

# NORMAL AND STARLIKE TILINGS IN SEPARABLE BANACH SPACES

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ABSTRACT. In this note, we provide a starlike and normal tiling in any separable Banach space. That means, there are positive constants  $r$  and  $R$  (not depending on the separable Banach space) such that every tile of this tiling is starlike, contains a ball of radius  $r$  and is contained in a ball of radius  $R$ .

## 1. INTRODUCTION

A family  $\{T_i\}_{i \in I}$  of subsets of a Banach space  $X$  is a *tiling of  $X$*  if  $\bigcup_{i \in I} T_i = X$ , each tile  $T_i$  is a closed subset of  $X$  with nonempty interior and  $C_i \cap C_j$  has empty interior for all  $i \neq j$ . The tiling is said to be convex (starlike) whether the tiles  $T_i$  are convex (starlike, respectively).

In the following, all Banach spaces are considered over the reals. We use the standard notation in a Banach space  $X$  with norm  $\|\cdot\|$ . The closed ball of center  $x \in X$  and radius  $r > 0$  in  $X$  is denoted by  $B(x, r)$  (also by  $B_{\|\cdot\|}(x, r)$  or  $B_X(x, r)$  if it is necessary to precise the norm or the space, respectively, we are referring to).

The study of tilings in infinite dimensional Banach space started with V. Klee in [12] and [13]. V. Klee constructed in every separable Banach space  $X$  by means of biorthogonal systems a convex tiling by “parallelotopes” that are *finitely bounded* (bounded intersections with each finite dimensional subspace) but norm-unbounded. He also studied the existence of tilings based on *discrete proximal* sets in some Banach spaces. Suppose that a set  $A \subset X$  is a discrete proximal set and define the associated “Voronoi cells”

$$T_a = \{x \in X : \|x - a\| \leq \|x - a'\| \text{ for all } a' \in A\}.$$

V. Klee proves that the family of sets  $\{T_a\}_{a \in A}$  is a starlike tiling whenever the norm is strictly convex. Moreover, if  $X$  is a Hilbert space and the norm  $\|\cdot\|$  is a given by a scalar product in  $X$ , then the tiling is convex.

**Definition 1.1.** A tiling  $\{T_i\}_{i \in I}$  of a Banach space  $(X, \|\cdot\|)$  is said to be  $K$ -normal if there are constants  $r > 0$ ,  $K > 0$  and points  $x_i \in T_i$  satisfying  $B(x_i, r) \subset T_i \subset B(x_i, rK)$  for all  $i$ .

For any  $K$ -normal tiling  $\{T_i\}_{i \in I}$  and for every  $s > 0$  the homothetic family  $\{sT_i\}_{i \in I}$  satisfies  $B(sx_i, sr) \subset sT_i \subset B(sx_i, srK)$  for all  $i$  and thus  $\{sT_i\}_{i \in I}$  is a  $K$ -normal tiling of  $(X, \|\cdot\|)$  as well. Notice that the constant  $K$  of normality depends on the norm. By renorming with an equivalent norm  $\|\|\cdot\|\|$  such that  $\|\|\cdot\|\| \leq \|x\| \leq M\|\|\cdot\|\|$  for some  $M > 0$  and all  $x \in X$ , then  $B_{\|\|\cdot\|\|}(x_i, \frac{r}{M}) \subset T_i \subset B_{\|\|\cdot\|\|}(x_i, rK)$  for all  $i$  and the constant of normality of the tiling becomes  $KM$ . A tiling is said to be normal if it is  $K$ -normal for some  $K > 0$ .

For example, for every  $n \in \mathbb{N}$ , in  $\ell_2^n$  with the euclidean norm, the “Voronoi cells” associated with a maximal 2-separated set is a 2-normal convex tiling. R. Deville and M. García-Bravo proved that in any finite dimensional space there is a 2-normal starlike tiling [3]. Also, the space  $(c_0(\Gamma), \|\cdot\|_\infty)$  (for any non empty set  $\Gamma$ ) has a tiling by closed balls of radius 1 with centers at the points having coordinates in  $2\mathbb{Z}$  with only a finite number of non-null coordinates. The Banach space  $(\ell_\infty(\Gamma), \|\cdot\|_\infty)$  (for any non empty set  $\Gamma$ ) has a tiling by closed balls of radius 1 with centers at points having coordinates in  $2\mathbb{Z}$  as well. V. Klee proved the existence in the space  $(\ell^p(\Gamma), \|\cdot\|_p)$  with  $1 \leq p < \infty$

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and for any infinite cardinal  $\Gamma$  satisfying  $\Gamma^{\aleph_0} = \Gamma$  the existence of a  $2^{1/p}$ -separated proximal set  $A$  such that the ‘‘Voronoi cells’’ associated with the set  $A$  form a  $2^{(p-1)/p}$ -normal starlike tiling of the space  $\ell^p(\Gamma)$ . Moreover, when  $p = 2$  the tiling is convex and when  $p = 1$  the tiles are pairwise disjoint translates of the closed unit ball  $B_{\|\cdot\|_1}(0, 1)$ .

Later on, V. Fonf, A. Pezzota and C. Zanco [7] constructed for any infinite set  $\Gamma$  a convex tiling  $\{T_i\}_{i \in \Gamma}$  in  $\ell_\infty(\Gamma)$  with bounded tiles which is universal in the sense that for any normed space  $X$  with a norming set  $S \subset X^*$  of norm one functionals of cardinal  $|\Gamma|$ , there is an isomorphic embedding  $I : X \rightarrow \ell_\infty(\Gamma)$  such that  $\{I^{-1}(T_i)\}_{i \in \Gamma}$  is a convex tiling of  $X$  with bounded tiles and inner radii uniformly bounded from below by a positive constant, i.e. there is a constant  $r > 0$  and there are points  $x_i \in T_i$  satisfying  $B(x_i, r) \subset T_i$  for all  $i$ . Nevertheless the diameter of the tiles are not uniformly bounded from above. So the tiling  $\{I^{-1}(T_i)\}_{i \in \Gamma}$  of  $X$  is not normal.

For more properties on tilings such as point-finite, locally finite, protective, smooth or strictly convex, see [1], [2], [5], [8], [9], [10] and references therein.

It was proved by V. Fonf and J. Lindenstrauss [6] that in the separable Hilbert space  $\ell_2$  any maximal 1-separated set is not proximal. So the construction of convex and normal tilings with ‘‘Voronoi cells’’ given by V. Klee is not possible in  $\ell_2$ . Later on, D. Preiss [15] proved the existence of convex and normal tilings in  $\ell_2$ . Recently, R. Deville and M. Garcıa-Bravo proved the existence of normal and starlike tilings in any separable Banach space with a Schauder basis [3]. In this note, we reexamine the construction of convex normal tilings given by D. Preiss in  $\ell_2$  and the construction of unbounded convex tilings with inner radii uniformly bounded from below by a positive constant in general separable Banach spaces given also by Preiss [15]. As a result we get normal and starlike tilings (with the same constant of normality) for any separable Banach space  $(X, \|\cdot\|)$ . Thus, the construction provides a universal constant of normality, i.e. it can be taken not to depend on the (separable) space nor on the norm. Also it is worth to mention that these normal and starlike tilings can be defined in such a way that some of the tiles are convex.

Recall that the Banach-Mazur distance from any  $n$ -dimensional space  $X$  to  $\ell_2^n$  satisfies  $\text{dist}(X, \ell_2^n) \leq \sqrt{n}$ . Since  $\ell_2^n$  has a 2-normal convex tiling, then  $X$  has a  $2\sqrt{n}$ -normal convex tiling. The question whether every separable Banach space  $X$  (finite or infinite dimensional) has a  $K$ -normal convex tiling with constant of normality  $K$  independent of the space and the norm remains open. We do not know whether a positive answer to the finite dimensional case provides a positive answer to the infinite dimensional case.

## 2. NORMAL AND STARLIKE TILINGS IN SEPARABLE BANACH SPACES

Let start with a lemma providing an auxiliary tiling in  $\mathbb{R}^2$ .

**Lemma 2.1.** *Let us consider the subsets of  $\mathbb{R}^2$ ,*

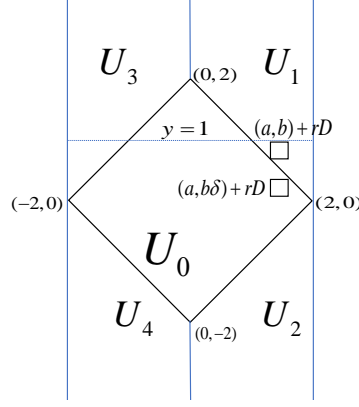
$$\begin{aligned} D &= [-1, 1] \times [-1, 1], & U_0 &= \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 2\}, \\ U_1 &= \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2; y \geq 0; x + y \geq 2\}, & U_2 &= \{(x, -y) \in \mathbb{R}^2 : (x, y) \in U_1\}, \\ U_3 &= -U_1, & U_4 &= -U_2. \end{aligned}$$

*Then, the family  $\{U_i : i = 0, 1, 2, 3, 4\}$  is a tiling of  $\{(x, y) \in \mathbb{R}^2 : |x| \leq 2\}$  and for any  $1 < a < 2$  and  $0 < b < 1$  with  $a + b > 2$  there are constants  $0 < r, \delta < 1$  satisfying*

- (L.a)  $D \subset U_0$ ,
- (L.b)  $(a, b) + rD \subset U_1 \cap \{(x, y) \in \mathbb{R}^2 : |y| \leq 1\}$ ,
- (L.c)  $(a, t) + rD \subset U_0$  whenever  $|t| \leq \delta b$ .

*Proof.* The computations on the conditions given for the squares  $(a, b) + rD$  and  $(a, t) + rD$  whenever  $|t| \leq \delta b$  (see Figure 1) yield the following conditions on  $r$  and  $\delta$ :  $0 < r \leq \min\{1 - b, \frac{a+b}{2} - 1, 2 - a\}$ ,  $r < 1 - \frac{a}{2}$  and  $0 < \delta \leq \frac{2-2r-a}{b}$ .  $\square$

**Theorem 2.2.** *Let  $(X, \|\cdot\|)$  be a separable Banach space. Then, there is a  $K$ -normal and starlike tiling of  $X$  for any constant  $K > 2 + \frac{1}{r}(a + 2b + \frac{8}{\delta})$  (where the constants  $a, b, r$  and  $\delta$  are given in Lemma 2.1).*


 FIGURE 1. Tiling in  $\mathbb{R}^2$ 

It is worth to mention that some tiles of this tiling are convex (see Remark 2.3, Propositions 2.5 and 2.7 and Remark 2.8 regarding the convexity of some tiles).

*Proof of Theorem 2.2.* Since it is already known that finite dimensional spaces have 2-normal starlike tilings [3], we may assume that  $X$  is infinite dimensional. The proof consists on a refinement of the convex tiling obtained by D. Preiss given in [15, Theorem 9] by constructing a starlike tiling in each of those tiles (Step 4). In order to give a complete proof of the theorem we will provide as well the proof of several lemmas given in [15] adapted to our particular case and with some modifications in the constants.

Let us start the proof by considering  $\{e_i, e_i^*\}_{i=1}^\infty$  a fundamental biorthogonal system in  $X$ . For any  $\varepsilon > 0$  we may assume up to a  $(1 + \varepsilon)$ -renorming that  $\|e_i^*\| = \|e_i\| = 1$  for all  $i$ . Indeed, from the result given in [11, Corollary 1.26], we get a fundamental biorthogonal system  $\{a_i, a_i^*\}_{i=1}^\infty$  such that  $\|a_i\| \|a_i^*\| \leq 1 + \varepsilon$  for all  $i$ . By defining  $e_i^* = \frac{a_i^*}{\|a_i^*\|}$  and  $e_i = \|a_i^*\| a_i$  for all  $i$  we get  $\|e_i^*\| = 1 = e_i^*(e_i)$  and  $1 \leq \|e_i\| \leq 1 + \varepsilon$  for all  $i$ . Considering the  $(1 + \varepsilon)$ -equivalent norm  $|\cdot|$  whose closed unit ball is  $B_{|\cdot|} = \overline{\text{conv}}(B_{\|\cdot\|} \cup \{\pm e_i\})$  we get the normalized fundamental biorthogonal system  $\{e_i^*, e_i\}$  for the norm  $|\cdot|$ . In the following we use same notation  $\|\cdot\|$  for the new norm.

Let us define for  $k \in \mathbb{N} \cup \{0\}$  the finite dimensional subspaces,  $V_k = [\{e_1, \dots, e_k\}]$  the linear span of  $e_1, \dots, e_k$  with  $V_0 = \{0\}$ . Also, we define the quotients  $Z_k = X/V_k$  with the corresponding quotient norm  $\|\cdot\|_{Z_k}$  for every  $k \in \mathbb{N} \cup \{0\}$ . If there is no ambiguity on the norm we are referring to we shall write only  $\|\cdot\|$  for the quotient norm. Consider the natural projections

$$Q_k : X \rightarrow Z_k, \quad Q_k(\hat{x}) = \hat{x},$$

$$Q_{k,k+1} : Z_k \rightarrow Z_{k+1}, \quad Q_{k,k+1}(\hat{x}) = \hat{x},$$

For simplicity, we refer to the corresponding equivalence classes in  $Z_k$  for all  $k$  in the same way.

**Step 1.** Consider  $Z_k = X/V_k$  with the canonical quotient norm and a constant  $0 < \delta < 1$ . Then, there is a family  $\{v_{j,k}^*, v_{j,k}\}_{j=0}^\infty \subset Z_k^* \times Z_k$  such that

$$(1.a) \quad \|v_{j,k}^*\| = \|v_{j,k}\| = 1 = v_{j,k}^*(v_{j,k}) = 1 \text{ for all } j, k.$$

$$(1.b) \quad |v_{j,k}^*(v_{j',k})| \leq \delta \text{ if } j < j'.$$

$$(1.c) \quad \sup_j |v_{j,k}^*(v)| \geq \delta \|v\| \text{ for all } v \in Z_k.$$

*Proof.* Several proofs are given in [15, Lemma 7] and [3, Lemma 2.5]. Let us give here the following proof. Consider a dense set  $\{x_n\}_{n=0}^\infty$  in the unit sphere of  $Z_k$  and  $\{x_n^*\}_{n=0}^\infty$  the associated functionals with norm one such that  $x_n^*(x_n) = 1$ . Define  $v_0 = x_0$ ,  $v_0^* = x_0^*$  and  $n_0 = 0$ . Take the first natural (if it exists) such that  $n_1 > n_0$  and  $|v_0^*(x_{n_1})| \leq \delta$  and define  $v_1 = x_{n_1}$  and  $v_1^* = x_{n_1}^*$ . In general, take  $n_{j+1}$  the first natural (if it exists) greater than  $n_j$  such that  $|v_i^*(x_{n_{j+1}})| \leq \delta$  for  $i \leq j$  and define  $v_{j+1} = x_{n_{j+1}}$  and  $v_{j+1}^* = x_{n_{j+1}}^*$ . If the space  $Z_k$  is finite dimensional the above process will stop at some step  $j_0$  and the family  $\{v_j^*, v_j\}_{j=1}^{j_0}$  would satisfy properties (1.a)-(1.c). In our case  $Z_k$  is infinite dimensional

and the above process continues for every  $j$  (notice that  $\cap_{i \leq j} \ker v_i^*$  is a non-trivial subspace) and it produces an infinite sequence  $\{v_j^*, v_j\}_{j=0}^\infty$  satisfying properties (1.a)-(1.c). Indeed, it is clear that properties (1.a) and (1.b) hold. To prove property (1.c) it is enough to see that (1.c) holds for the points of the sequence  $\{x_m\}_{m=0}^\infty$  because this sequence is dense in the sphere. For any  $m \geq 0$  there is  $i \geq 0$  such that  $n_i \leq m < n_{i+1}$ . If  $n_i = m$ , then clearly  $\sup_j |v_j^*(x_m)| = |v_i^*(x_m)| = 1 > \delta \|x_m\|$ . If  $n_i < m < n_{i+1}$ , there is  $s \leq i$  such that  $|v_s^*(x_m)| > \delta = \delta \|x_m\|$  and thus  $\sup_j |v_j^*(x_m)| > \delta \|x_m\|$ . This yields property (1.c). Finally, we relabel  $\{v_j^*, v_j\}_{j=0}^\infty$  as  $\{v_{j,k}^*, v_{j,k}\}_{j=0}^\infty$  and finish the proof of Step 1.

**Step 2.** [15, Lemma 8] *There is a convex tiling  $\{H_j^k\}_{j=0}^\infty$  of  $Z_k = X/V_k$  and points  $h_j^k \in H_j^k$  with  $h_0^k = 0$  such that*

- (2.a)  $B_{Z_k}(0, 1) \subset H_0^k \subset B_{Z_k}(0, \frac{4}{\delta})$ ,
- (2.b)  $\|Q_{k,k+1} h_j^k\| \leq 1 - r$ ,
- (2.c)  $B_{Z_k}(h_j^k, r) \subset H_j^k$ ,
- (2.d)  $\|h - h_j^k\| \leq a + 2b + 2\|Q_{k,k+1} h\|$  for every  $h \in H_j^k$ ,

where the positive constants  $\delta$ ,  $r$ ,  $a$  and  $b$  are those considered in Lemma 2.1 and Step 1.

*Proof.* It can be easily checked that  $Q_{k,k+1}(B_{Z_k}(0, 1)) = B_{Z_{k+1}}(0, 1)$ , where  $B_{Z_k}(0, 1)$  denotes the closed unit ball of  $Z_k$ . Indeed, on the one hand  $\|Q_{k,k+1}(\hat{x})\| = \text{dist}(x, V_{k+1}) \leq \text{dist}(x, V_k) = \|\hat{x}\|$  for all  $x \in X$  and thus  $Q_{k,k+1}(B_{Z_k}(0, 1)) \subset B_{Z_{k+1}}(0, 1)$ . On the other hand, clearly  $Q_{k,k+1} \circ Q_k = Q_{k+1}$  for all  $k \geq 0$  and  $Q_k(B(0, 1)) = B_{Z_k}(0, 1)$  for all  $k \geq 0$ . Therefore,  $Q_{k,k+1}(B_{Z_k}(0, 1)) = B_{Z_{k+1}}(0, 1)$  for all  $k \geq 0$ . Also clearly  $\ker Q_{k,k+1} = \{\hat{x} \in Z_k : x \in V_{k+1} \setminus V_k\} \cup \{0\}$  so the dimension of  $\ker Q_{k,k+1}$  is 1.

Let us define for  $j \geq 0$  and  $k \geq 0$ , the functions

$$\pi_{j,k} : Z_k \rightarrow \mathbb{R}^2, \quad \pi_{j,k}(z) = (e_{k+1}^*(z), v_{j,k+1}^* \circ Q_{k,k+1}(z)), \quad \text{for } z \in Z_k.$$

Here, we identify isometrically  $(Z_k)^* = V_k^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in V_k\}$ . In particular, notice that  $\hat{e}_{k+1}$  has norm 1 as a point of  $Z_k$  and  $e_{k+1}^*$  has norm 1 as a functional on  $Z_k$ .

For every  $z \in Z_k$  we select a point  $x \in X$  such that  $z = \hat{x}$  and a linear combination  $\sum_{i=1}^{k+1} \lambda_i e_i \in V_{k+1}$  such that

$$\begin{aligned} \|Q_{k,k+1}(z)\|_{Z_{k+1}} &= \text{dist}(x, V_{k+1}) = \|x - \sum_{i=1}^{k+1} \lambda_i e_i\| = \|(x - \lambda_{k+1} e_{k+1}) - \sum_{i=1}^k \lambda_i e_i\| \geq \\ &\geq \text{dist}(x - \lambda_{k+1} e_{k+1}, V_k) = \|z - \lambda_{k+1} \hat{e}_{k+1}\|_{Z_k}. \end{aligned}$$

Then,

$$\begin{aligned} \|z\|_{Z_k} &\leq \|e_{k+1}^*(z) \hat{e}_{k+1}\|_{Z_k} + \|\lambda_{k+1} \hat{e}_{k+1} - e_{k+1}^*(z) \hat{e}_{k+1}\|_{Z_k} + \|z - \lambda_{k+1} \hat{e}_{k+1}\|_{Z_k} \leq \\ &\leq |e_{k+1}^*(z)| + |e_{k+1}^*(z - \lambda_{k+1} \hat{e}_{k+1})| + \|Q_{k,k+1}(z)\|_{Z_{k+1}} \leq \\ &\leq |e_{k+1}^*(z)| + \|z - \lambda_{k+1} \hat{e}_{k+1}\|_{Z_k} + \|Q_{k,k+1}(z)\|_{Z_{k+1}} \leq \\ &\leq |e_{k+1}^*(z)| + 2\|Q_{k,k+1}(z)\|_{Z_{k+1}}. \end{aligned} \tag{2.1}$$

Consider the following sets

$$\begin{aligned} T_k &= \{z \in Z_k : |e_{k+1}^*(z)| \leq 2\}, \\ H_0^k &= \bigcap_{j \geq 0} \pi_{j,k}^{-1}(U_0), \\ H_j^{k,p} &= \pi_{j,k}^{-1}(U_p) \cap \bigcap_{j' < j} \pi_{j',k}^{-1}(U_0), \quad \text{for } 1 \leq p \leq 4 \text{ and } j \geq 0. \end{aligned}$$

where the sets  $U_0, U_1, \dots, U_4$  are the subsets of  $\mathbb{R}^2$  defined in Lemma 2.1. Then, let us check that the family  $\{H_0^k, H_j^{k,p} : j \geq 0, p = 1, 2, 3, 4\}$  is a tiling of  $T_k$  satisfying conditions (2.a)-(2.d) in the statement of Step 2:

*Tiling.* Let us check that the above family is a convex tiling. Clearly the sets of the family  $\{H_0^k, H_j^{k,p} : j \geq 0, p = 1, 2, 3, 4\}$  are closed and convex because they are the inverse image by continuous linear functions of closed and convex sets. Also this family covers  $T_k$ : Indeed, Suppose that  $z \in T_k$  and that there is  $j \geq 0$  with  $\pi_{j,k}(z) \notin U_0$  and take the minimum with this property. Then,  $z \in H_j^{k,p}$  for some  $p \in \{1, 2, 3, 4\}$ . Otherwise  $\pi_{j,k}(z) \in U_0$  for all  $j \geq 0$  and then  $z \in H_0^k$ .

Also, the sets of the family  $\{H_0^k, H_j^{k,p} : j \geq 0, p = 1, 2, 3, 4\}$  have pairwise disjoint interiors. Suppose there is a ball  $B_{Z_k}(z, \rho)$  (with  $\rho > 0$ ) contained in  $H_0^k \cap H_j^{k,p}$  for some  $j \geq 0$  and  $p \in \{1, 2, 3, 4\}$ . Then  $\pi_{j,k}(B_{Z_k}(z, \rho)) \subset U_0 \cap U_p$ . Since the continuous linear function  $\pi_{j,k}$  is surjective, it is open and thus  $\pi_{j,k}(B_{Z_k}(z, \rho))$  is a nonempty open set of  $\mathbb{R}^2$ , which is impossible (because the set  $U_0 \cap U_p$  has empty interior in  $\mathbb{R}^2$ ). The proof for any pair of sets  $H_j^{k,p}$  and  $H_{j'}^{k,p'}$  with  $(j, p) \neq (j', p')$  is similar.

(2.a) If  $z \in B_{Z_k}(0, 1)$ , then for every  $j \geq 0$ ,

$$\max\{|e_{k+1}^*(z)|, |v_{j,k+1}^* \circ Q_{k,k+1}(z)|\} \leq \max\{\|e_{k+1}^*\| \|z\|, \|v_{j,k+1}^*\| \|Q_{k,k+1}\| \|z\|\} = \|z\| \leq 1.$$

and thus  $\pi_{j,k}(B_{Z_k}(0, 1)) \subset D = [-1, 1] \times [-1, 1] \subset U_0$  for all  $j \geq 0$ . This means that

$$B_{Z_k}(0, 1) \subset H_0^k.$$

For the second inclusion, if  $z \in H_0^k$  and  $j \geq 0$ ,

$$|e_{k+1}^*(z)| + |v_{j,k+1}^* \circ Q_{k,k+1}(z)| \leq 2,$$

and thus

$$\sup_j |v_{j,k+1}^* \circ Q_{k,k+1}(z)| \leq 2 - |e_{k+1}^*(z)|. \quad (2.2)$$

Also recall that

$$\delta \|Q_{k,k+1}(z)\|_{Z_{k+1}} \leq \sup_j |v_{j,k+1}^* \circ Q_{k,k+1}(z)|.$$

Thus from inequalities (2.1) and (2.2) we get

$$\begin{aligned} \|z\|_{Z_k} &\leq |e_{k+1}^*(z)| + 2 \|Q_{k,k+1}(z)\|_{Z_{k+1}} \leq |e_{k+1}^*(z)| + \frac{2}{\delta} \sup_j |v_{j,k+1}^* \circ Q_{k,k+1}(z)| \leq \\ &\leq |e_{k+1}^*(z)| + \frac{2}{\delta} (2 - |e_{k+1}^*(z)|) = (1 - \frac{2}{\delta}) |e_{k+1}^*(z)| + \frac{4}{\delta} \leq \frac{4}{\delta}. \end{aligned}$$

The last inequality holds because  $0 < \delta < 1$  and then  $1 - \frac{2}{\delta} < 0$ . This yields

$$H_0^k \subset B_{Z_k}(0, \frac{4}{\delta}).$$

(2.b) Let us select a point  $u_{j,k+1} \in Z_k$  such that  $Q_{k,k+1}(u_{j,k+1}) = v_{j,k+1}$  and  $e_{k+1}^*(u_{j,k+1}) = 0$ . This can be done by taking  $x \in X$  with  $Q_{k+1}(x) = v_{j,k+1}$ . Then  $v_{j,k+1} = Q_{k+1}(x) = Q_{k,k+1}(Q_k(x)) = Q_{k,k+1}(Q_k(x + V_{k+1})) = Q_{k,k+1}(Q_k(x) + \lambda \widehat{e}_{k+1})$  for any  $\lambda \in \mathbb{R}$ . Clearly there is  $\lambda_0$  such that  $e_{k+1}^*(Q_k(x) + \lambda_0 \widehat{e}_{k+1}) = 0$ . Then, we can take  $u_{j,k+1} = Q_k(x) + \lambda_0 \widehat{e}_{k+1}$ . If we define  $h_j^{k,1} = a \widehat{e}_{k+1} + b u_{j,k+1} \in Z_k$ , we have

$$\|Q_{k,k+1}(h_j^{k,1})\| = b \|v_{j,k+1}\| = b \leq 1 - r.$$

(2.c) Notice that  $\pi_{j,k}(h_j^{k,1}) = (a, b)$  for all  $j \geq 0$  and thus

$$\pi_{j,k}(B_{Z_k}(h_j^{k,1}, r)) \subset (a, b) + r([-1, 1] \times [-1, 1]) \subset U_1.$$

Moreover, if  $j' < j$ , then

$$\pi_{j',k}(B_{Z_k}(h_j^{k,1}, r)) \subset (a, b v_{j',k+1}^*(v_{j,k+1})) + r([-1, 1] \times [-1, 1]) \subset U_0$$

because the second coordinate satisfies  $|b v_{j',k+1}^*(v_{j,k+1})| \leq b \delta$  (we apply condition (L.c)). Thus

$$B_{Z_k}(h_j^{k,1}, r) \subset H_j^{k,1}, \quad \text{for all } j \geq 0.$$

(2.d) Set  $h_0^k = 0$ . If  $z \in H_0^k$  then by (2.1),

$$\|z\| \leq |e_{k+1}^*(z)| + 2\|Q_{k,k+1}(z)\| \leq 2 + 2\|Q_{k,k+1}(z)\|.$$

Now, if  $z \in H_j^{k,1}$  then by (2.1),

$$\begin{aligned} \|z - h_j^{k,1}\| &\leq |e_{k+1}^*(z - h_j^{k,1})| + 2\|Q_{k,k+1}(z - h_j^{k,1})\| \leq \\ &\leq a + 2\|Q_{k,k+1}(h_j^{k,1})\| + 2\|Q_{k,k+1}(z)\| = \\ &= a + 2\|bv_{j,k+1}\| + 2\|Q_{k,k+1}(z)\| = a + 2b + 2\|Q_{k,k+1}(z)\|. \end{aligned}$$

Since  $2 \leq a + 2b$  we get (2.d).

By symmetry, the proof of the above properties (2.a)-(2.c) for  $H_j^{k,2}$ ,  $H_j^{k,3}$ ,  $H_j^{k,4}$  is similar.

Finally to get a tiling of the entire space  $Z_k$  we consider the additional tiles

$$\{\tilde{H}_n^k = 4n\hat{e}_{k+1} + T_k : n \in \mathbb{Z} \setminus \{0\}\}$$

with associated points  $\tilde{h}_n^k := 4n\hat{e}_{k+1}$ . Then  $Q_{k,k+1}(\tilde{h}_n^k) = 0$ ,  $B_{Z_k}(\tilde{h}_n^k, r) \subset \tilde{H}_n^k$ . Also by (2.1),  $\|h - \tilde{h}_n^k\| \leq |e_{k+1}^*(h - \tilde{h}_n^k)| + 2\|Q_{k,k+1}(h - \tilde{h}_n^k)\| \leq 2 + 2\|Q_{k,k+1}(h)\|$  for all  $h \in \tilde{H}_n^k$ .

Thus the family

$$\{H_0^k, H_j^{k,p}, \tilde{H}_n^k : j \geq 0; p = 1, 2, 3, 4; n \in \mathbb{Z} \setminus \{0\}\}$$

is a tiling of  $Z_k$ . We relabel the tiles of this tiling and their associated points by  $\{H_j^k\}$  and  $\{h_j^k\}$  respectively, and in such a way that  $H_0^k$  remains the same tile. Then, this tiling has conditions (2.a)-(2.d). Notice that  $H_0^k$  is a bounded neighborhood of 0.

**Step 3.** [15, Theorem 9] *There is a tiling  $\{C_j^k\}_{j,k}$  of  $X$  (where  $k > 0 \Rightarrow j > 0$ ) and associated points  $\{x_j^k\}_{j,k} \subset X$  satisfying*

$$B(x_j^k + V_k, r) \subset C_j^k \subset B(x_j^k + V_k, R)$$

with  $x_0^0 = 0$  and  $R := a + 2b + \frac{8}{\delta}$ , (for the constants  $a, b, \delta$  and  $r$  considered in the previous steps).

*Proof.* Let us consider the tiling  $\{H_j^k\}_{j \geq 0}$  of  $Z_k$  given in Step 2 for every  $k \geq 0$  and define the sets, where  $k > 0 \Rightarrow j > 0$ , as follows

$$C_j^k = Q_k^{-1}(H_j^k) \cap \bigcap_{m=k+1}^{\infty} Q_m^{-1}(H_0^m).$$

The family  $\{C_j^k\}$  satisfies:

(3.a) It is a covering of  $X$ : for every  $x \in X$  we consider the smallest index  $k$  such that  $Q_m(x) \in H_0^m$  for all  $m > k$ . Notice that  $\|Q_k(x)\|_{Z_k} = \text{dist}(x, V_k) \rightarrow 0$  and thus  $x \in B_{Z_k}(0, 1) \subset H_0^k$  for  $k$  large enough. If  $k = 0$  then there is  $j \geq 0$  such that  $Q_0(x) = x \in H_j^0$  and thus  $x \in C_j^0$ . If  $k > 0$  then  $Q_k(x) \notin H_0^k$  and thus there is  $j > 0$  such that  $Q_k(x) \in H_j^k$ . Therefore  $x \in C_j^k$  with  $j > 0$ .

(3.b) The sets  $C_j^k$  have pairwise disjoint interiors. Indeed, suppose that there is an open ball  $\overset{\circ}{B}(x, \rho)$  (with  $\rho > 0$ ) such that  $\overset{\circ}{B}(x, \rho) \subset C_j^k \cap C_{j'}^k$  for some  $j \neq j'$ . Then  $Q_k(\overset{\circ}{B}(x, \rho)) \subset H_j^k \cap H_{j'}^k$ . Since  $Q_k$  is continuous, linear and surjective, it is open and thus  $Q_k(\overset{\circ}{B}(x, \rho))$  is a nonempty open set contained in  $H_j^k \cap H_{j'}^k$ , which is impossible, because  $H_j^k \cap H_{j'}^k$  has empty interior. Now, suppose that there is an open ball  $\overset{\circ}{B}(x, \rho)$  (with  $\rho > 0$ ) such that  $\overset{\circ}{B}(x, \rho) \subset C_j^k \cap C_{j'}^{k'}$  for some  $k < k'$ . Then,  $Q_k(\overset{\circ}{B}(x, \rho)) \subset H_j^k$  and  $Q_m(\overset{\circ}{B}(x, \rho)) \subset H_0^m$  for all  $m > k$ . In particular,  $Q_{k'}(\overset{\circ}{B}(x, \rho)) \subset H_0^{k'}$ . Since  $0 \leq k < k'$ , we have  $k' > 0$  and thus the associated  $j' > 0$ .

Also  $Q_{k'}(\overset{\circ}{B}(x, \rho)) \subset H_{j'}^{k'}$  and thus the nonempty open set  $Q_{k'}(\overset{\circ}{B}(x, \rho)) \subset H_0^{k'} \cap H_{j'}^{k'}$ , which is impossible because  $H_0^{k'} \cap H_{j'}^{k'}$  has empty interior.

(3.c) Select points  $x_j^k \in X$  such that  $x_j^k + V_k = Q_k^{-1}h_j^k$ . Notice that in particular we can choose  $x_0^0 = 0$ . Let us check that

$$B(x_j^k + V_k, r) \subset C_j^k \subset B(x_j^k + V_k, R).$$

Firstly, From condition (2.d) we have  $B_{Z_k}(h_j^k, r) \subset H_j^k$  and thus

$$\begin{aligned} Q_k^{-1}(B_{Z_k}(h_j^k, r)) &= \{x \in X : \text{dist}(x - x_j^k, V_k) \leq r\} = \\ &= x_j^k + V_k + B(0, r) = B(x_j^k + V_k, r) \subset Q_k^{-1}(H_j^k). \end{aligned}$$

Moreover, from condition (2.b) and for  $m > k$  we have that  $\|Q_m(x_j^k)\| \leq \|Q_{k+1}(x_j^k)\| = \|Q_{k,k+1}(h_j^k)\| \leq 1 - r$ . Therefore

$$Q_m(x_j^k + V_k + B(0, r)) = Q_m(x_j^k) + B_{Z_m}(0, r) \subset B_{Z_m}(0, 1) \subset H_0^m.$$

Thus  $x_j^k + V_k + B(0, r) \subset Q_m^{-1}(H_0^m)$ . This yields  $B(x_j^k + V_k, r) \subset C_j^k$ .

Secondly, if  $x \in C_j^k$  then  $Q_{k+1}(x) \in H_0^{k+1}$  and by condition (2.a) we have  $\|Q_{k+1}(x)\| \leq \frac{4}{\delta}$ . Since  $Q_k(x) \in H_j^k$  and applying condition (2.d) we have that  $\|Q_k(x - x_j^k)\| \leq a + 2b + 2\|Q_{k+1}(x)\| \leq a + 2b + 2\frac{4}{\delta} = a + 2b + \frac{8}{\delta}$ . This yields

$$C_j^k \subset B(Q_k^{-1}h_j^k, R) = B(x_j^k + V_k, R) = x_j^k + V_k + B(0, R) \quad \text{with } R = a + 2b + \frac{8}{\delta}.$$

**Step 4.** There is a tiling  $\{C_{i,j}^k\}_{i,j,k}$  of  $X$  (where “ $k > 0 \Rightarrow j > 0$ ” and “ $k = 0 \Rightarrow i = 0$ ”) and associated points  $\{d_i^k + x_j^k\}_{i,j,k} \subset X$  satisfying

$$B(d_i^k + x_j^k, r) \subset C_{i,j}^k \subset B(d_i^k + x_j^k, R'),$$

with  $d_0^0 = x_0^0 = 0$  and  $R' = a + 2b + \frac{8}{\delta} + 2r$ , (for the constants  $a, b, \delta$  and  $r$  considered in the previous steps).

*Proof.* In the last step of the proof of Theorem 2.2 we are going to define a refinement of the tiling of  $X$  obtained in Step 3 by tiling each tile. Note that each tile is a sort of “cylinder” around a finite dimensional subspace  $V_k$ . In order to do that, first consider a family  $\{d_i^k\}_i \subset V_k$  being maximal in  $V_k$  with respect to the property  $\|d_i^k - d_{i'}^k\| \geq 2r$  for  $i \neq i'$ . Notice that for  $k = 0$  we have  $V_0 = \{0\}$  and only one point  $d_i^0$  which is  $d_0^0 = 0$ . For  $k > 0$ ,  $i$  runs over  $\mathbb{N} \cup \{0\}$ . For every  $k \geq 0$  fixed, consider the “modified Voronoi cells”  $\{D_i^k\}_i$  in  $X$  associated with the set  $\{d_i^k\}_i$  and defined by

$$\begin{aligned} V_i^k &= \{x \in X : \|x - d_i^k\| \leq \|x - d_{i'}^k\| \text{ for all } i'\}, \\ D_i^k &= \overline{(V_i^k \setminus \bigcup_{i' < i} V_{i'}^k)} \subset V_i^k. \end{aligned}$$

(4.a) For every  $k$  fixed, the family  $\{D_i^k\}_i$  is a tiling of  $X$ . The proof is similar to the one given in [3, Lemma 2.2]) and [13, Theorem 3.1]. For  $k = 0$ , there is just one tile  $D_0^0 = X$ . For  $k > 0$ , since the family  $\{d_i^k\}_i \subset V_k$  is  $2r$ -separated and  $V_k$  is finite dimensional, the limit  $\lim_{i \rightarrow \infty} \|d_i^k\| = \infty$  and thus for every  $x \in X$  the  $\inf_i \|x - d_i^k\|$  is attained. Then  $x \in D_i^k$  being  $i$  the first  $i$  where the infimum is attained. Thus, the family  $\{D_i^k\}_i$  covers  $X$ . Also, the sets of the family  $\{D_i^k\}_i$  have pairwise disjoint interiors. Indeed, if  $i' < i$  and there is an open ball  $\overset{\circ}{B}(x, \rho) \subset D_i^k \cap D_{i'}^k$  (with  $\rho > 0$ ). Then  $\overset{\circ}{B}(x, \rho) \subset D_{i'}^k \subset V_{i'}^k$  and  $\overset{\circ}{B}(x, \rho) \subset D_i^k \subset \overline{V_i^k \setminus V_{i'}^k}$ . Thus,  $V_i^k \setminus \overset{\circ}{B}(x, \rho) \supset V_i^k \setminus V_{i'}^k$ . Since the set  $V_i^k \setminus \overset{\circ}{B}(x, \rho)$  is closed we have  $V_i^k \setminus \overset{\circ}{B}(x, \rho) \supset \overline{V_i^k \setminus V_{i'}^k}$ . This is a contradiction with the fact that  $\overset{\circ}{B}(x, \rho) \subset \overline{V_i^k \setminus V_{i'}^k}$ .

(4.b) In addition, let us see that the sets of the family  $\{D_i^k\}_i$  are starlike. In fact, we will check that  $D_i^k$  is starlike with respect to  $d_i^k$ . Since the closure of a starlike set is starlike, it is enough to check that  $V_i^k \setminus \bigcup_{i' < i} V_{i'}^k$  is starlike. If  $x \in V_i^k \setminus \bigcup_{i' < i} V_{i'}^k$ , then

$$\|x - d_i^k\| \leq \|x - d_{i'}^k\| \quad \text{for all } i' \quad \text{and} \quad \|x - d_i^k\| < \|x - d_{i'}^k\| \quad \text{for } i' < i.$$

For all  $t \in [0, 1]$  we have that  $x_t = tx + (1-t)d_i^k$  satisfies

$$\|x_t - d_i^k\| = \|x - d_i^k\| - \|x - x_t\| \quad \text{and} \quad \|x - d_{i'}^k\| \leq \|x - x_t\| + \|x_t - d_{i'}^k\| \quad \text{for all } i'.$$

Thus,

$$\begin{aligned} \|x_t - d_i^k\| &= \|x - d_i^k\| - \|x - x_t\| \leq \|x - d_i^k\| + \|x_t - d_{i'}^k\| - \|x - d_{i'}^k\| \leq \|x_t - d_{i'}^k\| \quad \text{for all } i', \\ \|x_t - d_i^k\| &= \|x - d_i^k\| - \|x - x_t\| \leq \|x - d_i^k\| + \|x_t - d_{i'}^k\| - \|x - d_{i'}^k\| < \|x_t - d_{i'}^k\| \quad \text{for } i' < i. \end{aligned}$$

Thus  $x_t \in V_i^k \setminus \bigcup_{i' < i} V_{i'}^k$ .

Also, it is worth noting that in the case that the norm is strictly convex we could consider the sets  $V_i^k$  in place of the sets  $D_i^k$  because the sets  $V_i^k$  have pairwise disjoint interior ([13, Theorem 3.1]). Also if the norm is locally uniformly rotund then the sets  $D_i^k$  coincide with the sets  $V_i^k$  because for any fixed  $k$  the sets of the family  $\{V_i^k\}_i$  have pairwise disjoint interiors and each  $V_i^k$  is the closure of its interior ([13, Theorem 3.1]).

(4.c) Moreover,  $B(d_i^k, r) \subset D_i^k$ . Indeed, if  $x \in X$  and  $\|x - d_i^k\| < r$  then for  $i' \neq i$  we have

$$\|x - d_{i'}^k\| = \|(d_i^k - d_{i'}^k) + (x - d_i^k)\| \geq \|d_i^k - d_{i'}^k\| - \|x - d_i^k\| > 2r - r = r > \|x - d_{i'}^k\|$$

and

$$x \in V_i^k \setminus \bigcup_{i' \neq i} V_{i'}^k \subset V_i^k \setminus \bigcup_{i' < i} V_{i'}^k.$$

Thus the closed ball

$$B(d_i^k, r) \subset \overline{V_i^k \setminus \bigcup_{i' < i} V_{i'}^k} = D_i^k.$$

(4.d) Take the intersection  $C_{i,j}^k = (x_j^k + D_i^k) \cap C_j^k$  and the family  $\{C_{i,j}^k\}_{i,j,k}$  (where “ $k > 0 \Rightarrow j > 0$ ” and “ $k = 0 \Rightarrow i = 0$ ”). Let us see some of the properties of this family:

★ The family  $\{C_{i,j}^k\}_{i,j,k}$  covers  $X$ : if  $x \in X$ , then there is  $C_j^k$  such that  $x \in C_j^k$  (because  $\{C_j^k\}_{j,k}$  is a covering of  $X$ ). Now, for  $k, j$  fixed,  $\{(x_j^k + D_i^k) \cap C_j^k\}_i$  is a covering of  $C_j^k$  and thus there is  $i$  such that  $x \in (x_j^k + D_i^k) \cap C_j^k$ .

★ Each set  $C_{i,j}^k$  is starlike with respect to  $x_j^k + d_i^k$ : Suppose that  $x \in C_{i,j}^k$ . Since  $C_j^k$  is convex,  $x \in C_j^k$  and  $x_j^k + d_i^k \in x_j^k + V_k \subset C_j^k$ , we have that  $t(x_j^k + d_i^k) + (1-t)x \in C_j^k$  for every  $0 \leq t \leq 1$ . Also, the translated set  $x_j^k + D_i^k$  is starlike with respect to  $x_j^k + d_i^k$  and thus  $t(x_j^k + d_i^k) + (1-t)x \in x_j^k + D_i^k$  for all  $0 \leq t \leq 1$ . Therefore  $t(x_j^k + d_i^k) + (1-t)x \in C_{i,j}^k$  for all  $0 \leq t \leq 1$ .

★  $B(x_j^k + d_i^k, r) \subset C_{i,j}^k$ . Indeed, from Step 3 we get

$$B(x_j^k + d_i^k, r) = x_j^k + d_i^k + B(0, r) \subset x_j^k + V_k + B(0, r) \subset C_j^k.$$

Also, from (4.c) we get

$$B(x_j^k + d_i^k, r) = x_j^k + B(d_i^k, r) \subset x_j^k + D_i^k.$$

★  $C_{i,j}^k \subset B(x_j^k + d_i^k, R + 2r)$ . Indeed, suppose there is  $x \in C_{i,j}^k$  satisfying that  $\|x - (x_j^k + d_i^k)\| > R + 2r$ . Since  $x \in C_j^k \subset x_j^k + V_k + B(0, R)$ , there is a vector  $v \in V_k$  such that  $\|x - (x_j^k + v)\| \leq R$ . For this  $v \in V_k$  there is  $d_{i'}^k \in V_k$  such that  $\|v - d_{i'}^k\| \leq 2r$ . Now,

$$\begin{aligned} \|(x - x_j^k) - d_{i'}^k\| &= \|x - x_j^k - d_{i'}^k\| = \|x - x_j^k - v + v - d_{i'}^k\| \leq \|x - x_j^k - v\| + \|v - d_{i'}^k\| \leq R + 2r < \\ &< \|x - (x_j^k + d_i^k)\| = \|(x - x_j^k) - d_i^k\|. \end{aligned}$$



Therefore  $x - x_j^k \notin V_i^k$  and thus  $x - x_j^k \notin D_i^k$ , i.e.  $x \notin x_j^k + D_i^k$  and  $x \notin (x_j^k + D_i^k) \cap C_j^k = C_{i,j}^k$ , which is not true.

★ The sets of the family  $\{C_{i,j}^k\}_{i,j,k}$  have pairwise disjoint interiors. Indeed, this assertion follows from the fact that every  $k \geq 0$  fixed, the sets  $\{D_i^k\}_i$  have pairwise disjoint interiors and the sets  $\{C_j^k\}_{j,k}$  have pairwise disjoint interiors. Suppose there is a ball  $B(x, s) \subset C_{i,j}^k \cap C_{i',j'}^{k'}$  with  $x \in X$  and  $s > 0$ .

(a) If  $(j, k) \neq (j', k')$ , then  $B(x, s) \subset C_j^k \cap C_{j'}^{k'}$  and thus the interior of  $C_j^k \cap C_{j'}^{k'} \neq \emptyset$ , which is not true because of (3.b).

(b) If  $(j, k) = (j', k')$  and  $i \neq i'$  then  $B(x, s) \subset (x_j^k + D_i^k) \cap (x_j^k + D_{i'}^k) = x_j^k + D_i^k \cap D_{i'}^k$  and thus the interior of  $D_i^k \cap D_{i'}^k \neq \emptyset$ , which is not true because of (4.a).

Notice that the constant of normality  $K$  can be chosen any number  $K > \frac{R+2r}{r} = 2 + \frac{1}{r}(a + 2b + \frac{8}{\delta})$ .

This finishes the proof of Theorem 2.2.  $\square$

**Remark 2.3.** Note that in the proof of Theorem 2.2, for  $k = 0$  the vector space  $V_0 = \{0\}$  and the sets  $C_j^0$  verify that  $B(x_j^0, r) \subset C_j^0 \subset B(x_j^0, R)$ . Also, there is just one  $d_i^0$  which is  $d_0^0 = 0$ . Therefore the tiles  $C_j^0$  of Step 3 remain the same after the refinement given in Step 4 and thus, the tiles  $C_{0,j}^0 = C_j^0$  are convex and satisfy  $B(x_j^0, r) \subset C_{0,j}^0 = C_j^0 \subset B(x_j^0, R)$ . In particular, since  $x_0^0 = 0$  we get that  $B(0, r) \subset C_{0,0}^0 = C_0^0 = B(0, R)$  and thus the “first tile”  $C_{0,0}^0$  is a bounded convex neighborhood of 0.

**Remark 2.4.** Let us give some estimations of the constant of normality  $K$ . For example, for  $a = 1.3$ ,  $b = 0.9$ ,  $r = 0.1$  and  $\delta = \frac{5}{9}$  we get  $\frac{R+2r}{r} = 177$ . Thus, we get a constant of normality  $K > 177$ . For other choices of  $U_0$  in Lemma 2.1 it is possible to obtain better constants of normality. For example, let us redefine  $U_0$  as  $U_0 = \{(x, y) \in \mathbb{R}^2 : |x| + 2|y| \leq 3, |x| \leq 2\}$  and modify conveniently the sets  $U_i$  for  $i = 1, 2, 3, 4$ . That allows to have  $H_0^k \subset B(0, \frac{3}{\delta})$  for all  $k$ , which yields to a constant  $R = a + 2b + \frac{6}{\delta}$ . Now, for example for  $a = 1.8$ ,  $b = 0.8$ ,  $r = \frac{0.4}{3}$  and  $\delta = 0.5$ , we get  $\frac{R+2r}{r} = 117.5$  and thus a constant of normality  $K > 117.5$ .

In the next Corollary, instead of using “modified Voronoi cells” to slice every “cylinder”  $C_j^k$  of Theorem 2.2 we could fix a natural number  $N$  and use the linear projections  $P_k : X \rightarrow V_k$ ,  $P_k(x) = \sum_{m=1}^k e_m^*(x)e_m$  for  $x \in X$  and  $0 < k \leq N$  to slice the tiles  $C_j^k$  for  $0 < k \leq N$  and all  $j$ . For  $k = 0$  and  $k > N$  we use the “modified Voronoi cells” as before. Since the constants  $R_N := \sup_{0 < k \leq N} \|P_k\| < \infty$ , the refinement of the tiling  $\{C_j^k\}$  would produce a normal tiling  $\{\tilde{C}_{i,j}^k\}$  defined in the following proposition.

**Proposition 2.5.** Let  $(X, \|\cdot\|)$  be a separable infinite dimensional Banach space. Following the notation in the proof of Theorem 2.2, consider for any fixed natural number  $N$  the linear projections  $P_k : X \rightarrow V_k$ ,  $P_k(x) = \sum_{m=1}^k e_m^*(x)e_m$  for  $0 < k \leq N$  and all  $x \in X$ . Then, there is a tiling  $\{\tilde{C}_{i,j}^k\}_{i,j,k}$  of  $X$  (where “ $k > 0 \Rightarrow j > 0$ ” and “ $k = 0 \Rightarrow i = 0$ ”) so that:

- The tiles  $\tilde{C}_{i,j}^k$  are convex for  $k \leq N$  and all  $i, j$ . In particular for  $0 < k \leq N$ ,  $\tilde{C}_{i,j}^k = P_k^{-1}(\tilde{D}_i^k) \cap C_j^k$  where  $\{\tilde{D}_i^k\}_i$  is a normal and convex tiling of  $V_k$ . Also,  $\tilde{C}_{0,j}^0 = C_j^0$  for all  $j$  and the tile  $C_{0,0}^0 = C_0^0$  is a bounded and convex neighborhood of 0.
- The tiles  $\tilde{C}_{i,j}^k = C_{i,j}^k = (x_j^k + D_i^k) \cap C_j^k$  are starlike for  $k > N$  and all  $i, j$ , where the points  $\{x_j^k\}_{j,k}$  are defined in Step 3 (of Theorem 2.2) and the normal and starlike tiling  $\{D_i^k\}_i$  of  $V_k$  is defined in Step 4 (of Theorem 2.2).
- The tiling  $\{\tilde{C}_{i,j}^k\}_{i,j,k}$  of  $X$  is normal.

*Proof.* (a) Let us consider the tiling  $\{C_j^k\}_{j,k}$  of  $X$  constructed in Step 3 in the proof of Theorem 2.2 and follow the notation of that proof. Define  $\tilde{C}_{i,j}^k = P_k^{-1}(\tilde{D}_i^k) \cap C_j^k$  for  $0 < k \leq N$  and all  $i, j$  where  $\{\tilde{D}_i^k\}_i$  is a normal and convex tiling of  $V_k$  such that  $B_{V_k}(\tilde{d}_i^k, r) \subset \tilde{D}_i^k \subset B_{V_k}(\tilde{d}_i^k, r_k)$  for some  $r_k \geq r$  and suitable points  $\tilde{d}_i^k \in V_k$ . Since  $\{e_m\}_m$  is a fundamental biorthogonal system of  $X$ , we

have that  $X = [\{e_m\}_{m=1}^k] \oplus [\{e_m\}_{m=k+1}^\infty]$ , where  $[H]$  denotes the closed linear span of the set  $H$ . Thus, can assume that the points  $x_j^k$  in the proof of the Theorem 2.2 have been chosen so that  $x_j^k \in [\{e_i : i > k\}]$  for  $0 < k \leq N$ . Now if  $x \in \tilde{C}_{i,j}^k$ , let us check that

$$B(x_j^k + \tilde{d}_i^k, \frac{r}{\|P_k\|}) \subset \tilde{C}_{i,j}^k \subset B(x_j^k + \tilde{d}_i^k, R(1 + \|P_k\|) + r_k).$$

Recall that  $R = a + 2b + \frac{8}{\delta}$ , where  $a, b, \delta$  are the positive constants chosen in Lemma 2.1. Firstly, take  $x \in \tilde{C}_{i,j}^k \subset C_j^k \subset x_j^k + V_k + B(0, R)$  and  $v \in V_k$  such that  $\|x - x_j^k - v\| = \text{dist}(x - x_j^k, V_k) \leq R$ . Then,

$$\begin{aligned} \|x - (x_j^k + \tilde{d}_i^k)\| &= \|(x - x_j^k - v) + (v - \tilde{d}_i^k)\| \leq R + \|(v - P_k(x)) + (P_k(x) - \tilde{d}_i^k)\| \leq \\ &R + \|P_k(v - x)\| + r_k = R + \|P_k(v - x + x_j^k)\| + r_k \leq \\ &\leq R + \|P_k\|R + 2r_k \leq R(1 + \|P_k\|) + r_k. \end{aligned}$$

Secondly, if  $x \in B(x_j^k + \tilde{d}_i^k, \frac{r}{\|P_k\|})$ , then  $P_k(x) \in B_{V_k}(\tilde{d}_i^k, r)$  and  $Q_k(x) \in B_{Z_k}(\hat{x}_j^k, \frac{r}{\|P_k\|}) \subset B_{Z_k}(\hat{x}_j^k, r)$ . Thus,  $P_k(x) \in \tilde{D}_i^k$  and  $x \in B(x_j^k + V_k, r) \subset C_j^k$ . This yields  $x \in \tilde{C}_{i,j}^k$ . Therefore if  $t_N := \sup\{r_k : 0 < k \leq N\}$  and  $R_N := \sup_{0 < k \leq N} \|P_k\| < \infty$  we have

$$B(x_j^k + \tilde{d}_i^k, \frac{r}{R_N}) \subset \tilde{C}_{i,j}^k \subset B(x_j^k + \tilde{d}_i^k, R(1 + R_N) + t_N) \quad \text{for } 0 < k \leq N. \quad (2.3)$$

For  $k = 0$  and  $k > N$ , the sets  $\tilde{C}_{i,j}^k$  are the sets  $C_{i,j}^k$  defined in the proof of Theorem 2.2. Also, the points  $x_j^k$  and  $d_i^k$  are the points defined in the proof of Theorem 2.2. Recall that they satisfy

$$B(x_j^k + d_i^k, r) \subset \tilde{C}_{i,j}^k \subset B(x_j^k + d_i^k, R + 2r) \quad \text{for } k = 0 \text{ and } k > N. \quad (2.4)$$

The fact that  $\{\tilde{C}_{i,j}^k\}_{i,j,k}$  is a tiling of  $X$  can be proved as in Theorem 2.2. Assertion (b) is a direct consequence of the proof of Theorem 2.2. Because of the fact that  $\frac{r}{R_N} \leq r < R + 2r < R + 2 < R + R \leq R(1 + R_N) < R(1 + R_N) + t_N$  and the inclusions in (2.3) and (2.4), it follows that the tiling  $\{\tilde{C}_{i,j}^k\}_{i,j,k}$  is  $K$ -normal for any constant  $K > \frac{R_N}{r}(R(1 + R_N) + t_N)$  and assertion (c) is proved.  $\square$

**Remark 2.6.** A result of M.I. Kadec and M.G. Snobar [4, pg. 243 and pg. 320] establishes that for any  $n$ -dimensional space  $V$  and any Banach space  $X$  containing  $V$  the constant

$$\lambda(V, X) = \inf\{\|P\| : P : X \rightarrow V \text{ is a linear projection from } X \text{ to } V\} \leq \sqrt{n}.$$

Thus, replacing the projections  $\{P_k : X \rightarrow V_k\}_{k=1}^N$  with suitable projections  $\{\tilde{P}_k : X \rightarrow V_k\}_{k=1}^N$  in Proposition 2.5, we can get  $R_N$  as close as needed to  $\sqrt{N}$ . Also recall that in the introduction it was mentioned that any  $n$ -dimensional space  $V$  has a  $2\sqrt{n}$ -normal convex tiling and thus the constant  $t_N$  in the proof of Proposition 2.5 can be taken to satisfy  $t_N \leq 2\sqrt{N}$ . In this case (by modifying the points  $x_j^k$  for  $0 < k \leq N$  to be in  $\ker \tilde{P}_k$ ) a coarse estimation for the constant of normality  $K > \frac{R_N}{r}(R(1 + R_N) + t_N)$  given in the proof of Proposition 2.5 will be  $K > \frac{1}{r}[R(N + \sqrt{N}) + 2N]$  with  $R = a + 2b + \frac{8}{\delta}$ .

Recall that in any finite dimensional space  $(X, \|\cdot\|)$  there is a 2-normal and starlike tiling [3]. Recall also that it is an open problem whether there is  $S > 0$  such that every finite dimensional Banach space  $(X, \|\cdot\|)$  has  $S$ -normal and convex tilings. We finish this section with a review of the steps 2 to 4 in the proof of Theorem 2.2 for the finite dimensional setting. We adapt it to get  $K$ -normal tilings such that some tiles are convex and the rest of them are starlike being  $K = 2 + \frac{1}{r}(a + 2b + \frac{8}{\delta})$  independently of the dimension of the space.

**Proposition 2.7.** For every space  $(X, \|\cdot\|)$  of dimension  $M$  there is a  $K$ -normal starlike tiling  $\{C_{i,j}^k\}_{i,j,k}$  (indexed with  $0 \leq k \leq M - 1$ , where “ $k > 0 \Rightarrow j > 0$ ” and “ $k = 0 \Rightarrow i = 0$ ”) such that

- (a) the tiles  $\{C_{0,j}^0\}_j$  are convex,  
 (b) the tile  $C_{0,0}^0$  is a convex neighborhood of 0,  
 (c) the constant  $K$  can be taken as  $K = 2 + \frac{1}{r}(a + 2b + \frac{8}{\delta})$  independently of  $M$  (the constants  $a, b, r, \delta$  are given by Lemma 2.1 in the proof of Theorem 2.2).

*Proof.* Let us review the 4 steps of the proof of Theorem 2.2 for the  $M$ -dimensional space  $X$ . In the following we can assume  $M \geq 2$ . Firstly, we consider an Auerbach basis, i.e. a normalized basis  $\{e_i\}_{i=1}^M$  of  $X$  with associated normalized biorthogonal functionals  $\{e_i^*\}_{i=1}^M$ . We consider the subspaces  $V_0 = \{0\}$  and  $V_k = [\{e_1, \dots, e_k\}]$  for  $0 < k \leq M$ , the quotient spaces  $Z_k = X/V_k$  and the natural quotients  $Q_k : X \rightarrow Z_k$ . Also, consider  $Q_{k,k+1} : Z_k \rightarrow Z_{k+1}$  for  $k+1 \leq M$ .

- (1) Step 1 in the proof of Theorem 2.2 provides a finite and normalized family  $\{v_{j,k}^*, v_{j,k}\}_{j=0}^{m_k} \subset Z_k^* \times Z_k$  for  $0 \leq k \leq M-1$  and certain integer numbers  $m_k \geq 0$  satisfying conditions (1.a)-(1.c).  
 (2) In Step 2 of the proof of Theorem 2.2, the surjective functions  $\pi_{j,k} : Z_k \rightarrow \mathbb{R}^2$ , the sets  $T_k, H_0^k, H_j^{k,p}, \tilde{H}_n^k$  and the associated points  $h_j^{k,p}, \tilde{h}_n^k$  are defined in the same way for  $0 \leq k \leq M-2$ ,  $p = 1, 2, 3, 4$ ,  $n \in \mathbb{Z} \setminus \{0\}$  and  $0 \leq j \leq m_k$ . The only difference is that the sets  $H_0^k$  are defined by the finite intersection

$$H_0^k = \bigcap_{0 \leq j \leq m_k} \pi_{j,k}^{-1}(U_0) \quad \text{for } 0 \leq k \leq M-2.$$

By relabeling  $\{H_0^k, H_j^{k,p}, \tilde{H}_n^k\}$  and their associated points we get the convex tiling  $\{H_j^k\}_{j=0}^\infty$  of  $Z_k$  with associated points  $\{h_j^k\}_{j=0}^\infty$  (where  $h_0^k = 0$ ) satisfying properties (2.a) to (2.d). Now, for  $k = M-1$ , we define

$$\begin{aligned} \pi_{0,M-1} : Z_{M-1} &\rightarrow \mathbb{R}, & \pi_{0,M-1}(z) &= e_M^*(z) \quad \text{for } z \in Z_{M-1}, \\ T_{M-1} &= \{z \in Z_{M-1} : |e_M^*(z)| \leq 2\} = B_{Z_{M-1}}(0, 2), \\ \tilde{H}_n^{M-1} &= 4n\hat{e}_M + T_{M-1} \quad \text{for } n \in \mathbb{Z}, \end{aligned}$$

where  $\tilde{H}_n^{M-1}$  has the associated point  $4n\hat{e}_M$  for all  $n \in \mathbb{Z}$ . Again, by relabeling  $\{\tilde{H}_n^{M-1}\}$  and the associated points, we get the convex tiling  $\{H_j^{M-1}\}_{j=0}^\infty$  of  $Z_{M-1}$  with associated points  $\{h_j^{M-1}\}_{j=0}^\infty$  (where  $h_0^{M-1} = 0$ ) satisfying the properties

- (2.a')  $B_{Z_{M-1}}(0, 2) = H_0^{M-1}$ ,  
 (2.b')  $Q_{M-1,M}(h_j^{M-1}) = 0 = Q_{M-1,M}(h)$  for all  $h \in H_j^{M-1}$ ,  
 (2.c')  $B_{Z_{M-1}}(h_j^{M-1}, 2) = H_j^{M-1}$  and thus  
 (2.d')  $\|h - h_j^{M-1}\| \leq 2$  for every  $h \in H_j^{M-1}$ .

This is just a tiling in the 1-dimensional space  $Z_{M-1}$  by closed intervals of length 4.

- (3) Step 3 in the proof of Theorem 2.2 provides a tiling  $\{C_j^k\}_{j,k}$  of  $X$  (where  $0 \leq k \leq M-1$  and  $j \geq 0$ ; also “ $k > 0 \Rightarrow j > 0$ ”) and associated points  $\{x_j^k\}_{j,k} \subset X$  such that

$$B(x_j^k + V_k, r) \subset C_j^k \subset B(x_j^k + V_k, R) \quad \text{for } 0 \leq k \leq M-1,$$

with same  $x_0^0 = 0$  and same constants  $r > 0$  and  $R = a + 2b + \frac{8}{\delta}$ . In this case, the only difference is the definition of the sets  $C_j^k$  as the finite intersection

$$C_j^k = Q_k^{-1}(H_j^k) \cap \bigcap_{m=k+1}^M Q_m^{-1}(H_0^m), \quad \text{for } 0 \leq k \leq M-1,$$

where we define  $H_0^M = \{0\}$ . Clearly, for  $k = M-1$  and  $j > 0$  the sets  $C_j^{M-1} = Q_{M-1}^{-1}(H_j^{M-1}) = \{x \in X : |e_M^*(x) - 4n_j| \leq 2\}$  for certain  $n_j \in \mathbb{Z} \setminus \{0\}$ . So for  $k = M-1$  and  $j > 0$ ,

$$C_j^{M-1} = B(x_j^{M-1} + V_{M-1}, 2) = x_j^{M-1} + V_{M-1} + B(0, 2).$$

- (4) Step 4 in the proof of Theorem 2.2 provides a starlike tiling  $\{C_{i,j}^k\}_{i,j,k}$  of  $X$  (where  $0 \leq k \leq M-1$  and  $j \geq 0$ ; also “ $k > 0 \Rightarrow j > 0$ ” and “ $k = 0 \Rightarrow i = 0$ ”) and associated points  $\{d_i^k + x_j^k\} \subset X$  satisfying

$$B(d_i^k + x_j^k, r) \subset C_{i,j}^k \subset B(d_i^k + x_j^k, R'), \quad (2.5)$$

with same points  $d_0^0 = x_0^0 = 0$  and same constants  $r$  and  $R' = a + 2b + \frac{8}{\delta} + 2r$ . Indeed, in order to get the above tiling, the points  $\{d_i^k\}_i \subset V_k$ , the sets  $\{V_i^k\}_i$  and  $\{D_i^k\}_i$  are defined in the same way for  $0 \leq k \leq M-1$ . Moreover, the tiles  $C_{i,j}^k = (x_j^k + D_i^k) \cap C_j^k$  are defined in the same way for  $0 \leq k \leq M-1$ .

Finally, this tiling  $\{C_{i,j}^k\}_{i,j,k}$  verifies that the tiles  $C_{0,j}^0 = (x_j^0 + D_0^0) \cap C_j^0 = C_j^0$  are convex for all  $j$  and the tile  $C_{0,0}^0$  is a convex neighborhood of 0 because  $d_0^0 + x_0^0 = 0$ . So we get (a) and (b) in the statement of Proposition 2.7.

From the inclusions in (2.5), we can take the constant of normality as  $K = 2 + \frac{1}{r}(a + 2b + \frac{8}{\delta})$  and get (c) in the statement of Proposition 2.7. □

**Remark 2.8.** *An analogous result to Proposition 2.5 can be established in the case of any finite dimensional space  $(X, \|\cdot\|)$  if we fix a natural number  $N$  and consider any space of dimension  $M > N$ . By slicing the “cylinders”  $C_j^k$  in the proof of Proposition 2.7 for  $0 < k \leq N$  with similar projections  $P_k$  to those given in the proof of Proposition 2.5 and for  $k > 0$  and  $N < k < M$  by using “modified Voronoi cells” as in the proof of Theorem 2.2, we get a normal tiling  $\{\tilde{C}_{i,j}^k\}_{i,j,k}$  (labelled in the same way as in the proof of Proposition 2.7) with normality constant not depending on the dimension  $M > N$  (although it depends on  $N$ ), where the tiles  $C_{i,j}^k$  are convex for  $0 \leq k \leq N$  and the rest of them are starlike.*

**Remark 2.9.** *It is worth mentioning a result of G. Pisier [14], asserting that there is a separable infinite dimensional Banach space  $(X, \|\cdot\|)$  and a number  $s > 0$  such that every finite rank projection  $P$  from  $X$  into  $X$  satisfies  $\|P\| \geq s\sqrt{\text{rank } P}$ . So in this particular situation it is not clear how to adapt the construction of Preiss by using projections over finite dimensional subspaces of  $X$ .*

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