\phi(x^*) = \|x^*\|^{*2} + |Q(x^*)|^{*2} + \sum_k 2^{-k}\phi_k(x^*),$$

the Minkowski functional $|||\cdot|||^*$ of the set $\{x^* \in X^* : \phi(x^*) + \phi(-x^*) \leq 4\}$ is an equivalent dual norm on X^*.

Consider the mapping (not necessarily linear) $S : X^* \to Y^\perp$, defined as $S(x^*) = x^* - B(Q(x^*))$. It is proved in [15] that x^* is a locally uniformly rotund point for $|||\cdot|||^*$ provided $Q(x^*)$ is locally uniformly rotund for $|\cdot|^*$ and $S(x^*)$ is locally uniformly rotund for $\|\cdot\|^*$ in $Y^\perp$. We claim that x^* is a weak*-norm point of continuity (extreme, weak* denting) for $|||\cdot|||^*$ whenever $Q(x^*)$ is a locally uniformly rotund point for $|\cdot|^*$ and $S(x^*)$ is weak*-norm point of continuity (extreme, weak* denting respectively) for $\|\cdot\|^*$ in $Y^\perp$.

Indeed, consider $x^* \in X^*$ such that $Q(x^*)$ is a locally uniformly rotund point for $|\cdot|^*$ and $S(x^*)$ is weak*-norm point of continuity for $\|\cdot\|^*$ in $Y^\perp$. Take a net $\{x^*_\alpha\}$ weak* converging to x^* with $|||x^*_\alpha||| = |||x^*|||$. In particular, we have that $\lim_\alpha \frac{1}{2}|||x^*|||^{*2} + \frac{1}{2}|||x^*_\alpha|||^{*2} - |||(x^* + x^*_\alpha)/2|||^{*2} = 0$. Then, from convexity arguments,

(i) $\lim_\alpha \frac{1}{2}||x^*||^{*2} + \frac{1}{2}||x^*_\alpha||^{*2} - ||(x^* + x^*_\alpha)/2||^{*2} = 0$,

(ii) $\lim_\alpha \frac{1}{2}|Q(x^*)|^{*2} + \frac{1}{2}|Q(x^*_\alpha)|^{*2} - |Q(x^* + x^*_\alpha)/2|^{*2} = 0$,

(iii) $\lim_\alpha \frac{1}{2}\phi_k(x^*) + \frac{1}{2}\phi_k(x^*_\alpha) - \phi_k((x^* + x^*_\alpha)/2)$, for every $k \in \mathbb{N}$.

We can suppose that $|Q(x^*)|^* = 1$. From the fact that $Q(x^*)$ is a locally uniformly rotund point in Y^* for the norm $|\cdot|^*$ and ii) we obtain that $\lim_\alpha |Q(x^*_\alpha) - Q(x^*)|^* = 0$ and thus, $x^*_\alpha - B(Q(x^*_\alpha))$ weak* converges to $x^* - B(Q(x^*))$. Since $x^* - B(Q(x^*))$ is a weak*–norm point of continuity of $\|\cdot\|^*$ in $Y^\perp$, we need only to show that

$$\lim_\alpha ||x^*_\alpha - B(Q(x^*_\alpha))||^* = ||x^* - B(Q(x^*))||^* \tag{3.1}$$

and then, we will obtain that $\lim_\alpha x^*_\alpha = x^*$. Given $\varepsilon > 0$, take δ_1, $0 < \delta_1 < \varepsilon$, so that $||B(Q(x^*)) - B(y^*)||^* < \varepsilon$ whenever $|Q(x^*) - y^*|^* < \delta_1$. Now, pick δ_2, $0 < \delta_2 < \delta_1$ such that $|(Q(x^*) + y^*)/2|^* > 1 - \delta_2$ implies $|Q(x^*) - y^*|^* < \delta_1$. Consider $k_0 \in \mathbb{N}$, $k_0 > 10/\delta_2$, and a net $\{y^*_\alpha\} \in S_{|\cdot|^*}$ with

$$\phi_{k_0}((x^* + x^*_\alpha)/2) - \Big(\varphi_{y^*_\alpha}((x^* + x^*_\alpha)/2)^2 + \tfrac{1}{k_0}\psi_{y^*_\alpha}((x^* + x^*_\alpha)/2)^2\Big) \le \delta_2/4k_0 .$$

With the same arguments of [15], we obtain an α_0 such that, for $\alpha \ge \alpha_0$,
$|(Q(x^*) + y^*_\alpha)/2|^* > 1 - \delta_2$ and

$$\frac{1}{2}||x^* - P_{y^*_\alpha}(x^*)||^{*2} + \frac{1}{2}||x^*_\alpha - P_{y^*_\alpha}(x^*_\alpha)||^{*2} - \Big\|\frac{x^* + x^*_\alpha}{2} - P_{y^*_\alpha}\Big(\frac{x^* + x^*_\alpha}{2}\Big)\Big\|^{*2} \le \varepsilon. \tag{3.2}$$

Also, we can suppose $|Q(x^*_\alpha) - Q(x^*)|^* \leq \delta_1$ for $\alpha \geq \alpha_0$, so

$$|B(Q(x^*)) - B(y^*_\alpha)|^* \leq \varepsilon \quad \text{and} \quad |B(Q(x^*)) - B(Q(x^*_\alpha))|^* \leq \varepsilon. \tag{3.3}$$

Therefore, from the definition of $P_{y^*_\alpha}$ and (3.3), we have that

$$||P_{y^*_\alpha}(x^*) - B(Q(x^*))||^* \leq (2||B|| + 1)\varepsilon \tag{3.4}$$

$$||P_{y^*_\alpha}(x^*_\alpha) - B(Q(x^*_\alpha))||^* \leq 2(2||B|| + 1)\varepsilon, \tag{3.5}$$

for $\alpha \geq \alpha_0$ and $||B|| = \sup\{||B(y^*)||^* : y^* \in S_{|\cdot|^*}\}$. Finally, from (3.2), (3.4) and (3.5) we obtain that

$$\frac{1}{4}\Big(||x^* - B(Q(x^*))||^* - ||x^*_\alpha - B(Q(x^*_\alpha))||^*\Big)^2 \leq \frac{1}{2}||x^* - B(Q(x^*))||^{*2}$$
$$+\frac{1}{2}||x^*_\alpha - B(Q(x^*_\alpha))||^{*2} - \|\frac{1}{2}(x^* + x^*_\alpha) - \frac{1}{2}(B(x^*) + B(x^*_\alpha))\|^{*2} \leq C\varepsilon,$$

where $\alpha \geq \alpha_0$ and $C > 0$ depends only on $||B||$. Thus, (3.1) is proved.

Note that we have the same conclusion if $S(x^*) = 0$ and $x^* \in L_{|\cdot|^*}$. Also, if $Q(x^*) = 0$ and $x^* \in L_{\|\cdot\|^*}$ we have that i) implies $\lim_\alpha |Q(x^*_\alpha)|^* = 0$ and thus, there are $\{z^*_\alpha\} \in Y^\perp$ with $\lim_\alpha ||z^*_\alpha - x^*_\alpha||^* = 0$. This implies that $\lim_\alpha \frac{1}{2}||x^*||^{*2} + \frac{1}{2}||z^*_\alpha||^{*2} - ||(x^* + z^*_\alpha)/2||^{*2} = 0$ and, since x^* is a weak*–norm point of continuity for the norm $\|\cdot\|^*$ in $Y^\perp$ we have that $\lim_\alpha z^*_\alpha = \lim_\alpha x^*_\alpha = x^*$.

The corresponding assertions for extreme and, thus, for weak* denting points can be obtained in a similar way.

To conclude the proof, observe that the mappings S and Q are continuous and open. Then, the sets

$$L_{|\cdot|^*} = \{x^* \in X^* : \ |\cdot|^* \text{ is locally uniformly rotund at } Q(x^*)\},$$
$$L_{\|\cdot\|^*} = \{x^* \in X^* : \ S(x^*) \text{ is weak* denting of } \|\cdot\|^* \text{ in } Y^\perp\}$$

and therefore $L = L_{|\cdot|^*} \cap L_{\|\cdot\|^*}$ are G_δ dense sets of X^*. Hence, the space $(X, |||\cdot|||)$ has the Mazur intersection property. $\square$

Next results show that the properties "X^* admits a dual locally uniformly rotund norm" and "X^* can be renormed with the weak* Mazur intersection property" are *three space* properties.

Corollary 3.11. *Let X a Banach space and Y a closed subspace of X such that both Y and X/Y admit norms whose dual norms are locally uniformly rotund. Then X admits a norm whose dual norm is locally uniformly rotund.*

Proposition 3.12. *Let X be a Banach space and Y a subspace of X such that both Y^* and $(X/Y)^*$ can be renormed to have the weak* Mazur intersection property. Then X^* can be renormed to have the weak* Mazur intersection property.*

Proof. A dual norm has the weak* Mazur intersection property if and only if its predual norm has a dense set of denting points in its unit sphere [11]. In order to prove the proposition we just need the following renorming result of Troyanski (see [4], Ch.IV.3):

Consider the set D of all denting points of the unit ball of a Banach space X. Then, X admits an equivalent norm which is locally uniformly rotund at each point of D.

Therefore, Y and X/Y have equivalent norms $\|\cdot\|$ and $|\cdot|$ respectively, with a dense set of locally uniformly rotund points. We denote by Q the canonical proyection $Q:\ X \longrightarrow X/Y$, and by $B:\ X/Y \longrightarrow X$ the Bartle-Graves continuous selection mapping. We denote by $|||\cdot|||$ the norm constructed in X, in the same way as [4, Thm. VII.3.1]. It is proved as in Proposition 3.10 that whenever $Q(x_0)$ is a locally uniformly rotund point for the norm $|\cdot|$ and $x_0 - B(Q(x_0))$ is a locally uniformly rotund point (extreme, point of continuity, denting) for the norm $\|\cdot\|$, then x_0 is a locally uniformly rotund point (extreme, point of continuity, denting, respectively) for the norm $|||\cdot|||$ (as in Proposition 3.10, it does not matter if $Q(x_0) = 0$ or $x_0 - B(Q(x_0)) = 0$). We conclude that $|||\cdot|||^*$ has the weak* Mazur intersection property. □

Recall that a Banach space is said to have the *Kadec property* if the norm topology and the weak topology coincide on the unit sphere. A Banach space is said to have the *Kadec-Klee property* if every sequence on the unit sphere which is weakly convergent is also norm convergent. From the proof of Proposition 3.12 we obtain the following corollary.

Corollary 3.13. *Let X be a Banach space and Y a subspace of X such that X/Y has an equivalent locally uniformly rotund norm. Then, X has an equivalent Kadec norm if and only if Y has an equivalent Kadec norm.*

We mention that the corresponding result for the Kadec-Klee property have been obtained by Lin and Zhang in [22].

Haydon gave in [17] an example of an Asplund space admitting no equivalent Gâteaux differentiable norm, namely the space $C_0(L)$ of all continuous functions vanishing at the infinity over the following tree L: denote by ω_1 the smallest uncountable ordinal, α an ordinal number and consider $L = \bigcup_{\alpha<\omega_1} \omega_1^\alpha$ which is called the full uncountable branching tree of height ω_1. Therefore, it is a natural question to ask whether the space $C_0(L)$ admits an equivalent norm with the Mazur intersection property. The answer is affirmative. Moreover, we will show that, for every tree T, the space $C_0(T)$ admits a norm with the Mazur intersection property, solving a problem posed by Deville, Godefroy and Zizler in [4, Ch. VII]. First, we need the following lemmas.

Lemma 3.14. *Let K be a compact Hausdorff scattered space such that $\operatorname{card} K = \operatorname{card} I$, being I the set of isolated points of K. Then, the Banach space $C(K)$ admits an equivalent norm with the Mazur intersection property. Moreover, its dual norm has a dense set of locally uniformly rotund points.*

Proof. The space $C(K)$ is an Asplund space, so its dual space is identifiable with $\ell_1(K)$. For every $\omega \in K' = K\backslash I$, we can consider disjoint subsets of different points $\{t_n^\omega\}_{n=1}^\infty \subset I$ and $A = I \setminus \{t_n^\omega : \ \omega \in K', \ n \in \mathbb{N} \ \}$. Denote by $\delta_t \in \ell_1(K)$ the evaluation at the point $t \in K$ and by χ_t the characteristic function at the point t. Clearly $\chi_t \in C(K)$ if and only if t is an isolated point in K. Let us consider the biorthogonal system $\{y_n^\omega, f_n^\omega\}_{n\in\mathbb{N},\,\omega\in\mathbb{K}'} \subset C(K)^* \times C(K)$, where $y_n^\omega = (1/n)\delta_{t_n^\omega}$ and $f_n^\omega = n\chi_{t_n^\omega}$. Then, the system

$$\mathcal{S} = \{\delta_\omega + \dagger_\setminus^\omega, \{_\setminus^\omega\}_{\setminus\in\mathbb{N},\,\omega\in\mathbb{K}'} \cup \{\delta_\sqcup, \chi_\sqcup\}_{\sqcup\in\mathcal{A}} \subset \mathcal{C}(\mathcal{K})^* \times \mathcal{C}(\mathcal{K})$$

is a fundamental biorthogonal system in $C(K)^*$. We apply now Theorem 2.4 to finish the proof. □

Remark 3.15. Note that the Banach space $C_0(L)$ verifies the above assertion.

Proposition 3.16. *The Banach space $C_0(T)$ admits a norm with the Mazur intersection property whenever T is a tree space.*

Proof. For any $t \in T$ we denote by t^+ the set of immediate successors of t and consider the subset of T

$$H = \{t \in T' \,:\, t^+ = \emptyset\},$$

where T' is the set of all accumulation points of T and the closed subspace of $C_0(T)$

$$Y = \{f \in C_0(T) \,:\, f(t) = 0, \quad \text{if} \quad t \in H\}.$$

Observe that $Y \approx C_0(T \setminus H)$, the set of all continuous functions on the Alexandrov compactification $\widehat{T \setminus H} = (T \setminus H) \cup \{\infty\}$ vanishing at the infinity. The space $T \setminus H$ is locally compact, Hausdorff, scattered and verifies that the cardinal of its isolated points is equal to card$(T \setminus H)$, so by Lemma 3.14 we obtain a norm on Y such that its dual norm has a dense set of locally uniformly rotund points. On the other hand, it can be easily verified using the fact that H is an antichain and the Tietze's extension theorem that $C_0(T)/Y$ is isomorphic to $c_0(H)$, and then, $C_0(T)/Y$ admits a norm such that its dual norm has a dense set of locally uniformly rotund points. Now the assertion follows from Proposition 3.10. □

4. An Asplund space admitting no equivalent norm with the Mazur intersection property

There arises now the question of whether every Asplund space admits a norm with the Mazur intersection property. To answer this question by the negative, the candidate should be an Asplund space without Fréchet differentiable norms. The difficulty lies in the fact that, by Lemma 3.14, Haydon's example admits equivalent norms with the Mazur intersection property.

The following proposition provides a necessary condition for a Banach space to have the Mazur intersection property. As a consequence, if continuum hypothesis is assumed, we find an Asplund space admitting no equivalent norm with the above property, namely the Kunen space $\mathcal{K}$ [27].

Proposition 4.1. *Let X be an infinite dimensional Banach space with the Mazur intersection property or whose dual space has the weak* Mazur intersection property. Then, there is a bounded subset $(x_\alpha)_{\alpha\in I} \subset X$ with* $\operatorname{card} I = \operatorname{dens} X$ *such that for every $\alpha \in I$,* $x_\alpha \notin \overline{\operatorname{conv}}\Big(\{x_\beta\}_{\beta\in I\setminus\{\alpha\}}\Big)$.

Proof. First, consider a Banach space $(X, |\cdot|)$ with the Mazur intersection property. Then, by Theorem 2.1, the norm $|\cdot|^*$ has a dense set of weak* denting points in its unit sphere. Consider $0<\delta<1$ and find a family of weak* denting points $(f_\alpha)_{\alpha\in I} \subset S_{|\cdot|^*}$ with $\operatorname{card} I = \operatorname{dens} X^* = \operatorname{dens} X$ such that

$$|f_\alpha - f_\beta| \geq \delta, \qquad \text{for } \alpha \neq \beta. \tag{4.1}$$

Then, there is a family of slices $S(B_{|\cdot|^*},\ y_\alpha,\ \rho_\alpha)$, for $\alpha \in I$, with $|y_\alpha| = 1$, $f_\alpha(y_\alpha) > \rho_\alpha > 0$, and

$$S(B_{|\cdot|^*},\ y_\alpha,\ \rho_\alpha) \cap S(B_{|\cdot|^*},\ y_\beta,\ \rho_\beta) = \emptyset, \qquad \text{for } \alpha \neq \beta. \tag{4.2}$$

We denote $x_\alpha = (1/\rho_\alpha)y_\alpha$ for every $\alpha \in I$. It follows from (4.2) that $f_\alpha(x_\alpha) > 1$ and $|f_\alpha(x_\beta)| \leq 1$ for $\alpha, \beta \in I$, $\beta \neq \alpha$. Consequently,

$$x_\alpha \notin \overline{\text{conv}}\Big(\{x_\beta\}_{\beta \in I \setminus \{\alpha\}}\Big).$$

For the second assertion, consider the Banach space $(X^*, |\cdot|^*)$ with the weak* Mazur intersection property. Then, the norm $|\cdot|$ in X has a dense set of denting points in its unit sphere [11]. Take $0 < \delta < 1$ and find a family of denting points $(x_\alpha)_{\alpha \in I}$ in X, $|x_\alpha| = 1$, with $\operatorname{card} I = \operatorname{dens} X$ such that

$$|x_\alpha - x_\beta| \geq \delta, \qquad \text{for } \alpha \neq \beta. \tag{4.3}$$

From the fact that the points $(x_\alpha)_{\alpha \in I}$ are denting in $B_{|\cdot|}$ and condition (4.3), we get that

$$x_\alpha \notin \overline{\text{conv}}\left(\{x_\beta\}_{\beta \in I \setminus \{\alpha\}}\right). \ \square$$

Let us present now two examples of Banach spaces failing the above properties. It was Shelah [30] the first to construct an example of a non-separable Banach space $\mathcal{S}$ such that for every uncountable set $\{x_i\}_{i \in I}$ in the space, there exists $i_0 \in I$ such that

$$x_{i_0} \in \overline{\text{conv}}\left(\{x_i\}_{I \setminus \{i_0\}}\right). \tag{4.4}$$

Shelah's example was constructed assuming the diamond principle for $\aleph_1$. Later, Kunen [27] provided an example, assuming the continuum hypothesis, of a non-separable space $\mathcal{K}$ satisfying the mentioned property of Shelah's space. Some properties of the Kunen space can be found in [7]. These examples have an additional property: they are *Asplund* spaces.

Corollary 4.2. *Let $\mathcal{X}$ be $\mathcal{S}$ or $\mathcal{K}$. There is neither equivalent norm on $\mathcal{X}$ with the Mazur intersection property nor equivalent dual norm on $\mathcal{X}^*$ with the weak* Mazur intersection property.*

Corollary 4.3. *Let $\mathcal{X}$ be $\mathcal{S}$ or $\mathcal{K}$. Then, the set of denting (weak* denting) points of the unit ball of every equivalent norm (dual norm) on $\mathcal{X}$ ($\mathcal{X}^*$) lies in a separable subspace.*

Proof. Let $\|\cdot\|$ ($\|\cdot\|^*$) be an equivalent norm on $\mathcal{X}$ ($\mathcal{X}^*$) and take a set A_n of denting (weak* denting) points of $S_{\|\cdot\|}$ ($S_{\|\cdot\|^*}$) which is maximal with respect to the condition (4.3) (respectively (4.1)) for $\delta = 1/n$. Then, by a similar argument as in Proposition 4.1 and using (4.4), we obtain that the set A_n is countable and therefore, the set of denting (weak* denting) point of $S_{\|\cdot\|}$ ($S_{\|\cdot\|^*}$) is included in the separable subspace $H = [\cup_{n\in\mathbb{N}} A_n]$. ☐

Observe that Corollary 4.2 implies, in particular, that $\mathcal{K}$ admits no locally uniformly rotund norm. The first example of a space $C(T)$, with T a scattered compact space, admitting no locally uniformly rotund renorming was Haydon's space $C_0(L)$ [17], cited in Remark 3.15. Actually, Haydon's space admits no rotund norm. On the other hand, $C_0(L)$ admits a fundamental biorthogonal system so $C_0(L)^*$ can be renormed with the weak* Mazur intersection property [26].

A generalization of the Mazur intersection property has been considered by Godefroy and Kalton in [13]: a Banach space $(X, \|\cdot\|)$ has the *ball generated property* if every closed, bounded and convex set is an intersection of finite unions of balls.

Recall that a norm closed subspace H of a dual Banach space $(X^*, \|\cdot\|^*)$ is called *1–norming* ([13], [14]) if for every $x \in X$ we have

$$\|x\| = \sup\{f(x) : \ f \in B_{\|\cdot\|^*} \cap H\}. \tag{4.5}$$

The existence in X^* of a proper 1–norming subspace implies that the set of weak* denting points of $B_{\|\cdot\|^*}$ is not a dense set in $S_{\|\cdot\|^*}$ and thus X has not the Mazur intersection property. Further results can be deduced from the lack of proper 1–norming subspaces ([2], [13], [14]). A norm $\|\cdot\|$ on X is called *nicely* *smooth* if $\cap_{x\in X}(x+\|x^{**}-x\|B_{\|\cdot\|^*})=\{x^{**}\}$ for every $x^{**}\in X^{**}$. This concept has been introduced by Godefroy in [12]. It has been proved in [16] that a norm is nicely smooth if and only if there are no proper 1–norming closed subspaces in X^*. The class of nicely smooth norms includes Fréchet differentiable norms, strongly subdifferentiable norms and others ([2], [13]).

Corollary 4.4. *Let $\mathcal{X}$ be $\mathcal{S}$ or $\mathcal{K}$. Then,*

(i) *for every equivalent norm on $\mathcal{X}$, the dual $\mathcal{X}^*$ contains a separable, and thus a proper, 1–norming subspace.*
(ii) *there is no equivalent norm on $\mathcal{X}$ with the ball generated property.*
(iii) *there is no equivalent nicely smooth norm on $\mathcal{X}$.*

Proof. In order to prove (i), observe that the space $\mathcal{X}$ is Asplund and then $B_{\|\cdot\|^*}=\overline{\mathrm{conv}}^{w^*}(H\cap B_{\|\cdot\|^*})$, where H is the separable subspace in $\mathcal{X}^*$ generated by the set of weak* denting points of $B_{\|\cdot\|^*}$. It means that the norm $\|\cdot\|$ verifies (4.5) and H is a proper separable 1–norming subspace. Note that H is a minimal norming subspace.

The assertion of (ii) follows from (i) and [13, Theorem 8.3] which states that if a Banach space $(X,\|\cdot\|)$ has the ball generated property, then $(X^*,\|\cdot\|^*)$ has no proper 1–norming subspaces. Analogously, (iii) follows from (i) and the above mentioned characterization of nicely smooth norms given in [16]. □

Corollary 4.5. *Every nonseparable subspace and quotient of $\mathcal{S}$ and $\mathcal{K}$ embeds isometrically into $\ell_\infty(\mathbb{N})$ for any equivalent norm on $\mathcal{S}$ and $\mathcal{K}$.*

Proof. The results stated in Corollaries 4.2, 4.3 and 4.4 for $\mathcal{S}$ and $\mathcal{K}$ are also verified for every nonseparable subspace and quotient of $\mathcal{S}$ and $\mathcal{K}$ since they have the property given in (4.4). In particular, the assertion follows from Corollary 4.4(i). □

This result yields a positive answer to a question raised by Finet and Godefroy ([7], Question IV.5) . Also, by a transfer argument [4, p. 46], Corollary 4.5 assures the existence of a rotund norm on $\mathcal{S}$ and $\mathcal{K}$. This implies that there is no Kadec norm on these spaces. Indeed, the "squaring" of a rotund norm $\|\cdot\|$ with a Kadec norm $|\cdot|$ produces a norm $|||\cdot|||^2 = \|\cdot\|^2 + |\cdot|^2$ with the property that every point of the unit sphere is denting.

Finally, using Theorem 3.1, $\mathcal{S}$ and $\mathcal{K}$ can be embedded into an Asplund space with the Mazur intersection property. Therefore, not every subspace of an Asplund space with the Mazur intersection property can be equivalently renormed to have this property. It can be useful to compare this observations with the results obtained in [1]. Observe also that the existence of a new Asplund space without Fréchet differentiable norms, provides the opportunity to investigate some related questions as, for instance, the existence of smooth bump functions or Gâteaux smooth norms in this space. Let us mention here that, by a result of Moltó and Troyanski [25], there is no uniformly Gâteaux differentiable norm on $\mathcal{S}$ or $\mathcal{K}$.

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