OPERATOR RANGES AND ENDOMORPHISMS WITH A PRESCRIBED BEHAVIOUR ON BANACH SPACES

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Abstract. We obtain several extensions of a theorem of Shevchik which asserts that if R is a proper dense operator range in a separable Banach space E, then there exists a compact, one-to-one and dense-range operator $T : E \to E$ such that $T(E) \cap R = \{0\}$, and some results of Chalendar and Partington concerning the existence of compact, one-to-one and dense-range endomorphisms on a separable Banach space E which leave invariant a given closed subspace $Y \subset E$, or more generally, a countable increasing chain of closed subspaces of E.

1. Introduction

A linear subspace R of a Banach space E is said to be an **operator range** (in E) if there exist a Banach space X and a bounded linear operator $T : X \to E$ such that $R = T(X)$. If $X = E$, we say that R is an **endomorphism range** in E. These subspaces possess rather surprising disjointness properties. A classical theorem of von Neumann, reformulated by Dixmier in terms of operator ranges (see e.g. [11, Theorem 3.6]), yields that if R is a non-closed endomorphism range in a separable Hilbert space H, then there exists a unitary operator $T : H \to H$ such that R and (the endomorphism range) $T(R)$ are essentially disjoint, that is, $R \cap T(R) = \{0\}$. This theorem implies that every separable Banach space contains a couple of dense essentially disjoint operator ranges [5, Proposition 2.6]. A strengthening of this result was obtained by Shevchik [19], who proved that if R is a proper dense operator range in a separable Banach space E, then there exists a compact, one-to-one and dense-range operator $T : E \to E$ such that $T(E) \cap R = \{0\}$. Such operator is actually **nuclear**, that is, there exist sequences $\{e_n\}_n \subset E$ and $\{f_n\}_n \subset E^*$ (the dual space of E) such that $\sum_{n\geq 1} \|e_n\|\|f_n\| < \infty$ and $T(x) = \sum_{n\geq 1} f_n(x)e_n$ for all $x \in E$. It is worth to mention that, according to a result of Bennet and Kalton [2], the operator range R in this result can be replaced with non-barrelled dense subspaces of E. However, this assertion is not true if R is an arbitrary (infinite-codimensional) linear subspace of E: as it was pointed out by Drewnowski [8], every Banach space E contains a dense (barrelled) subspace $V \subset E$ which is not essentially disjoint with respect to the range of any one-to-one operator $T : X \to E$ defined on an infinite-dimensional Banach space X. For further developments related to essential disjointness of operator ranges, we refer to the works [6], [9], [16] and [18].

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Some other kinds of constructions, concerning the existence of compact, one-to-one and dense-range endomorphisms on separable Banach spaces which leave invariant closed infinite-dimensional subspaces of such spaces, were carried out by Yahagi [23], and by Chalendar and Partington [7]. In particular, Theorem 2.1 in [7] yields that if Y is a closed infinite-dimensional subspace of a separable Banach space E, then there exists a compact (in fact nuclear) one-to-one and dense-range operator $T : E \to E$ such that $T(Y)$ is a dense subspace of Y. A generalization of this theorem, where the subspace Y is replaced with a sequence $\{Y_n\}_n$ of closed subspaces of E such that $Y_n \subset Y_{n+1}$ for all $n \geq 1$, was also obtained in [7, Theorem 2.2]. For more information on the existence of operators with other striking properties on separable Banach spaces we refer to the paper of Grivaux [14].

In this work, we establish several extensions of the aforementioned theorems of Shevchik and Chalendar and Partington. The proofs of these results, to be stated in sections 3, 4 and 5, rely on the existence of M-bases with some special features in separable Banach spaces, and some stability properties of minimal sequences in Banach spaces with a "disjoint behaviour" with respect to infinite-codimensional operator ranges, or more generally, countable unions of infinite-codimensional operator ranges in those spaces, to which we devote the next section. In the sequel, we denote by $\mathcal{R}(E)$ the family made up of all infinite-codimensional operator ranges in a real Banach space E, and by $\mathcal{S}(E)$ the class of sets of the form $\bigcup_{n\geq 1} R_n$, where R_n are elements of $\mathcal{R}(E)$. It is worth to mention (see e.g. [1, Corollary 2.17]) that if R is a proper dense operator range in a Banach space E, then $R \in \mathcal{R}(E)$.

Section 3 deals with some generalizations of Shevchik's theorem. The first result of that section ensures the existence of a nuclear and dense-range endomorphism T on a separable Banach space E such that $T(E)$ is essentially disjoint with respect to a given set $R \in \mathcal{S}(E)$ and $\overline{T^*(E^*)}$ (being T^* the adjoint operator of T) fills a given closed separable total subspace $Z \subset E^*$. Recall that a linear subspace F of the dual of a Banach space E is said to be **total** if F is w^*-dense in E^*. The construction of the operator T in that result also yields that $T^*(E^*) \cap V = \{0\}$, for a given set $V \in \mathcal{S}(Z)$. Next, we obtain two more extensions of Shevchik's theorem, both of which imply that if E is a separable Banach space, then for any $R \in \mathcal{S}(E)$ and $V \in \mathcal{S}(E^*)$ there exists a nuclear, one-to-one and dense-range endomorphism $T : E \to E$ such that $T(E) \cap R = \{0\}$, $T^*(E^*) \cap V = \{0\}$ and $T^*(E^*)$ is (not only total but) λ-norming for E, for any given number $\lambda \in (0, 1)$. Recall that if $\lambda \in (0, 1]$, then a linear subspace $F \subset E^*$ is said to be **λ-norming** (for E) if $\sup\{f(x) : f \in F, \|f\| \leq 1\} \geq \lambda\|x\|$ for each $x \in E$. The latter result yields that if E is a separable Banach space, then for every proper dense operator range $R \subset E^*$ there is a dense operator range $V \subset E^*$ which is isomorphic to R and satisfies $R \cap V = \{0\}$. This (non-separable) weak version of Shevchik's theorem provides a refinement of a result of Plichko [18], who proved the existence of two dense essentially disjoint operator ranges in ℓ^∞, answering a question possed by Borwein and Tingley [3].

In Sections 4 and 5, we obtain several extensions of the aforementioned theorems of Chalendar and Partington, which yield as well some disjointness properties of the involved operators. In the first result of Section 4, we consider a closed infinite-dimensional and infinite-codimensional subspace Y of a separable Banach space E, and provide conditions on two sets $R \in \mathcal{S}(E)$ and $V \in \mathcal{S}(E^*)$ to ensure the existence of a nuclear, one-to-one and dense-range operator $T : E \to E$ which, among other properties, satisfies that $\overline{T(Y)} = Y$, $T(E) \cap R = \{0\}$, $T^*(E^*) \cap V = \{0\}$ and the subspace $F = T^*(E^*)|_Y \subset Y^*$ (the restriction to Y of $T^*(E^*)$) is λ-norming for Y, for any given number $0 < \lambda < 1$. We also prove that if X and Y are **quasicomplemented subspaces** of a separable Banach space E (that is, if $X \cap Y = \{0\}$ and $X + Y$ is dense in E), then there exists a nuclear, one-to-one and dense-range operator $T : E \to E$ such that $\overline{T(X)} = X$, $\overline{T(Y)} = Y$ and $T(E)$ and $T^*(E^*)$ are essentially disjoint with respect to certain sets in $\mathcal{S}(E)$ and $\mathcal{S}(E^*)$. In Section 5, we consider an increasing sequence $\{Y_n\}_n$ of closed subspaces of a separable Banach space E such that $\dim(Y_n) = \dim(Y_{n+1}/Y_n) = \infty$ for each n, and establish conditions on two sets $R \in \mathcal{S}(E)$ and $W \in \mathcal{S}(E^*)$ to guarantee the existence of a nuclear, one-to-one and dense-range operator $T : E \to E$ which, among other things, preserves the subspaces of the chain (that is, $\overline{T(Y_n)} = Y_n$ for each n) and satisfies the disjointness properties $T(E) \cap R = \{0\}$ and $T^*(E^*) \cap W = \{0\}$ (the second property whenever the subspace $\overline{\cup_n Y_n}$ is infinite-codimensional in E).

The notation we use is standard. We consider real normed spaces. The symbols S_E, B_E and I_E stand respectively for the unit sphere, the closed unit ball and the identity operator of a normed space E. If A is a subset of E, we denote by $\operatorname{span}(A)$ (or $\operatorname{span} A$), $\operatorname{co}(A)$ and $\overline{\operatorname{co}}(A)$ the linear span, the convex hull and the closed convex hull of A, respectively. If $\{x_n\}_n$ is a sequence in E, we write $[\{x_n\}_n]$ for its closed linear span. The symbol $A^\perp$ refers to the annihilator subspace of a set $A \subset E$, that is, $A^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in A\}$. Analogously, given a set $F \subset E^*$, we write $F_\perp = \{x \in E : f(x) = 0 \text{ for all } f \in F\}$.

2. Minimal sequences and countable unions of operator ranges

A key ingredient in the proofs of the aforementioned results is the following lemma concerning the existence of minimal sequences in a Banach space with a special behaviour with respect to countable unions of infinite-codimensional operator ranges in that space. Recall that a sequence $\{x_n\}_n$ in a Banach space E is said to be **minimal** whenever there exists a sequence $\{f_n\}_n \subset E^*$ such that $\{x_n, f_n\}_n$ is a biorthogonal system in E, that is, $f_n(x_m) = \delta_{n,m}$, for all $n, m \geq 1$. If all the functionals f_n lie in a given subspace $F \subset E^*$, then we say that $\{x_n\}_n$ is an **F-minimal** sequence. It will be convenient to introduce the following notation: A sequence $\{x_n\}_n$ in a Banach space E is said to have **property $(*)$ with respect to a subset** $V \subset E$ if the conditions $\{a_n\}_n \in \ell^1$ and $\sum_n a_n x_n \in V$ imply $a_n = 0$ for all $n \geq 1$.

Lemma 2.1. *Let $(E, \|\cdot\|)$ be a Banach space, let $X \subset E$ and $F \subset E^*$ be closed subspaces and $S \in \mathcal{S}(X)$. If $\{x_n\}_n$ is an F-minimal sequence in X then, for every $\varepsilon \in (0,1)$ there exist an isomorphism $\varphi : E \longrightarrow E$ and an F-minimal sequence $\{y_n\}_n \subset B_X$ such that*

(1) $\|\varphi - I_E\| \leq \varepsilon$.
(2) $\varphi([\{x_n\}_n]) = [\{y_n\}_n]$.
(3) $\varphi(X) = X$.
(4) $\varphi^*(F) = F$.
(5) $\operatorname{span}(\overline{\operatorname{co}}(\{\pm y_n\}_n)) \cap S = \{0\}$.
(6) $\{y_n\}_n$ *satisfies property $(*)$ with respect to S.*

In the proof of this lemma, we shall use the following strenghtening of a result of Fonf [12, Lemma 3].

Lemma 2.2. *Let $\{V_m\}_m$ be a sequence of symmetric closed convex and bounded sets in a Banach space $(E, \|\cdot\|)$ such that* $\operatorname{codim}_E(\operatorname{span} V_m) = \infty$ *for all $m \geq 1$. Then, for every pair of sequences $\{v_n\}_n \subset S_E$ and $\{\varepsilon_n\}_n \subset (0, \infty)$ there exist sequences $\{w_n\}_n \subset S_E$ and $\{\gamma_n\}_n \subset (0, \infty)$ satisfying $\sum_n \gamma_n < 1$,*

$$\|v_n - w_n\| < \varepsilon_n, \quad \textit{for all} \quad n \geq 1$$

and

$$\operatorname{span}(\overline{\operatorname{co}}(\{\pm\gamma_n w_n\}_{n=1}^{\infty})) \cap \operatorname{span} V_m = \{0\}, \quad \textit{for all} \quad m \geq 1.$$

Proof. We may assume that $V_m \subset \frac{1}{m} B_E$ for all $m \geq 1$, and thus $\bigcup_m V_m$ is closed and symmetric. Let us write, for each $m \geq 1$, $L_m := \operatorname{span} V_m$. Because of the convexity and symmetry of V_m we have $L_m = \bigcup_{k \geq 1} k V_m$. Since $\bigcup_m L_m = \bigcup_{m,k} k V_m$ is a countable union of closed sets with empty interior in E, thanks to Baire's category theorem the set $E \setminus (\bigcup_m L_m)$ is dense in E. Notice that if $\lambda \neq 0$ and $w \in E$, then $\lambda w \in E \setminus (\bigcup_m L_m)$ if and only if $w \in E \setminus (\bigcup_m L_m)$. Thus, the set $S_E \setminus (\bigcup_m L_m)$ is dense in S_E. So, there is $w_1 \in S_E \setminus (\bigcup_m L_m)$ such that $\|w_1 - v_1\| < \varepsilon_1$.

Now, for every $m \geq 1$ we define $L_{m,1} = L_m$ and $L_{m,2} = \operatorname{span}(\{w_1\} \cup L_{m,1})$. By hypothesis, $\operatorname{codim}_E(L_{m,1}) = \infty = \operatorname{codim}_E(L_{m,2})$, so $L_{m,2} \neq E$. It can be checked that

$$L_{m,2} = \bigcup_{k \geq 1} k \operatorname{co}(\{\pm w_1\} \cup V_m)$$

and $\operatorname{co}(\{\pm w_1\} \cup V_m)$ is a symmetric closed convex and bounded set with empty interior. Thus $\bigcup_m L_{m,2}$ is a countable union of symmetric closed convex and bounded sets with empty interior in E. A new appeal to Baire's category theorem yields that the set $E \setminus (\bigcup_m L_{m,2}) = E \setminus (\bigcup_{m,k} k \operatorname{co}(\{\pm w_1\} \cup V_m))$ is dense in E. Also, if $\lambda \neq 0$ and $w \in E$, then $\lambda w \in E \setminus (\bigcup_m L_{m,2})$ if and only if $w \in E \setminus (\bigcup_m L_{m,2})$. That is, the set $S_E \setminus (\bigcup_m L_{m,2})$ is dense in S_E and there is $w_2 \in S_E \setminus (\bigcup_m L_{m,2})$ such that $\|w_2 - v_2\| < \varepsilon_2$. Notice that, because of the construction, we have $\operatorname{span}\{w_1, w_2\} \cap (\bigcup_m L_m) = \{0\}$.

By induction, we get a sequence $\{w_n\}_n \subset S_E$ such that, for every n,

$$\|w_n - v_n\| < \varepsilon_n \quad \text{and} \quad w_n \in S_E \setminus \bigcup_m L_{m,n},$$

where $L_{m,n}$ is defined for each $m \geq 1$ as

$$L_{m,1} := L_m \text{ and } L_{m,n} := \operatorname{span}(\{w_{n-1}\} \cup L_{m,n-1}) = \operatorname{span}(\{w_1, \dots, w_{n-1}\} \cup L_{m,n-1}), \text{ for } n \geq 2.$$

Arguing as before we can deduce that

- $L_{m,n} = \bigcup_{k=1}^{\infty} k \operatorname{co}(\{\pm w_1, \dots, \pm w_{n-1}\} \cup V_m)$,
- the closed convex bounded and symmetric set $\operatorname{co}(\{\pm w_1, \dots, \pm w_{n-1}\} \cup V_m)$ has empty interior in E,
- $\operatorname{codim}_E(L_{m,n}) = \infty$ and
- $\operatorname{span}\{w_1, \dots, w_n\} \bigcap (\bigcup_m L_m) = \{0\}$.

Now, for each n we set $W_n := \operatorname{span}\{w_1, \ldots, w_n\}$ and consider the function

$$r_n(t) := \operatorname{dist}(tS_{W_n}, \bigcup_m V_m) = \inf\{\|x - y\| : x \in tS_{W_n},\, y \in \bigcup_m V_m\}, \qquad \text{for } t > 0.$$

Since $\bigcup_m V_m$ is closed and $tS_{W_n} \cap (\bigcup_m V_m) = \emptyset$ we have $\operatorname{dist}(x, \bigcup_m V_m) > 0$, whenever $x \in tS_{W_n}$ and $t > 0$. As tS_{W_n} is compact we get $r_n(t) = \inf\{\operatorname{dist}(x, \bigcup_m V_m) : x \in tS_{W_n}\} > 0$ for all $t > 0$. Also, since $\{W_n\}_n$ is a strictly increasing sequence of sets, we have $r_{n+1}(t) \leq r_n(t)$ for all $n \geq 1$ and all $t > 0$.

We claim that r_n is strictly increasing. Indeed, let us fix $0 < t < t'$. For every $x \in S_{W_n}$ and $y \in \bigcup_m V_m$ we have $\|t'x - y\| = t'\|x - \frac{y}{t'}\| = \frac{t'}{t}\left\|tx - \frac{t}{t'}\, y\right\|$, and bearing in mind that $tx \in tS_{W_n}$ and $\frac{t}{t'}\, y \in \frac{t}{t'}(\bigcup_m V_m) \subset \bigcup_m V_m$ it follows that

$$r_n(t') \geq \frac{t'}{t} r_n(t) > r_n(t).$$

Let us define, for each $k \geq 1$,

$$\gamma_k = 2^{-k} r_k(\frac{1}{k}).$$

Notice that $r_k(\frac{1}{k}) \leq \frac{1}{k}$ for all $k \geq 1$, and thus $\sum_{k\geq 1} \gamma_k \leq \sum_{k\geq 1} \frac{2^{-k}}{k} < 1$. Moreover, for all $n \geq 1$ we have

$$\sum_{k=n+1}^{\infty} \gamma_k = \sum_{k=n+1}^{\infty} 2^{-k} r_k(\frac{1}{k}) \leq \sum_{k=n+1}^{\infty} 2^{-k} r_n(\frac{1}{k}) \leq \sum_{k=n+1}^{\infty} 2^{-k} r_n(\frac{1}{n}) = 2^{-n} r_n(\frac{1}{n}).$$

Now, we shall prove that

$$\overline{\operatorname{co}}(\{\pm\gamma_k w_k\}_k) \cap (\bigcup_m V_m) = \{0\}.$$

Suppose that $z \neq 0$ and $z \in \overline{\operatorname{co}}(\{\pm\gamma_k w_k\}_k)$. Let us fix a natural number n large enough such that $\|z\| > \frac{1}{n} + 2^{-n} r_n(\frac{1}{n})$. Then, for every $0 < \varepsilon < \|z\| - \frac{1}{n} - 2^{-n} r_n(\frac{1}{n})$ there is a sum $\sum_{k=1}^{s} \lambda_k \gamma_k w_k$ such that $\sum_{k=1}^{s} |\lambda_k| \leq 1$ and $\|z - \sum_{k=1}^{s} \lambda_k \gamma_k w_k\| < \varepsilon$, where we may assume that $s > n$. In particular,

$$\begin{aligned}
\left\|\sum_{k=1}^{n} \lambda_k \gamma_k w_k\right\| &\geq \left\|\sum_{k=1}^{s} \lambda_k \gamma_k w_k\right\| - \left\|\sum_{k=n+1}^{s} \lambda_k \gamma_k w_k\right\| \\
&\geq \|z\| - \varepsilon - \sum_{k=n+1}^{s} \gamma_k \geq \|z\| - \varepsilon - 2^{-n} r_n(\frac{1}{n}) > \frac{1}{n}.
\end{aligned}$$

Then, for every $y \in \bigcup_m V_m$,

$$\begin{aligned}
\|z - y\| &\geq \left\|\sum_{k=1}^{n} \lambda_k \gamma_k w_k - y\right\| - \left\|\sum_{k=n+1}^{s} \lambda_k \gamma_k w_k\right\| - \left\|z - \sum_{k=1}^{s} \lambda_k \gamma_k w_k\right\| \geq \\
&\geq r_n\left(\|\sum_{k=1}^{n} \lambda_k \gamma_k w_k\|\right) - \sum_{k=n+1}^{s} \gamma_k - \varepsilon \\
&\geq r_n(\frac{1}{n}) - 2^{-n} r_n(\frac{1}{n}) - \varepsilon.
\end{aligned}$$

Since this inequality holds for every $0 < \varepsilon < \|z\| - \frac{1}{n} - 2^{-n} r_n(\frac{1}{n})$, we get

$$\|z - y\| \geq r_n(\frac{1}{n}) - 2^{-n} r_n(\frac{1}{n}) > 0$$

for all $y \in \bigcup_m V_m$. Therefore, $z \notin \bigcup_m V_m$.

The above implies that

$$\operatorname{span}(\overline{\mathrm{co}}(\{\pm\gamma_k w_k\}_{k=1}^{\infty})) \cap (\bigcup_m L_m) = \{0\}.$$

Indeed, observe that

$$\operatorname{span}(\overline{\mathrm{co}}(\{\pm\gamma_k w_k\}_{k=1}^{\infty})) = \bigcup_{n\geq 1} n\,\overline{\mathrm{co}}(\{\pm\gamma_k w_k\}_{k=1}^{\infty}).$$

Thus, if $z \in \operatorname{span}(\overline{\mathrm{co}}(\{\pm\gamma_k w_k\}_{k=1}^{\infty})) \cap (\bigcup_m L_m) = \operatorname{span}(\overline{\mathrm{co}}(\{\pm\gamma_k w_k\}_{k=1}^{\infty})) \cap (\bigcup_{m,s\geq 1} sV_m)$, then there exist natural numbers n_0, s_0 and m_0 such that $z \in (n_0\,\overline{\mathrm{co}}(\{\pm\gamma_k w_k\}_{k=1}^{\infty})) \cap (s_0\, V_{m_0})$. Therefore,

$$\frac{z}{s_0\, n_0} \in (\frac{1}{s_0}\overline{\mathrm{co}}(\{\pm\gamma_k w_k\}_{k=1}^{\infty})) \cap (\frac{1}{n_0} V_{m_0}) \subset \overline{\mathrm{co}}(\{\pm\gamma_k w_k\}_{k=1}^{\infty}) \cap V_{m_0} = \{0\},$$

and this yields $z = 0$. □

Proof of Lemma 2.1. We may assume without loss of generality that $\|x_n\| = 1$ for all $n \geq 1$. Let $\{f_n\}_n \subset F$ be a sequence such that $\{x_n, f_n\}_n$ is a biorthogonal system in E and fix a sequence $\{\varepsilon_n\}_n$ of positive numbers with $\sum_{n\geq 1} \varepsilon_n < \varepsilon$. Let $\{R_m\}_m$ a sequence in $\mathcal{R}(E)$ such that $S = \bigcup_{m\geq 1} R_m$. We will define a sequence of subsets $\{V_m\}_m$ of E satisfying the hypothesis of Lemma 2.2 and such that $R_m \subset \operatorname{span} V_m$ for every m. Fix a natural number $m \geq 1$. If the operator range R_m is closed, we consider the symmetric closed convex and bounded set $V_m = R_m \cap B_E$, which obviously satisfies $\operatorname{span} V_m = R_m$, and so $\operatorname{codim}_E(\operatorname{span} V_m) = \infty$. If R_m is not closed we define $E_m = \overline{R_m}$. Then, R_m is a proper dense operator range in E_m. Consider a Banach space Z_m and a one-to-one operator $A_m : Z_m \to E_m$ such that $R_m = A(Z_m)$, and set $V_m = \overline{A_m(B_{Z_m})}$. Notice that V_m has empty interior in E_m (otherwise, by the open mapping theorem, $A_m(Z_m) = E_m$, which is not true). Consider the vector space $W_m = \operatorname{span} V_m = \bigcup_{j\geq 1} jV_m$, and let $\|\cdot\|_m$ the norm on W_m whose closed unit ball is V_m. Then, $(W_m, \|\cdot\|_m)$ is a Banach space (see e.g. [10, Exercise 2.22]). Moreover, because of the boundedness of the set V_m, there is a constant $C_m > 0$ such that $\|x\| \leq C_m$ whenever x lies in $V_m = B_{(W_m, \|\cdot\|_m)}$ (the closed unit ball of $(W_m, \|\cdot\|_m)$). Therefore, the inclusion operator $I_m : (W_m, \|\cdot\|_m) \to (E_m, \|\cdot\|)$, being $\|\cdot\|$ the norm in E_m inherited from E, is continuous. Consequently, $W_m = \operatorname{span} V_m$ is a proper dense operator range in E_m, which yields that $\operatorname{codim}_{E_m}(\operatorname{span} V_m) = \infty$, and hence $\operatorname{codim}_E(\operatorname{span} V_m) = \infty$.

Applying now Lemma 2.2, we deduce the existence of a normalized sequence $\{w_n\}_n \subset X$ and a sequence of positive numbers $\{\gamma_n\}_n$ such that $\sum_{n\geq 1} \gamma_n < 1$,

$$\|x_n - w_n\| < \varepsilon_n \|f_n\|^{-1} \quad \text{for all} \quad n \geq 1$$

and

$$\operatorname{span}(\overline{\mathrm{co}}(\{\pm\gamma_n w_n\}_n)) \cap (\bigcup_m \operatorname{span} V_m) = \{0\}.$$

In particular,

$$\operatorname{span}(\overline{\operatorname{co}}(\{\pm\gamma_n w_n\}_n)) \cap S = \{0\}. \tag{2.1}$$

Since $\sum_{n\geq 1} \|f_n\|\|w_n - x_n\| \leq \sum_{n\geq 1} \varepsilon_n < \varepsilon < 1$, the formula

$$\varphi(x) = x + \sum_{n\geq 1} f_n(x)(w_n - x_n), \quad x \in E$$

defines an isomorphism $\varphi : E \to E$ with $\|\varphi - I_E\| < \varepsilon$. Moreover, taking into account that $\{x_n, f_n\}_n$ is a biorthogonal system we get

$$\varphi(x_n) = x_n + (w_n - x_n) = w_n \quad \text{for all} \quad n \geq 1.$$

Since φ is an isomorphism we have $\varphi([\{x_n\}_n]) = [\{w_n\}_n]$. We claim that $\varphi(X) = X$. As X is a closed subspace of E and $\{x_n\}_n$, $\{w_n\}_n \subset X$ we get $\varphi(X) \subset X$. Moreover, if $x \in X$ then there is $y \in E$ such that $\varphi(y) = y + \sum_{n\geq 1} f_n(y)(w_n - x_n) = x$. Therefore $y = x - \sum_{n\geq 1} f_n(y)(w_n - x_n) \in X$, which yields $X \subset \varphi(X)$

Now, we shall prove that $\varphi^*(F) = F$. Indeed, for each $f \in E^*$ we have

$$\varphi^*(f) = f + \sum_{n\geq 1} f(w_n - x_n) f_n.$$

In particular, as F is a closed subspace and $\{f_n\}_n \subset F$, we obtain $\varphi^*(f) \in F$ whenever $f \in F$, that is $\varphi^*(F) \subset F$. On the other hand, since φ is an isomorphism on E we have that φ^* is an isomorphism on E^*. Thus, for every $f \in F$ we can find $g \in E^*$ such that $f = \varphi^*(g) = g + \sum_{n\geq 1} g(w_n - x_n) f_n$. Hence, $g = f - \sum_{n\geq 1} g(w_n - x_n) f_n$, and so $g \in F$. Consequently, $\varphi^*(F) \subset F$.

Now, for every $n \geq 1$ we define

$$y_n = \gamma_n w_n \quad \text{and} \quad g_n = \gamma_n^{-1} (\varphi^{-1})^*(f_n).$$

It is clear that $y_n \in \overline{\operatorname{co}}(\{\pm\gamma_n w_n\}_n)$, $g_n \in F$ and $g_n(y_k) = \delta_{n,k}$ for all $n, k \geq 1$. So, $\{y_n\}_n$ is an F-minimal sequence in B_X, and because of (2.1), it satisfies (5).

It remains to check that $\{y_n\}_n$ enjoys property $(*)$ with respect to S. Let $\{a_n\}_n$ be a sequence in ℓ^1 such that the vector $y = \sum_{n\geq 1} a_n y_n$ lies in S. Since $y \in \operatorname{span}(\overline{\operatorname{co}}(\{\pm\gamma_n w_n\}_n))$, thanks to (5) we get $y = 0$. Therefore, $a_n = g_n(y) = 0$ for all $n \geq 1$. □

Remark 2.3. It is worth noticing that in Lemma 2.1, in the case that E is a dual space, say $E = X^*$ for some Banach space X, the isomorphism φ is the adjoint of certain isomorphism $\phi : X \to X$. In particular, φ is (w^*, w^*)-continuous.

3. Essential disjointness

The purpose of this section is to establish some strenghtenings of Shevchik's theorem. We observe that, because of the injectivity of the operator $T : E \to E$ provided by that result, the subspace $T^*(E^*) \subset E^*$ is total (over E). Moreover, the compactness of T guarantees that $T^*(E^*)$ is also separable. Thus, it is natural to wonder on the existence of a nuclear dense-range operator $T : E \to E$ such that $T^*(E^*)$ fills a given closed total separable subspace $Z \subset E^*$, and $T(E) \cap R = \{0\}$ for a given proper dense (or an infinite-codimensional) operator range $R \subset E$. More generally, we have the following result.

Theorem 3.1. *Let E be an (infinite dimensional) separable Banach space and Z be a closed, separable and total subspace of E^*. Then, for any $R \in \mathcal{S}(E)$ and any $V \in \mathcal{S}(Z)$ there exists a nuclear operator $T : E \to E$ such that*

(1) $\overline{T(E)} = E$.
(2) $T^*(E^*) \subset Z$.
(3) $\overline{T^*(Z)} = Z$ *(in particular, T is one-to-one).*
(4) $T(E) \cap R = \{0\}$.
(5) $T^*(E^*) \cap V = \{0\}$.

In the proof of this theorem, as well as of almost all the results in this work, we shall make use of the existence of M-bases with some special features on separable Banach spaces. Reall that a biorthogonal system $\{e_n, e_n^*\}_n$ in a (separable) Banach space E is said to be a **Markushevich basis** (in short, an **M-basis**) of E if $[\{e_n\}_n] = E$ and $\overline{\text{span}(\{e_n^*\}_n)}^{w^*} = E^*$. For a detailed account on such bases, we refer to the monographs [15] and [22].

Proof of Theorem 3.1. According to [22, pg. 224, Theorem 8.1] (see also [15, pg. 8, Lemma 1.21]), there is an M-basis $\{x_n, z_n\}_n$ of E with $[\{z_n\}_n] = Z$. By Lemma 2.1, applied to the sequence $\{x_n\}_n$, the set $R \in \mathcal{S}(E)$ and the subspaces $X = E$ and $F = Z$, there exist a Z-minimal sequence $\{y_n\}_n \subset B_E$ satisfying property $(*)$ with respect to R and an isomorphism $\varphi : E \to E$ such that $\varphi([\{x_n\}_n]) = [\{y_n\}_n]$. Therefore, $[\{y_n\}_n] = E$.

Applying again Lemma 2.1, now in the space E^*, to the E-minimal sequence $\{z_n\}_n$, the set $V \in \mathcal{S}(Z)$ and the subspaces $X = Z$ and $F = E$, we deduce the existence of a sequence $\{f_n\}_n \subset B_Z$ which is E-minimal and has property $(*)$ with respect to V, and an isomorphism $\psi : E^* \to E^*$ such that $[\{\psi(z_n)\}_n] = [\{f_n\}_n]$ and $\psi(Z) = Z$. Hence, $[\{f_n\}_n] = Z$.

Let us write, for each $x \in E$,

$$T(x) = \sum_{n \geq 1} 2^{-n} f_n(x) y_n.$$

Since the sequences $\{y_n\}_n$ and $\{f_n\}_n$ are bounded, this formula defines a nuclear endomorphism on E. The E-minimality of $\{f_n\}_n$ entails that $y_n \in T(E)$ for all $n \geq 1$. Indeed, if $\{e_n\}_n$ is a sequence in E such that $\{e_n, f_n\}_n$ is a biorthogonal system then $T(e_n) = 2^{-n} y_n$, that is, $y_n \in T(E)$ for all $n \geq 1$. Therefore, $E = [\{y_n\}_n] \subset \overline{T(E)}$, and property (1) is fulfilled.

Moreover, as

$$T^*(f) = \sum_{n \geq 1} 2^{-n} f(y_n) f_n \quad \text{for all} \quad f \in E^*,$$

we immediately get (2), and taking into account that the sequence $\{y_n\}_n$ is Z-minimal, it follows that $f_n \in T^*(Z)$ for all $n \geq 1$, so $Z = [\{f_n\}_n] = \overline{T^*(E^*)}$.

It remains to show (4) and (5). Pick $x \in E$ such that $T(x) \in R$. Since the sequence $\{y_n\}_n$ satisfies property $(*)$ with respect to R and $\{2^{-n} f_n(x)\}_n \in \ell^1$ we get $f_n(x) = 0$ for all $n \geq 1$, and therefore $Tx = 0$. Consequently, $T(E) \cap R = \{0\}$. Bearing in mind that $\{f_n\}_n$ satisfies $(*)$ with respect to V, the same argument entails that $T^*(E^*) \cap V = \{0\}$. □

It is well-known (see e.g. [22, pg. 225, Lemma 8.1] or [15, pg. 8, Theorem 1.22]) that every separable Banach space admits a closed, 1-norming and separable subspace $Z \subset E^*$. This fact and Theorem 3.1 lead to the following result.

Corollary 3.2. *If E is a separable Banach space then, for every $R \in \mathcal{S}(E)$ there exists a nuclear, one-to-one and dense-range operator $T : E \to E$ such that $T(E) \cap R = \{0\}$ and the subspace $T^*(E^*)$ is 1-norming for E.*

The next result provides another extension of Shevchik's theorem.

Theorem 3.3. *If Y is a closed infinite-dimensional subspace of a separable Banach space E then, for every $R \in \mathcal{S}(Y)$, every $V \in \mathcal{S}(Y^*)$ and every $\lambda \in (0, 1)$ there exists a nuclear one-to-one operator $T : E \to E$ such that*

(1) $T(E) \subset Y$.
(2) $\overline{T(Y)} = Y$.
(3) $T(E) \cap R = \{0\}$.
(4) $T^*(E^*)|_Y \cap V = \{0\}$.
(5) $T^*(E^*)|_Y$ is λ-norming for Y.
(6) In addition, if Y^* is separable, then the operator T can be built so that $\overline{T^*(E^*)|_Y} = Y^*$.
(7) In addition, if E^* is separable, then the operator T can be built so that $\overline{T^*(E^*)} = E^*$.

In the proof of this theorem we shall use a result of Singer [20, Theorem 2], which asserts that, if Y is an infinite-dimensional and infinite-codimensional closed subspace of a separable Banach space E and $\{y_n^*\}_n \subset Y^*$ is a total sequence (that is, such that $\mathrm{span}(\{y_n^*\}_n)$ is total over Y), then there exists a sequence $\{f_n\}_n \subset E^*$ such that $f_n|_Y = y_n^*$ for each n and $\{f_n\}_n$ is total over E. We shall also need the following variant of this result.

Lemma 3.4. *Let E be a Banach space with separable dual, let Y be a closed subspace of E and $\{y_n^*\}_n$ a sequence in Y^* such that $[\{y_n^*\}_n] = Y^*$. Then, there exists a sequence $\{f_n\}_n$ in E^* such that $f_n|_Y = y_n^*$ for each n and $[\{f_n\}_n] = E^*$.*

Proof. Since $(E/Y)^*$ identifies with $Y^\perp \subset E^*$ and E^* is separable so is $Y^\perp$, and there is a dense sequence $\{z_n^*\}_n$ in $Y^\perp$. For every n, let us consider a linear continuous extension to E of the functional y_n^*, and denote it by $h_n^* \in E^*$. For convenience, let us relabel the sequence $\{h_n^*\}_n$ in the form $\{h_{m,j}^*\}_{m,j\in\mathbb{N}}$. Since for every fixed m,

$$z_m^* + \frac{h_{m,j}^*}{j\|h_{m,j}^*\|} \xrightarrow{j\to\infty} z_m^*,$$

we have $z_m^* \in [\{z_k^* + \frac{h_{k,j}^*}{j\|h_{k,j}^*\|} : k, j \in \mathbb{N}\}]$, thus for every m, j we get

$$h_{m,j}^* \in [\{z_k^* + \frac{h_{k,i}^*}{i\|h_{k,i}^*\|} : k, i \in \mathbb{N}\}].$$

Let us define, for every m, j,

$$f_{m,j} := j\|h_{m,j}^*\|z_m^* + h_{m,j}^*.$$

Clearly, $f_{m,j}|_Y = h_{m,j}^*|_Y$ and $[\{f_{k,j} : k, j \in \mathbb{N}\}] = [\{z_k^* + \frac{h_{k,j}^*}{j\|h_{k,j}^*\|} : k, j \in \mathbb{N}\}]$. Let us check that $[\{z_k^*\}_k \cup \{h_{k,j}^*\}_{k,j}] = E^*$. Let us rewrite $\{h_{m,j}^*\}_{m,j}$ as $\{h_n^*\}_n$ by reverting the first relabeling.

Fix $f \in E^*$ and take $\varepsilon > 0$. Since $Y^* = [\{y_n^*\}_n]$, there is a finite linear combination $\sum_i a_i y_i^*$ such that $\|f|_Y - \sum_i a_i y_i^*\| < \varepsilon$, that is,

$$\|f|_Y - \sum_i a_i h_i^*|_Y\| < \varepsilon.$$

Since Y^* is isometrically isomorphic to the quotient space $E^*/Y^\perp$, there exists a functional $g \in Y^\perp$ such that $\|f - \sum_i a_i h_i^* - g\| < \varepsilon$. On the other hand, as $\{z_n^*\}_n$ is dense in $Y^\perp$, there is a finite linear combination $\sum_j b_j z_j^*$ with $\|g - \sum_j b_j z_j^*\| < \varepsilon$. Therefore,

$$\|f - \sum_i a_i h_i^* - \sum_j b_j z_j^*\| < 2\varepsilon.$$

Thus, $f \in [\{z_k^*\}_k \cup \{h_{k,j}^*\}_{k,j}]$. Finally, we rewrite $\{f_{m,j}\}_{m,j}$ as $\{f_n\}_n$ by reverting the first relabeling, which yields $f_n|_Y = h_n^*|_Y = y_n^*$ for each n, and the proof is finished. ☐

Proof of Theorem 3.3. According to [15, pg. 8, Theorem 1.22] (see also [22, pg. 226, Corollary 8.1]) there is an M-basis $\{e_n, e_n^*\}_n$ of Y such that the subspace $[\{e_n^*\}_n] \subset Y^*$ is 1-norming for Y. Fix a number $0 < \varepsilon < 1$ such that $(1-3\varepsilon)/(1+\varepsilon) > \lambda$. Thanks to Lemma 2.1, there exist an isomorphism $\varphi : Y \to Y$ with $\|\varphi - I_Y\| < \varepsilon$ and a minimal sequence $\{y_n\}_n \subset B_Y$ which enjoys property $(*)$ with respect to R and satisfies $[\{y_n\}_n] = [\{\varphi(e_n)\}_n] = Y$.

Evidently, the functionals $y_n^* := (\varphi^{-1})^*(e_n^*) \in Y^*$ $(n \geq 1)$ constitute a Y-minimal sequence. Applying Lemma 2.1 to this sequence, we deduce the existence of an isomorphism $\psi : Y^* \to Y^*$ with $\|\psi - I_{Y^*}\| < \varepsilon$ and a Y-minimal sequence $\{v_n^*\}_n \subset B_{Y^*}$ satisfying property $(*)$ with respect to V and $[\{v_n^*\}_n] = [\{\psi(y_n^*)\}_n]$.

Since $\|\varphi^* - I_{Y^*}\| < \varepsilon$ we have $\|(\varphi^*)^{-1}\| \leq 1/(1-\varepsilon)$. Thus, the isomorphism $\tau : Y^* \to Y^*$ defined as $\tau = \psi \circ (\varphi^*)^{-1}$ satisfies

$$\|\tau\| \leq (1+\varepsilon)/(1-\varepsilon)$$

and

$$\|\tau - I_{Y^*}\| \leq \|(\varphi^*)^{-1}\|\|\psi - \varphi^*\| \leq \varepsilon(1-\varepsilon)^{-1}(\|\psi - I_{Y^*}\| + \|\varphi^* - I_{Y^*}\|) < 2\varepsilon(1-\varepsilon)^{-1}.$$

The last two inequalities imply that the subspace $[\{v_n^*\}_n] \subset Y^*$ is $\frac{1-3\varepsilon}{1+\varepsilon}$-norming, and hence λ-norming. In particular, the sequence $\{v_n^*\}_n$ is total over Y, and thanks to [20, Theorem 2], we obtain a total sequence $\{f_n\}_n \subset E^*$ such that $f_n|_Y = v_n^*$ for all $n \geq 1$.

At this point, we consider the nuclear operator $T : E \to E$ defined by the formula

$$T(x) = \sum_{n=1}^{\infty} 2^{-n}(1 + \|f_n\|)^{-1} f_n(x) y_n, \quad x \in E.$$

It is clear that T satisfies (1). Moreover, if $x \in \ker T$ then $\sum_{n\geq 1} 2^{-n}(1+\|f_n\|)^{-1} f_n(x) y_n = 0$. Because of the minimality of the sequence $\{y_n\}_n$ we obtain $f_n(x) = 0$ for all $n \geq 1$, and taking into account that $\{f_n\}_n$ is total over E we get $x = 0$. Therefore, T is one-to-one. Observe also that, since $\{f_n\}_n$ is Y-minimal (being an extension of $\{v_n^*\}_n$), we have $\{y_n\}_n \subset T(Y)$ and hence $[\{y_n\}_n] \subset \overline{T(Y)}$. As $[\{y_n\}_n] = Y$ and $T(Y) \subset Y$ we get $\overline{T(Y)} = Y$, and assertion (2) is proved.

To check (3), pick $x \in E$ such that $T(x) \in R$. Then, $\sum_{n\geq 1} 2^{-n}(1 + \|f_n\|)^{-1} f_n(x) y_n \in R$. As $\{y_n\}_n$ has property $(*)$ with respect to R we obtain $f_n(x) = 0$ for all $n \geq 1$, and thus $T(x) = 0$.

To prove (4) and (5), consider a sequence $\{g_n\}_n \subset E^*$ such that $\{y_n, g_n\}_n$ is a biorthogonal system in E (recall that $\{y_n\}_n$ is minimal). Since $T^*(f) = \sum_{n\geq 1} 2^{-n}(1+\|f_n\|)^{-1} f(y_n) f_n$ for all $f \in E^*$ we have $T^*(g_n) = 2^{-n}(1+\|f_n\|)^{-1} f_n$ for each $n \geq 1$, thus

$$\operatorname{span}(\{f_n\}_n) \subset T^*(E^*). \tag{3.2}$$

$f_n|_Y = v_n^*$ for all $n \geq 1$ we get $\operatorname{span}(\{v_n^*\}_n) \subset T^*(E^*)|_Y$, and therefore,

$$[\{v_n^*\}_n] \subset \overline{T^*(E^*)|_Y}. \tag{3.3}$$

Since $T^*(f)|_Y = \sum_{n\geq 1} 2^{-n}(1+\|f_n\|)^{-1} f(y_n) v_n^*$ for all $f \in E^*$, $\{y_n\}_n$ is bounded and the sequence $\{v_n^*\}_n$ satisfies property $(*)$ with respect to V, it follows that $T^*(E^*)|_Y \cap V = \{0\}$. Also, since $[\{v_n^*\}_n] \subset Y^*$ is λ-norming for Y, so is $T^*(E^*)|_Y$.

Now, assume that Y^* is separable. Then the initial M-basis $\{e_n, e_n^*\}_n$ of Y can be chosen to be shrinking, that is, $[\{e_n^*\}_n] = Y^*$ (see e.g. [15, pg. 8, Theorem 1.22]). Since φ^* and ψ are isomorphisms on Y^* and $[\{v_n^*\}_n] = [\{\psi \circ (\varphi^*)^{-1}(e_n^*)\}_n]$, taking into account (3.3) we obtain $Y^* = [\{v_n^*\}_n] \subset \overline{T^*(E^*)|_Y}$, and assertion (6) is proved.

To finish, suppose that E^* is separable. As before, we can assume that the sequence $\{e_n^*\}_n$ is linearly dense in Y^*, hence $\{v_n^*\}_n \subset Y^*$ is also linearly dense in Y^*. Therefore, by Lemma 3.4, the corresponding extension $\{f_n\}_n \subset E^*$ of $\{v_n^*\}_n$ can be chosen so that $[\{f_n\}_n] = E^*$, and using (3.2) we get $\overline{T^*(E^*)} = [\{f_n\}_n] = E^*$. □

Next, we establish a dual counterpart of the previous theorem.

Theorem 3.5. *Let E be a separable Banach space. If Z is a w^*-closed infinite-dimensional subspace of E^* then, for any $R \in \mathcal{S}(E)$ and any $V \in \mathcal{S}(Z)$ there exists a nuclear operator $T : E \to E$ such that*

(1) $\overline{T(E)} = E$.
(2) $T^*(E^*) \subset Z$.
(3) $T^*(Z)$ is w^*-sequentially dense in Z.
(4) $T(E) \cap R = \{0\}$.
(5) $T^*(E^*) \cap V = \{0\}$.
(6) In addition, if Z is separable, then property (3) may be replaced with $\overline{T^*(Z)} = Z$.

In the proof of this theorem, we shall use the following lemma.

Lemma 3.6. *Let E be a separable Banach space. If Z is a w^*-closed infinite-dimensional subspace of E^* then there exists a biorthogonal system $\{x_n, z_n\} \subset E \times Z$ such that $[\{x_n\}_n] = E$ and the subspace $[\{z_n\}_n]$ is w^*-sequentially dense in Z. If in addition, Z is separable, then $[\{z_n\}_n] = Z$.*

Proof. Let $Q : E \to E/Z_\perp$ be the quotient map and $\tau : Z \to (E/Z_\perp)^*$ the isometric (w^*, w^*)-isomorphism from Z onto $E/Z_\perp$ that assigns to each $z \in Z$ the functional $\tau(z) \in (E/Z_\perp)^*$ defined as $\langle \tau(z), Q(x) \rangle = z(x)$, $x \in E$. According to [15, pg. 8, Theorem 1.22] (see also [22, pg. 226, Corollary 8.1]) there exist sequences $\{e_n\}_n \subset E$ and $\{g_n\}_n \subset (E/Z_\perp)^*$ such that $\{Q(e_n), g_n\}_n$ is an M-basis of $E/Z_\perp$ and $[\{g_n\}_n]$ is norming for $E/Z_\perp$. In particular, thanks to a well-known result of Banach (see e.g. [4, Theorem V.12.11]), $[\{g_n\}_n]$ is w^*-sequentially dense in $(E/Z_\perp)^*$. Let us write, for each $n \geq 1$, $z_n = \tau^{-1}(g_n)$. Then, $[\{z_n\}_n]$ is

w^*-sequentially dense in Z. On the other hand, by a result of Singer [21, Theorem 3], there is a sequence $\{x_n\}_n \subset E$ such that $[\{x_n\}_n] = E$ and $Q(x_n) = Q(e_n)$ for all $n \geq 1$. It is clear that $z_n(x_m) = g_n(Q(e_m)) = \delta_{n,m}$ for each $n, m \geq 1$, that is, $\{x_n, z_n\}_n$ is a biorthogonal system. Moreover, in the case that Z is separable, the M-basis $\{Q(x_n), g_n\}_n$ of $E/Z_\perp$ can be chosen to be shrinking, that is, $[\{g_n\}_n] = (E/Z_\perp)^*$, and hence $[\{z_n\}_n] = Z$. □

Proof of Theorem 3.5. By the previous lemma, there is a biorthogonal system $\{x_n, z_n\}_n \subset E \times Z$ such that $[\{x_n\}_n] = E$ and $[\{z_n\}_n]$ is w^*-sequentially dense in Z. Applying now Lemma 2.1 to the sequence $\{x_n\}_n$, we can find a Z-minimal sequence $\{y_n\}_n \subset B_E$ which enjoys property $(*)$ with respect to R and satisfies $[\{y_n\}_n] = E$. Another appeal to that lemma (applied to the sequence $\{z_n\}_n$) yields a sequence $\{f_n\}_n \subset B_Z$ which is E-minimal, has property $(*)$ with respect to V and satisfies that $[\{f_n\}_n]$ is w^*-sequentialy dense in Z. Moreover, if Z is separable, the initial biorthogonal system $\{x_n, z_n\}_n$ can be chosen so that $[\{z_n\}_n] = Z$, and hence $[\{f_n\}_n] = Z$.

The arguments in the proof of Theorem 3.3 yield that the operator $T : E \to E$ defined by the formula

$$T(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x) y_n, \quad x \in E$$

satisfies the required properties. Indeed, the E-minimality of the sequence $\{f_n\}_n$ implies that $y_n \in T(E)$ for all $n \geq 1$, hence $E = [\{y_n\}_n] = \overline{T(E)}$. The inclusion $T^*(E^*) \subset Z$ is obvious. Moreover, by the Z-minimality of $\{y_n\}_n$ we get $f_n \in T^*(Z)$ for each $n \geq 1$. As $[\{f_n\}_n]$ is w^*-sequentially dense in Z, so is $T^*(Z)$, and in the case that Z is separable we get $Z = \overline{T^*(Z)}$. Finally, since $\{y_n\}_n$ and $\{f_n\}_n$ have property $(*)$ with respect to R and V respectively, we obtain (4) and (5). □

Applying Theorem 3.3 with $Y = E$ (or Theorem 3.5 with $Z = E^*$) we obtain at once the following result.

Corollary 3.7. *If E is an (infinite dimensional) separable Banach space then, for every couple of elements $R \in \mathcal{S}(E)$ and $V \in \mathcal{S}(E^*)$ and any number $0 < \lambda < 1$ there exists a nuclear one-to-one dense-range operator $T : E \to E$ such that*

(1) $T(E) \cap R = \{0\}$.
(2) $T^*(E^*)$ is λ-norming for E.
(3) $T^*(E^*) \cap V = \{0\}$.
(4) In addition, if E^* is separable, then assertion (2) may be replaced with $\overline{T^*(E^*)} = E^*$.

The former corollary leads to the following statement.

Corollary 3.8. *If E is an (infinite dimensional) separable Banach space then, for any two operator ranges $R \in \mathcal{R}(E)$, $V \in \mathcal{R}(E^*)$ and any $\varepsilon > 0$, there exists an isomorphism $\varphi : E \to E$ such that $\varphi(R) \cap R = \{0\}$, $\varphi^*(V) \cap V = \{0\}$ and $\|\varphi - I_E\| < \varepsilon$.*

Proof. The previous corollary guarantees the existence of a one-to-one and dense-range operator $T : E \to E$ such that $T(E) \cap R = \{0\}$ and $T^*(E^*) \cap V = \{0\}$. We can assume without loss of generality that $\|T\| < \varepsilon$. Hence, the operator $\varphi = I - T$ is an isomorphism on E and $\|\varphi - I_E\| < \varepsilon$. Pick $x \in R \cap \varphi(R)$. Then $x = u - T(u)$ for some $u \in R$, that is, $T(u) = u - x \in R$. Thus $T(u) \in R \cap T(R)$, therefore $T(u) = 0$, so (by the injectivity of T), $u = 0$, and consequently $x = 0$. Moreover, as $\overline{T(E)} = E$ we have that T^* is one-to-one, and arguing as before we obtain $\varphi^*(V) \cap V = \{0\}$. □

In [3, Problem 8], Borwein and Tingley asked if, for a given Banach space E, in particular for $E = \ell^\infty$, there exist two dense operator ranges R and V in E such that $R \cap V = \{0\}$. Plichko [18] proved that the answer is affirmative if $E = \ell^\infty(\Gamma)$ for any set Γ. Taking into account that isomorphisms in a Banach space carry proper dense operator ranges into proper dense operator ranges in that space, Corollary 3.8 yields the following refinement of Plichko's result in the case that Γ is countable.

Corollary 3.9. *If E is a separable Banach space, then for any proper dense operator range R in E^* there exists a dense operator range $V \subset E^*$ which is isomorphic to R and satisfies $V \cap R = \{0\}$.*

4. Endomorphisms preserving a closed subspace or a couple of quasicomplements

In this section, we provide two extensions of the result of Chalendar and Partington concerning the existence of nuclear, one-to-one and dense-range endomorphisms on a separable Banach space which leave invariant a closed subspace of that space [7, Theorem 2.1]. The first one reads as follows.

Theorem 4.1. *Let E be a separable Banach space, let Y be a closed infinite-dimensional and infinite-codimensional subspace of E, and let $R \in \mathcal{S}(E)$ and $V \in \mathcal{S}(E^*)$ such that*

$$R \cap Y \in \mathcal{S}(Y), \quad V|_Y \in \mathcal{S}(Y^*) \quad \text{and} \quad V \cap Y^\perp \in \mathcal{S}(Y^\perp).$$

Then, for any $\lambda \in (0,1)$ there exists a nuclear one-to-one dense-range operator $T : E \to E$ such that

(1) $T(Y) \subset Y$ and $\overline{T(Y)} = Y$.
(2) $T^*(Y^\perp) \subset Y^\perp$ and $T^*(Y^\perp)$ is w^*-sequentially dense in $Y^\perp$.
(3) $T(E) \cap R = \{0\}$.
(4) $T^*(E^*) \cap V = \{0\}$.
(5) $T^*(E^*)|_Y$ is λ-norming for Y.
(6) $T^*(E^*)|_Y \cap V|_Y = \{0\}$.
(7) In addition, if $(E/Y)^*$ or Y^* is separable, then properties (2) and (5) can be replaced respectively with

$$\overline{T^*(Y^\perp)} = Y^\perp \quad \text{and} \quad \overline{T^*(E^*)|_Y} = Y^*.$$

(8) In addition, if E^* is separable, then the operator T can be built so that $\overline{T^*(E^*)} = E^*$.

Proof. According to Theorem 3.3, there exists a nuclear one-to-one operator $A : E \to E$ with the following properties:

(a) $A(E) \subset Y$ and $\overline{A(Y)} = Y$.
(b) $A(Y) \cap R = \{0\}$.
(c) $A^*(E^*)|_Y$ is λ-norming for Y (and $\overline{A^*(E^*)|_Y} = Y^*$ is Y^* is separable).

(d) $A^*(E^*)|_Y \cap V|_Y = \{0\}$.

Notice that the requirements $R \cap Y \in \mathcal{S}(Y)$ and $V|_Y \in \mathcal{S}(Y^*)$ are essential to achieve that $A(Y) \cap R = \{0\}$ and $A^*(E^*)|_Y \cap V|_Y = \{0\}$.

Since $R \in \mathcal{S}(E)$, there are operator ranges $R_n \in \mathcal{R}(E)$ such that $R = \bigcup_n R_n$. By [5, Proposition 2.2], the subspace $A(E) + R_n$ is an operator range in E for every $n \geq 1$. Moreover, since $A(E) \cap R = \{0\}$ (thus $A(E) \cap R_n = \{0\}$ for all $n \geq 1$) and $A(E)$ is not closed, thanks to [5, Theorem 2.4] it follows that $A(E) + R_n$ is not closed for every $n \geq 1$. Hence, $\operatorname{codim}_E (A(E) + R_n) = \infty$ for all $n \geq 1$, and therefore $A(E) + R_n \in \mathcal{R}(E)$. This yields $A(E) + R = \bigcup_n (A(E) + R_n) \in \mathcal{S}(E)$.

Moreover, taking into account that $V \cap Y^\perp \in \mathcal{S}(Y^\perp)$, an appeal to Theorem 3.5, with $Z = Y^\perp$, entails the existence of another nuclear operator $B : E \to E$ such that

(e) $\overline{B(E)} = E$.
(f) $B(E) \cap (A(E) + R) = \{0\}$.
(g) $B^*(E^*) \subset Y^\perp$, $B^*(Y^\perp)$ is w^*-sequentially dense in $Y^\perp$ (and $\overline{B^*(Y^\perp)} = Y^\perp$ if $Y^\perp$ is separable).
(h) $B^*(E^*) \cap V = \{0\}$.

We shall prove that the operator $T = A + B$ satisfies the required properties. The nuclearity of A and B yields that T is nuclear as well. Moreover, because of (a) and (g) we get, respectively, $\ker A^* = Y^\perp$ and $\ker B = Y$. From the latter it follows that $T(y) = A(y)$ for all $y \in Y$, hence, by (a), $T(Y) \subset Y$ and $\overline{T(Y)} = Y$, and property (1) is checked. Analogously, as $\ker A^* = Y^\perp$, using (g) we deduce that $T^*(Y^\perp) = B^*(Y^\perp) \subset Y^\perp$ and $T^*(Y^\perp) = B^*(Y^\perp)$ is w^*-sequentially dense in $Y^\perp$, thus (2) is also fulfilled.

Before proving that T is one-to-one and dense-range, let us check that

$$T^{-1}(Y) \subset Y \tag{4.4}$$

and

$$(T^*)^{-1}(Y^\perp) \subset Y^\perp. \tag{4.5}$$

Firstly, if $x \in T^{-1}(Y)$ then, by (a), $B(x) \in Y$. Therefore, for all $f \in Y^\perp$ we have $0 = f(B(x)) = B^*(f)(x)$. So, $x \in [B^*(Y^\perp)]_\perp$, and bearing in mind that $\overline{B^*(Y^\perp)}^{w^*} = Y^\perp$ we get $x \in Y$, thus (4.4) is proved. Secondly, if $f \in (T^*)^{-1}(Y^\perp)$, then $T^*(f) \in Y^\perp$ and because of (g), $A^*(f) \in Y^\perp$. Now, by (a), $A^*(f) = 0$. Since $\ker A^* = Y^\perp$ we get $f \in Y^\perp$ and (4.5) is also checked.

Now, pick $x \in E$ with $T(x) = 0$. Then, by (4.4), $x \in T^{-1}(Y) \subset Y$, so $T(x) = A(x)$, hence $A(x) = 0$, and the injectivity of A yields $x = 0$. Analogously, if $f \in E^*$ and $T^*(f) = 0$, then $f \in (T^*)^{-1}(Y^\perp) \subset Y^\perp$. As $\ker A^* = Y^\perp$ we get $A^*(f) = 0$, and taking into account that, because of (e), B^* is injective, we get $f = 0$. So, T^* is injective as well, and consequently, $\overline{T(E)} = E$.

Let us check (3). Choose a vector $x \in E$ with $T(x) \in R$. Then $B(x) = -A(x) + T(x) \in A(E) + R$. Thus, by (f), $B(x) = 0$, that is, $x \in \ker B = Y$ and therefore, $A(x) = T(x) \in R$. A new appeal to (b) yields $A(x) = 0$, consequently $x = 0$.

To prove (4), pick $f \in E^*$ such that $T^*(f) \in V$. Then $T^*(f)|_Y \in V|_Y$, and taking into account that $B^*(f) \in Y^\perp$ we have $T^*(f)|_Y = A^*(f)|_Y \in V|_Y$. From (d) it follows that

$A^*(f)|_Y = 0$, thus $T^*(f)|_Y = 0$, i.e. $T^*(f) \in Y^\perp$, and thanks to (4.5) we get $f \in Y^\perp$. So, $T^*(f) = B^* f$ and hence $B^*(f) \in V$. Using now (h) we obtain $B^*(f) = 0$, and the inyectivity of B^* yields $f = 0$. Thus, (4) is fulfilled.

Now, since $B^*(E^*) \subset Y^\perp$ we get $T^*(E^*)|_Y = A^*|_Y$. So properties (5) and (6) follow immediately from (c) and (d).

Let us prove (7). If $(E/Y)^*$ is separable then (recall property (g)) the operator B can be constructed so that $\overline{B^*(Y^\perp)} = Y^\perp$. Taking into account that $\ker A^* = Y^\perp$, we get $\overline{T^*(Y^\perp)} = \overline{B^*(Y^\perp)} = Y^\perp$. Analogously, if Y^* is separable then, by property (c), we may assume that the operator A satisfies $\overline{A^*(E^*)|_Y} = Y^*$, thus $\overline{T^*(E^*)|_Y} = \overline{A^*(E^*)|_Y} = Y^*$.

Finally, if E^* is separable, then $Y^\perp$ and Y^* are separable as well. Thus, the operator T can be chosen to satisfy both $\overline{T^*(Y^\perp)} = Y^\perp$ and $\overline{T^*(E^*)|_Y} = Y^*$. Therefore, for any $f \in E^*$ and any $\varepsilon > 0$ there is $g \in E^*$ such that $\|(f - T^*(g))|_Y\| < \varepsilon$, and bearing in mind that Y^* is isometrically isomorphic to $E^*/Y^\perp$ and $\overline{T^*(Y^\perp)} = Y^\perp$ we can find $h \in Y^\perp$ with $\|f - T^*(g) - T^*(h)\| < \varepsilon$. Consequently, $\overline{T^*(E^*)} = E^*$, and property (8) is proved. ☐

Remark 4.2. Notice that if R is a subset of E of the form $R := R_1 + R_2$, where $R_1 \in \mathcal{S}(Y)$ and $R_2 \in \mathcal{S}(E)$ satisfy $R_2 \cap Y = \{0\}$, then $R \cap Y = R_1 \in \mathcal{S}(Y)$. Moreover, if $Q : E^* \to Y^*$ denotes the restriction map, then for any $V_1 \in \mathcal{S}(Y^\perp)$, any $V_2 \in \mathcal{S}(E^*)$ with $V_2 \cap Y^\perp = \{0\}$, and any $W \in \mathcal{S}(Y^*)$, the set

$$V := V_1 + V_2 \cap Q^{-1}(W)$$

satisfies that $V \cap Y^\perp = V_1 \in \mathcal{S}(Y^\perp)$ and $V|_Y = (V_2 \cap Q^{-1}(W))|_Y$. Moreover, $V \in \mathcal{S}(E^*)$. Indeed, since $V_1 = \bigcup_n V_{1,n}$, $V_2 = \bigcup_n V_{2,n}$ and $W = \bigcup_n W_n$ with $V_{1,n} \in \mathcal{R}(Y^\perp)$, $V_{2,n} \in \mathcal{R}(E^*)$ and $W_n \in \mathcal{R}(Y^*)$ respectively for all $n \geq 1$, we get

$$V = \bigcup_{k,n,m} (V_{1,k} + V_{2,n} \cap Q^{-1}(W_m)).$$

So it is enough to check that $V \in \mathcal{R}(E^*)$ whenever $V = V_1 + V_2 \cap Q^{-1}(W)$ for any $V_1 \in \mathcal{R}(Y^\perp)$, $V_2 \in \mathcal{R}(E^*)$ with $V_2 \cap Y^\perp = \{0\}$, and any $W \in \mathcal{R}(Y^*)$. First, it is clear that $\operatorname{codim}(V) = \infty$. Moreover, as W is an operator range in Y^*, thanks to [5, Proposition 2.1] we have that W admits a complete norm $\|\cdot\|_1$ such that $\|y^*\|_1 \geq \|y^*\|$ whenever $y^* \in W$. It is easy to check that the formula

$$||f|| = \|f\|_{E^*} + \|Qf\|_1, \quad f \in Q^{-1}(W)$$

defines a complete norm on the vector space $Q^{-1}(W)$. Thus, a new appeal to [5, Proposition 2.1] guarantees that $Q^{-1}(W)$ is an operator range in E^*. Therefore, $V \in \mathcal{R}(E^*)$.

We point out that the requirement $V|_Y \in \mathcal{S}(Y^*)$ in Theorem 4.1 has been used only to achieve property (6), more precisely, $T^*(E^*)|_Y \cap V|_Y = \{0\}$. Thus, if X and Y are infinite-dimensional quasicomplemented closed subspaces of the separable space E then, as $X \cap Y = \{0\}$ and $X^\perp \cap Y^\perp = \{0\}$, for any $R_1 \in \mathcal{S}(X)$, $R_2 \in \mathcal{S}(Y)$, $V_1 \in \mathcal{S}(X^\perp)$ and $V_2 \in \mathcal{S}(Y^\perp)$ there exists a nuclear, one-to-one and dense-range operator $T : E \to E$ such that

$$\overline{T(Y)} = Y, \quad T(E) \cap (R_1 + R_2) = \{0\} \quad \text{and} \quad T^*(E^*) \cap (V_1 + V_2) = \{0\}.$$

The next result guarantees the existence of an operator $T : E \to E$ which, in addition to these properties, satisfies $\overline{T(X)} = X$. Although in contrast with Theorem 4.1, we loss the normingness property of $T^*(E^*)|_Y$ over Y. Bearing in mind that, by the classical theorem of Murray and Mackey [17], every closed subspace of a separable Banach space admits a quasicomplement, this result provides another extension of [7, Theorem 2.1].

Theorem 4.3. *If X and Y are closed infinite-dimensional quasicomplemented subspaces of a separable Banach space E then, for any $R_1 \in \mathcal{S}(X)$, $R_2 \in \mathcal{S}(Y)$, $V_1 \in \mathcal{S}(X^\perp)$ and $V_2 \in \mathcal{S}(Y^\perp)$ there exists a nuclear one-to-one dense-range operator $T : E \to E$ such that*

(1) $T(X)$ and $T(Y)$ are dense subspaces of X and Y, respectively,
(2) $T^*(X^\perp)$ and $T^*(Y^\perp)$ are w^*-dense subspaces of $X^\perp$ and $Y^\perp$, respectively,
(3) $T(E) \cap (R_1 + R_2) = \{0\}$ and $T^*(E^*) \cap (V_1 + V_2) = \{0\}$.

Proof. Lemma 2.1, applied separately in the subspaces X and Y, ensures the existence of minimal sequences $\{x_n\}_n \subset B_X$ and $\{y_n\}_n \subset B_Y$ satisfying property $(*)$ with respect to R_1 and R_2 respectively, and such that $[\{x_n\}_n] = X$ and $[\{y_n\}_n] = Y$. Another use of Lemma 2.1 (in $X^\perp$ and $Y^\perp$) yields E-minimal sequences $\{f_n\}_n \subset B_{X^\perp}$ and $\{g_n\}_n \subset B_{Y^\perp}$ which enjoy property $(*)$ with respect to V_1 and V_2 respectively, and satisfy $\overline{\operatorname{span}\{f_n\}_n}^{w^*} = X^\perp$ and $\overline{\operatorname{span}\{g_n\}_n}^{w^*} = Y^\perp$. It is clear that the formulas

$$A(u) = \sum_{n=1}^{\infty} 2^{-n} g_n(u) x_n \quad \text{and} \quad B(u) = \sum_{n=1}^{\infty} 2^{-n} f_n(u) y_n, \quad u \in E$$

define nuclear endomorphisms on E.

We shall show that the operator $T = A + B$ has the specified properties. First, since A and B are nuclear, so is T. Let us check property (1). The construction of A and B immediately yields that the subspaces X and Y are T-invariant. Let us prove that $\overline{T(X)} = X$. Because of the E-minimality of $\{g_n\}_n$, there is a sequence $\{u_n\}_n$ in E such that $\{u_n, g_n\}_n$ is a biorthogonal system. Thus, for every fixed $n \geq 1$ we have

$$A(u_n) = 2^{-n} x_n. \tag{4.6}$$

Since the sum $X + Y$ is dense in E, there exist sequences $\{\widetilde{x}_k\}_k \subset X$ and $\{\widetilde{y}_k\}_k \subset Y$ such that $\lim_k \|u_n - (\widetilde{x}_k + \widetilde{y}_k)\| = 0$. Bearing in mind that the operator A vanishes on Y, we get $\lim_k \|A(u_n) - A(\widetilde{x}_k)\| = \lim_k \|A(u_n) - T(\widetilde{x}_k)\| = 0$, hence $A(u_n) \in \overline{T(X)}$, and because of (4.6) we have $x_n \in \overline{T(X)}$ for all $n \geq 1$. Taking into account that $[\{x_n\}_n] = X$ and $T(X) \subset X$ it follows that $\overline{T(X)} = X$. The same argument yields $\overline{T(Y)} = Y$.

Let us check (2). Notice that $A^*(f) = \sum_n 2^{-n} f(x_n) g_n$ and $B^*(f) = \sum_n 2^{-n} f(y_n) f_n$ for all $f \in E^*$. This yields $X^\perp$ and $Y^\perp$ are T^*-invariant subspaces of E^*. Moreover, using the minimality of the sequence $\{y_n\}_n$, arguing as before we obtain $f_n \in T^*(X^\perp) = B^*(X^\perp) \subset X^\perp$ for all $n \geq 1$. Thus $X^\perp = \overline{\operatorname{span}(\{f_n\}_n)}^{w^*} = \overline{T^*(X^\perp)}^{w^*}$. Proceeding identically we get $Y^\perp = \overline{T^*(Y^\perp)}^{w^*}$.

Notice also that property (1) and the denseness of $X + Y$ in E entail that $\overline{T(E)} = E$. Moreover, since $X \cap Y = \{0\}$ it follows that $X^\perp + Y^\perp$ is w^*-dense in E^*. This and assertion (2) imply that $\overline{T^*(E^*)}^{w^*} = E^*$, which is equivalent to the injectivity of T.

It remains to check (3). Take $u \in E$ such that $T(u) \in R_1 + R_2$. Then $A(u) + B(u) = r_1 + r_2$ for some $r_1 \in R_1$ and $r_2 \in R_2$. Hence $A(u) - r_1 = r_2 - B(u)$. Since $X \cap Y = \{0\}$, we get

$A(u)-r_1=0=r_2-B(u)$. Thus, $A(u)=r_1\in R_1$ and $B(u)=r_2\in R_2$. As $\{x_n\}_n$ and $\{y_n\}_n$ satisfy property $(*)$ with respect to R_1 and R_2 respectively, we obtain $A(u)=B(u)=0$. Consequently, $T(x)=0$, and so $T(E)\cap(R_1+R_2)=\{0\}$. The same argument yields that $T^*(E^*)\cap(V_1+V_2)=\{0\}$. □

5. Endomorphisms preserving chains of closed subspaces

In this section, we shall refine the previous techniques to give a partial extension of Theorem 4.1 when the subspace $Y\subset E$ is replaced with a countable chain of closed subspaces $\{Y_n\}_n$ of E such that $Y_n\subset Y_{n+1}$ for all $n\geq 1$. This result provides also a strenghtening of the result of Chalendar and Partington concerning the existence of a nuclear, one-to-one and dense range operator endomorphism on E preserving each Y_n [7, Theorem 2.2].

We point out that if $R\in\mathcal{R}(E)$ and $A:F\to E$ is any operator from a Banach space F to E such that $R=A(F)=\bigcup_{m\geq 1}mA(B_F)$, then the arguments given in the proof of Lemma 2.1 yield $\operatorname{codim}_E\big(\operatorname{span}\overline{A(B_F)}\big)=\infty$, and thus $\operatorname{span}\overline{A(B_F)}\in\mathcal{R}(E)$. Let us mention that the subspace $\operatorname{span}\overline{A(B_F)}$ is independent of the selection of F and A, and only depends on R. We omit the proof of this assertion because it is not needed in the proof of the next theorem. This fact allows us to define the operator range

$$R^+=\operatorname{span}\overline{A(B_F)}.$$

Evidently, $R\subset R^+$ for every $R\in\mathcal{R}(E)$. The reverse inclusion is fulfilled for instance in the following cases:

- If R is a closed infinite-codimensional subspace of E.
- If R is the image of an operator $A:F\to E$ defined on a reflexive Banach space F (in particular if R is an endomorphism range on a reflexive space).

Indeed, in the first case, as R is the image of the identity operator $I:R\to R$ we clearly have $R\in\mathcal{R}(E)$ and $R^+=R$. On the other hand, if there exist a reflexive Banach space F and an operator $T:F\to E$ such that $T(F)=R$ then, because of the weak compactness of the ball B_F, the set $T(B_F)$ is weakly closed in E, hence $\overline{T(B_F)}=T(B_F)$, and thus $T(F)^+=T(F)$.

Analogously, if $R\in\mathcal{S}(E)$ we will refer as R^+ to any countable union of subspaces U_n of E of the form $U_n=\operatorname{span}\overline{A_n(B_{F_n})}$, where $R_n=A_n(F_n)\in\mathcal{R}(E)$, F_n is a Banach space and $A_n:F_n\to E$ is an operator for every $n\geq 1$, and $R=\bigcup_n R_n\in\mathcal{S}(E)$, i.e. $R^+=\bigcup_n R_n^+$. Again, the set R^+ is independent of the selection of the operator ranges $R_n\in\mathcal{R}(E)$ satisfying $R=\bigcup_n R_n$, and only depends on R. We also omit the proof of this assertion because we will not need it.

The main result of this section reads as follows.

Theorem 5.1. *Let E be a separable Banach space, let $\{Y_n\}_n$ be a chain of closed subspaces of E such that $\dim(Y_1)=\dim(Y_{n+1}/Y_n)=\infty$ for all $n\geq 1$, let $Y=\overline{\bigcup_n Y_n}$ and consider sets $R\in\mathcal{S}(E)$ and $W\in\mathcal{S}(E^*)$ such that:*

(a) $R^+\cap Y_n\in\mathcal{S}(Y_n)$ for all $n\geq 1$,
(b) $W|_{Y_1}\in\mathcal{S}(Y_1^*)$, and $W|_{Y_n}\cap Y_{n-1}^{\perp}|_{Y_n}\in\mathcal{S}(Y_{n-1}^{\perp}|_{Y_n})$ if $n\geq 2$.
(c) In the case that E/Y is infinite-dimensional, $W\cap Y^{\perp}\in\mathcal{S}(Y^{\perp})$.

Then there exists a nuclear and one-to-one operator $T : E \to E$ with the following properties:

(1) $\overline{T(Y_n)} = Y_n$ *for all* $n \geq 1$.
(2) $\overline{T(Y)} = Y$.
(3) $\overline{T(E)} = E$.
(4) $T(E) \cap R = \{0\}$.
(5) $T^*(E^*)|_{Y_n} \cap W|_{Y_n} = \{0\}$ *for all* $n \geq 1$ *and* $T^*(E^*)|_Y \cap W|_Y = \{0\}$.
(6) If $\dim(E/Y) = \infty$ *then* $T^*(E^*) \cap W = \{0\}$.
(7) $T^*(Y^\perp)$ *is a* w^**-sequentially dense subspace of* $Y^\perp$, *and* $\overline{T^*(Y^\perp)} = Y^\perp$ *whenever* $Y^\perp$ *is separable.*
(8) If Y_n^* *is separable for some* $n \geq 1$ *then* $\overline{T^*(E^*)|_{Y_n}} = Y_n^*$.
(9) If Y^* *is separable then* $\overline{T^*(E^*)|_Y} = Y^*$.
(10) If E^* *is separable then* $\overline{T^*(E^*)} = E^*$.

If condition (a) is replaced with the weaker condition (a'): "$R \cap Y_n \in S(Y_n)$ *for all* $n \geq 1$*", then all assertions (1)-(10) remain true with the exception of (4), which becomes: (4')* $T(Y_n) \cap R = \{0\}$ *for all* n *and* $T(E \setminus Y) \cap R = \{0\}$.

In order to prove Theorem 5.1 we will need the following lemmas.

Lemma 5.2. *Let* $\{V_m\}_m$ *be a sequence of symmetric closed convex and bounded sets in a Banach space* E *and* $\{K_j\}_j \subset B_E$ *be a sequence of symmetric compact and convex sets satisfying*

$$(\bigcup_m \operatorname{span}(V_m)) \cap (\sum_{j=1}^n K_j) = \{0\}$$

for all $n \geq 1$. *Then, there exist constants* $0 < \gamma_j \leq 1$ *such that* $\sum_{j\geq 1} \gamma_j < \infty$ *and*

$$(\bigcup_m \operatorname{span} V_m) \cap \operatorname{span}(\sum_{j=1}^\infty \gamma_j K_j) = \{0\}.$$

Proof. We may assume that $K_1 \neq \{0\}$ and $V_m \subset \frac{1}{m} B_E$ for all $m \geq 1$, so $\bigcup_m V_m$ is closed (and symmetric). Denote $t_0 = \max\{\|v\| : v \in K_1\} > 0$. For each $n \geq 1$ we consider the function $r_n : (0, t_0] \to [0, \infty)$ defined as

$$r_n(t) = \inf \left\{ \|x - y\| : x \in (K_1 + \cdots + K_n) \cap tS_E,\, y \in \bigcup_m V_m \right\}, \quad 0 < t \leq t_0.$$

If $x \in (K_1 + \cdots + K_n) \cap tS_E$ for some $0 < t \leq t_0$ then, by the hypothesis, $x \notin \bigcup_m V_m$, and since $\bigcup_m V_m$ is closed we get $\operatorname{dist}(x, \bigcup_m V_m) = \inf\{\|x - y\| : y \in \bigcup_m V_m\} > 0$. As $(K_1 + \cdots + K_n) \cap tS_E$ is compact we derive that $r_n(t) > 0$ for each $0 < t \leq t_0$. Also, since $\{K_1 + \cdots + K_n\}_{n=1}^\infty$ is an increasing sequence of symmetric compact and convex sets, we have that $r_{n+1}(t) \leq r_n(t)$ for all $n \geq 1$ and all $0 < t \leq t_0$.

We claim that the function r_n is strictly increasing. Indeed, take $0 < t < t' \leq t_0$. For every $x \in (K_1 + \cdots + K_n) \cap t'S_E$ and every $y \in \bigcup_m V_m$ we have

$$\frac{t}{t'} x \in [\frac{t}{t'}(K_1 + \cdots + K_n)] \cap tS_E \subset (K_1 + \cdots + K_n) \cap tS_E$$

and $\frac{t}{t'}\,y \in \frac{t}{t'}(\bigcup_m V_m) \subset \bigcup_m V_m$. Therefore,

$$\|x-y\| = \frac{t'}{t}\left\|\frac{t}{t'}x - \frac{t}{t'}\,y\right\| \geq \frac{t'}{t}\,r_n(t),$$

and consequently, $r_n(t') \geq \frac{t'}{t}\,r_n(t) > r_n(t)$.

Now, let us fix the smallest natural number n_0 such that $\frac{1}{n_0} < t_0$, and define

$$\gamma_j = 3^{-j}t_0 \quad \text{if} \quad 1 \leq j \leq n_0 \quad \text{and} \quad \gamma_j = 2^{-j}r_j(\frac{1}{j}) \quad \text{for} \quad j > n_0.$$

Notice that, for every $j > n_0$, $r_j(\frac{1}{j}) \leq \|v_j - 0\| \leq \frac{1}{j}$, being v_j any point in $K_1 \cap \frac{1}{j}S_E$, and thus $\sum_{j\geq 1}\gamma_j \leq \sum_{j=1}^{n_0}\gamma_j + \sum_{j=n_0+1}^{\infty} 2^{-j}\frac{1}{j} < t_0$. Moreover, for every $n \geq n_0$ we have

$$\sum_{j=n+1}^{\infty}\gamma_j = \sum_{j=n+1}^{\infty} 2^{-j}r_j(\frac{1}{j}) \leq \sum_{j=n+1}^{\infty} 2^{-j}r_n(\frac{1}{j}) \leq \sum_{j=n+1}^{\infty} 2^{-j}r_n(\frac{1}{n}) = 2^{-n}r_n(\frac{1}{n}).$$

Now, we shall prove that

$$(\sum_{j\geq 1}\gamma_j K_j) \cap (\bigcup_{m\geq 1} V_m) = \{0\}. \tag{5.7}$$

Suppose that $z \neq 0$ and $z \in \sum_j \gamma_j K_j$. So there are points $k_j \in K_j$ such that $z = \sum_j \gamma_j k_j$. Now, let us fix a natural number $n \geq n_0$ such that $\|\sum_{j=1}^n \gamma_j k_j\| \geq \frac{1}{n}$. Notice that $\|\sum_{j=1}^n \gamma_j k_j\| \leq \sum_{j=1}^n \gamma_j\|k_j\| \leq \sum_{j=1}^n \gamma_j < t_0$ and $\sum_{j=1}^n \gamma_j k_j \in K_1 + \cdots + K_n$. Thus, for every $y \in \bigcup_m V_m$ we have

$$\begin{aligned}
\|z-y\| &\geq \left\|\sum_{j=1}^{n}\gamma_j k_j - y\right\| - \left\|\sum_{j=n+1}^{\infty}\gamma_j k_j\right\| \geq \\
&\geq r_n\left(\|\sum_{j=1}^{n}\gamma_j k_j\|\right) - \sum_{j=n+1}^{\infty}\gamma_j\|k_j\| \geq r_n\left(\|\sum_{j=1}^{n}\gamma_j k_j\|\right) - \sum_{j=n+1}^{\infty}\gamma_j \geq \\
&\geq r_n(\frac{1}{n}) - 2^{-n}r_n(\frac{1}{n}).
\end{aligned}$$

Therefore, $\|z-y\| \geq r_n(\frac{1}{n}) - 2^{-n}r_n(\frac{1}{n}) > 0$ for all $y \in \bigcup_m V_m$ and derive that $z \notin \bigcup_m V_m$.

To finish, pick $z \in \operatorname{span}(\sum_j \gamma_j K_j) \cap (\bigcup_m \operatorname{span}(V_m)) = (\bigcup_{p\in\mathbb{N}} p\sum_j \gamma_j K_j) \cap (\bigcup_{m,s\in\mathbb{N}} sV_m)$. Then, there are natural numbers p_0, s_0 and m_0 such that $z \in (p_0\sum_j K_j) \cap (s_0\,V_{m_0})$. Thus

$$\frac{z}{p_0 s_0} \in (\frac{1}{s_0}\sum_{j\geq 1}\gamma_j K_j) \cap \frac{1}{p_0}V_{m_0} \subset (\sum_{j\geq 1}\gamma_j K_j) \cap V_{m_0} = \{0\},$$

and thanks to (5.7) we get $z = 0$. □

By iterating Lemma 5.2 we can derive the next one.

Lemma 5.3. *Let E be a Banach space and let $\{K_j\}_j$ be a sequence of symmetric compact and convex sets of B_E satisfying $K_m \cap (\sum_{j=m+1}^n K_j) = \{0\}$ for all $m \geq 1$ and all $n > m$. Then, there exists a sequence $\{\gamma_j\}_{j\geq 2} \subset (0,1]$ such that $\sum_{j\geq 2}\gamma_j < \infty$ and*

$$\operatorname{span}(K_m) \cap \operatorname{span}(\sum_{j=m+1}^{\infty}\gamma_j K_j) = \{0\} \quad \textit{for all} \quad m \geq 1.$$

Proof. First, we notice that $\mathrm{span}(K_1) \cap (\sum_{j=2}^n K_j) = \{0\}$ for all $n \geq 2$. Indeed, if $z \in \mathrm{span}(K_1) \cap (\sum_{j=2}^n K_j)$ then, since $\mathrm{span}(K_1) = \bigcup_{s=1}^\infty sK_1$, there is a natural number s such that $z \in (sK_1) \cap (\sum_{j=2}^n K_j)$. So, $\frac{z}{s} \in K_1 \cap (\frac{1}{s}(\sum_{j=2}^n K_j)) \subset K_1 \cap (\sum_{j=2}^n K_j) = \{0\}$ and $z = 0$. Applying Lemma 5.2, we deduce the existence of a sequence $\{\gamma_{1,j}\}_{j \geq 2} \subset (0, 1]$ such that $\sum_{j \geq 2} \gamma_{1,j} < \infty$ and

$$\mathrm{span}(K_1) \cap \mathrm{span}(\sum_{j \geq 2} \gamma_{1,j} K_j) = \{0\}.$$

The previous argument yields that if $n \geq 3$ then $\mathrm{span}(K_2) \cap (\sum_{j=3}^n K_j) = \{0\}$, and taking into account that $\sum_{j=3}^n \gamma_{1,j} K_j \subset \sum_{j=3}^n K_j$, we obtain $\mathrm{span}(K_2) \cap (\sum_{j=3}^n \gamma_{1,j} K_j) = \{0\}$. A new appeal to Lemma 5.2 yields a sequence $\{\gamma_{2,j}\}_{j \geq 3} \subset (0, 1]$ such that $\sum_{j \geq 3} \gamma_{2,j} < \infty$ and

$$\mathrm{span}(K_2) \cap \mathrm{span}(\sum_{j \geq 3} \gamma_{2,j} \gamma_{1,j} K_j) = \{0\}.$$

We proceed inductively. In the m-th step, since $\mathrm{span}(K_m) \cap (\sum_{j=m+1}^{n} \gamma_{m-1,j} \cdot \ldots \cdot \gamma_{1,j} K_j) = \{0\}$ for all $n \geq m + 1$, we can apply Lemma 5.2 to get a sequence $\{\gamma_{m,j}\}_{j \geq m+1} \subset (0, 1]$ such that $\sum_{j \geq m+1} \gamma_{m,j} < \infty$ and

$$\mathrm{span}(K_m) \cap \mathrm{span}(\sum_{j \geq m+1} \gamma_{m,j} \cdot \ldots \cdot \gamma_{1,j} K_j) = \{0\}.$$

Finally, let us define, for each $j \geq 2$, $\gamma_j := \gamma_{j-1,j} \cdot \ldots \cdot \gamma_{1,j}$. Clearly, $\sum_{j \geq 2} \gamma_j < \infty$. If $m \geq 1$ then for each $j > m \geq 1$ we have $\gamma_j \leq \gamma_{m,j} \cdots \gamma_{1,j}$ and thus,

$$\mathrm{span}(K_m) \cap \mathrm{span}(\sum_{j=m+1}^{\infty} \gamma_j K_j) \subset \mathrm{span}(K_m) \cap (\sum_{j=m+1}^{\infty} \gamma_{m,j} \cdots \gamma_{1,j} K_j) = \{0\},$$

as we wanted. $\square$

We are able to prove the main result of this section.

Proof of Theorem 5.1. The operator $T : E \to E$ will be sum of a norm-convergent series of nuclear endomorphisms on E. The construction of these endomorphisms relies mainly on the next statement.

Claim. For each $n \geq 1$ there exist a minimal sequence $\{y_{n,j}\}_j \subset B_{Y_n}$ and a Y_n-minimal sequence $\{s^*_{n,j}\}_j \subset B_{Y_{n-1}^\perp}$ such that, if K_n and K'_n denote the symmetric compact convex sets

$$K_n = \overline{\mathrm{co}}(\{\pm 2^{-j} y_{n,j}\}_j) \subset B_{Y_n} \quad \text{and} \quad K'_n = \overline{\mathrm{co}}(\{\pm 2^{-j} s^*_{n,j}\}_j) \subset B_{Y_{n-1}^\perp},$$

and $T_n : E \to E$ is the nuclear operator defined by the formula

$$T_n(x) = \sum_{j \geq 1} 4^{-j} s^*_{n,j}(x) y_{n,j}, \quad x \in E,$$

then the following properties hold:

(a_n) $\mathrm{span}\,(K_n) \cap \left(R^+ + \mathrm{span}\,\left(\sum_{i=0}^{n-1} K_i\right)\right) = \{0\}$, where $K_0 = \{0\}$.
(b_n) $\mathrm{span}\,(K'_n|_{Y_n}) \cap \left(W|_{Y_n} + \mathrm{span}\,\left(\sum_{i=0}^{n-1} K'_i|_{Y_n}\right)\right) = \{0\}$, where $K'_0 = \{0\}$.

(c_n) $\overline{T_n(Y_n)} = Y_n$ and $\ker T_n = Y_{n-1}$, where $Y_0 = \{0\}$.

(d_n) If $(Y_n/Y_{n-1})^*$ is separable then $\overline{T_n^*(Y_{n-1}^\perp)|_{Y_n}} = Y_{n-1}^\perp|_{Y_n}$.

Moreover:

($\star$) If Y^* is separable and $q : E^* \to Y^*$ denotes the restriction map, then the operator $(q \circ T_1^*)^* : Y^{**} \to E^{**}$ is one-to-one.

($\star\star$) If E^* is separable, then the operator $T_1^{**} : E^{**} \to E^{**}$ is one-to-one.

The construction proceeds inductively. Since $R^+ \cap Y_1 \in \mathcal{S}(Y_1)$, thanks to Lemma 2.1 (applied in Y_1 to any minimal linearly dense sequence $\{e_{1,j}\}_j$ in Y_1), there exists a sequence $\{y_{1,j}\}_j \subset B_{Y_1}$ such that $[\{y_{1,j}\}_j] = Y_1$ and $R^+ \cap \operatorname{span}(\overline{\operatorname{co}}\,(\{\pm y_{1,j}\}_j)) = \{0\}$. From the latter we get

$$R^+ \cap \operatorname{span}\left(\overline{\operatorname{co}}(\{\pm 2^{-j} y_{1,j}\}_j)\right) = \{0\},$$

and property (a_1) is satisfied.

Another appeal to Lemma 2.1 (applied in Y_1^*, to any total Y_1-minimal sequence $\{e_{1,j}^*\}_j \subset Y_1^*$) gurantees the existence of a total and Y_1-minimal sequence $\{v_{1,j}^*\}_j \subset Y_1^*$ such that $W|_{Y_1} \cap \operatorname{span}(\overline{\operatorname{co}}(\{\pm v_{1,j}^*\}_j)) = \{0\}$, in particular

$$W|_{Y_1} \cap \operatorname{span}(\overline{\operatorname{co}}(\{\pm 2^{-j} v_{1,j}^*\}_j)) = \{0\}.$$

Moreover, if Y_1^* is separable the sequence $\{e_{1,j}^*\}_j$ can be chosen so that $[\{e_{1,j}^*\}_j] = Y_1^*$, thus $[\{v_{1,j}^*\}_j] = Y_1^*$.

Using now [20, Theorem 2], we obtain a sequence $\{g_{1,j}\}_j \subset E^*$ which is total over E and satisfies $g_{1,j}|_{Y_1} = v_{1,j}^*$ for all $j \geq 1$. Let us write, for each $j \geq 1$,

$$s_{1,j}^* = (1 + \|g_{1,j}\|)^{-1} g_{1,j}.$$

Then $\{s_{1,j}^*\}_j$ is a Y_1-minimal sequence in B_{E^*} that is total over E and satisfies

$$W|_{Y_1} \cap \operatorname{span}\left(\overline{\operatorname{co}}(\{\pm 2^{-j} s_{1,j}^*|_{Y_1}\}_j)\right) \subset W|_{Y_1} \cap \operatorname{span}\left(\overline{\operatorname{co}}(\{\pm 2^{-j} v_{1,j}^*\}_j)\right) = \{0\},$$

thus (b_1) is also fulfilled. Properties (c_1) and (d_1) can be achieved as in the proof of Theorem 3.3. Indeed, since $\{s_{1,j}^*\}_j$ is Y_1-minimal and $[\{y_{1,j}\}_j] = Y_1$ we get $\overline{T_1(Y_1)} = Y_1$. Furthermore, the minimality of $\{y_{1,j}\}_j$ and the fact that $\{s_{1,j}^*\}_j$ is total over E entail that $\ker T_1 = \{0\}$, and (c_1) is proved. Moreover, if Y_1^* is separable, the sequence $\{v_{1,j}^*\}_j$ satisfies $[\{v_{1,j}^*\}_j] = Y_1^*$, and so $[\{s_{1,j}^*|_{Y_1}\}_j] = Y_1^*$. In addition, by the minimality of $\{y_{1,j}\}_j$ we have $\{s_{1,j}^*\}_j \subset T_1^*(E^*)$, and thus $\overline{T_1^*(E^*)|_{Y_1}} = [\{s_{1,j}^*|_{Y_1}\}_j] = Y_1^*$.

Let us check ($\star$). First, we observe that if Y^* is separable, the sequence $\{s_{1,j}^*\}_j$ can be chosen so that (apart of being total over E) satisfies

$$[\{s_{1,j}^*|_Y\}_j] = Y^* \tag{5.8}$$

Indeed, according to Lemma 3.6, there is a sequence $\{h_{1,j}\}_j \subset Y^*$ such that $[\{h_{1,j}\}_j] = Y^*$ and $h_{1,j}|_Y = v_{1,j}^*$ for all $j \geq 1$. Thus, an appeal to [20, Theorem 2] yields a sequence $\{g_{1,j}\}_j \subset E^*$ which is total over E and satisfies $g_{1,j}|_Y = h_{1,j}$ (and hence $g_{1,j}|_{Y_1} = v_{1,j}^*$) for all $j \geq 1$. Therefore, the functionals $s_{1,j}^* = (1 + \|g_{1,j}\|)^{-1} g_{1,j} \subset B_{E^*}$ constitute a sequence satisfying (5.8).

Next, take $y^{**} \in Y^{**}$. For each $f \in E^*$ we have

$$\langle (q \circ T_1^*)^* (y^{**}),\, f \rangle = \sum_{j \geq 1} 4^{-j} f(y_{1,j})\, y^{**}\left(s_{1,j}^*|_Y\right),$$

hence

$$(q \circ T_1^*)^* (y^{**}) = \sum_{j \geq 1} 4^{-j} y^{**}\left(s_{1,j}^*|_Y\right) y_{1,j},$$

where each $y_{1,j}$ is identified with a functional in E^{**} via the canonical map $E \to E^{**}$. Thus, if $(q \circ T_1^*)^*(y^{**}) = 0$ then, thanks to the minimality of $\{y_{1,j}\}_j$ we obtain $y^{**}\left(s_{1,j}^*|_Y\right) = 0$ for all $j \geq 1$. Taking into account (5.8) we get $y^{**} = 0$, and assertion $(\star)$ is proved. The proof of property $(\star\star)$ follows in the same way.

Now, fix $n > 1$ and assume that for each $1 \leq i \leq n-1$ there exist sequences $\{y_{i,j}\}_j \subset B_{Y_i}$ and $\{s_{i,j}^*\}_j \subset B_{Y_{i-1}^\perp}$ satisfying the specified properties. An appeal to Lemma 3.6 (with Y_n instead of E and $Z = Y_{n-1}^\perp|_{Y_n} \subset Y_n^*$) ensures the existence of sequences $\{w_{n,j}\}_j \subset Y_n$ and $\{w_{n,j}^*\}_j \subset Y_{n-1}^\perp|_{Y_n}$ such that $\{w_{n,j},\, w_{n,j}^*\}_j$ is a biorthogonal system, $[\{w_{n,j}\}_j] = Y_n$ and $\operatorname{span}\{w_{n,j}^*\}_j$ is w^*-sequentially dense in $Y_{n-1}^\perp|_{Y_n}$. Moreover, in the case that $(Y_n/Y_{n-1})^*$ is separable, $[\{w_{n,j}^*\}_j] = Y_{n-1}^\perp|_{Y_n}$.

Since $\sum_{i=1}^{n-1} K_i$ is a symmetric compact and convex subset of Y_{n-1} and $R^+ \cap Y_n \in \mathcal{S}(Y_n)$, according to [13, Lemma 3.3] we have

$$\operatorname{span}\,(\sum_{i=1}^{n-1} K_i) + (R^+ \cap Y_n) \in \mathcal{S}(Y_n).$$

Notice that $\{w_{n,j}\}_j$ is a $Y_{n-1}^\perp$-minimal sequence in E. Therefore, by Lemma 2.1 (applied to that sequence and the subspaces $Y_n \subset E$ and $Y_{n-1}^\perp \subset E^*$), there is a $Y_{n-1}^\perp$-minimal sequence $\{y_{n,j}\}_j \subset B_{Y_n}$ such that $[\{y_{n,j}\}_j] = Y_n$ and

$$\left(R^+ + \operatorname{span}\,(\sum_{i=1}^{n-1} K_i) \right) \cap \operatorname{span}\,(\overline{\operatorname{co}}(\{\pm y_{n,j}\}_j)) = 0.$$

The latter clearly implies (a_n).

Now, bearing in mind that $W|_{Y_n} \in \mathcal{S}(Y_{n-1}^\perp|_{Y_n})$, a new appeal to Lemma 2.1 (applied to the Y_n-minimal sequence $\{w_{n,j}^*\}_j \subset Y_{n-1}^\perp|_{Y_n}$) yields a Y_n-minimal sequence $\{v_{n,j}^*\}_j \subset Y_{n-1}^\perp|_{Y_n}$ such that $\operatorname{span}(\{v_{n,j}^*\}_j)$ is w^*-sequentially dense in $Y_{n-1}^\perp|_{Y_n}$ and $W|_{Y_n} \cap \operatorname{span}\,(\overline{\operatorname{co}}(\{\pm v_{n,j}^*\}_j)) = \{0\}$. In particular,

$$W|_{Y_n} \cap \operatorname{span}\,(\overline{\operatorname{co}}(\{\pm 2^{-j} v_{n,j}^*\}_j)) = 0. \tag{5.9}$$

Moreover, if $(Y_n/Y_{n-1})^*$ is separable we can assume that $[\{v_{n,j}^*\}_j] = Y_{n-1}^\perp|_{Y_n}$.

Let $\{s_{n,j}^*\}_j$ be a sequence in E^* such that $s_{n,j}^*|_{Y_n} = v_{n,j}^*$ and $\|s_{n,j}^*\| = \|v_{n,j}^*\|$ for all $j \geq 1$. We shall show that the set $K_n' = \overline{\operatorname{co}}(\{\pm 2^{-j} s_{n,j}^*\}_j)$ satisfies (b_n). Pick $w \in W$ and $z \in \operatorname{span}\,(\sum_{i=1}^{n-1} K_i')$ such that

$$(w + z)|_{Y_n} \in \operatorname{span}\,(K_n'|_{Y_n}). \tag{5.10}$$

Take functionals $z_1 \in K_1', \ldots, z_{n-1} \in K_{n-1}'$ and scalars $a_1, \ldots, a_{n-1}$ such that $z = \sum_{i=1}^{n-1} a_i z_i$. Firstly, we shall prove that $z = 0$. We can assume that $a_i \neq 0$ for each $1 \leq i \leq n-1$. Since $K_i' \subset Y_{i-1}^\perp$ for each $2 \leq i \leq n-1$, it follows that

$$z|_{Y_1} = a_1 z_1|_{Y_1},\ z|_{Y_2} = a_1 z_1|_{Y_2} + a_2 z_2|_{Y_2},\ \dots,\ z|_{Y_{n-1}} = \sum_{i=1}^{n-1} a_i z_i|_{Y_{n-1}}.$$

On the other hand, from (5.10) we get $(w+z)|_{Y_1} = 0$, thus $z|_{Y_1} = -w|_{Y_1} \in W|_{Y_1}$. Since, by the inductive hypothesis, $\operatorname{span}\left(K_1'|_{Y_1}\right) \cap W|_{Y_1} = \{0\}$ we have $z|_{Y_1} = 0$, therefore $z_1|_{Y_1} = 0$. Moreover, as $\lim_j 2^{-j} s_{1,j}^* = 0$ we can find scalars λ_j such that $\sum_{j\geq 1} |\lambda_j| \leq 1$ and $z_1 = \sum_{j\geq 1} \lambda_j 2^{-j} s_{1,j}^*$ (see e.g. [10, Exercise 3.86]). Consequently

$$\sum_{j\geq 1} \lambda_j 2^{-j} s_{1,j}^*|_{Y_1} = z_1|_{Y_1} = 0.$$

Since $\{s_{1,j}^*|_{Y_1}\}_j$ is a Y_1-minimal sequence in Y_1^* it follows that $\lambda_j = 0$ for all $j \geq 1$, so $z_1 = 0$, and hence $z = \sum_{i=2}^{n-1} a_i z_i$. Iterating this process we obtain $z_i = 0$ for each $1 \leq i \leq n-1$, and thus $z = 0$. From this and (5.10) we get

$$w|_{Y_n} \in \operatorname{span}\left(K_n'|_{Y_n}\right) \cap W|_{Y_n} = \operatorname{span}\left(\overline{\operatorname{co}}(\{\pm 2^{-j} v_{n,j}^*\}_j)\right) \cap W|_{Y_n},$$

and thanks to (5.9) we have $w|_{Y_n} = 0$. Hence $(w+z)|_{Y_n} = 0$, as we wanted.

It remains to show that the operator $T_n : E \to E$ satisfies (c_n) and (d_n). This follows as in the proof of Theorem 3.5. Since $\{s_{n,j}^*\}_j$ is Y_n-minimal and $\{y_{n,j}\}_j$ is $Y_{n-1}^{\perp}$-minimal we obtain respectively $y_{n,j} \in T_n(Y_n)$ and $s_{n,j}^* \in T_n^*(Y_{n-1}^{\perp})$ for all $j \geq 1$, hence

$$\operatorname{span}(\{y_{n,j}\}_j) \subset T_n(Y_n) \quad \text{and} \quad \operatorname{span}(\{v_{n,j}^*\}_j) = \operatorname{span}(\{s_{n,j}^*|_{Y_{n-1}}\}_j) \subset T_n^*(Y_n^{\perp})|_{Y_{n-1}}.$$

As $[\{y_{n,j}\}_j] = Y_n$ and $\operatorname{span}(\{v_{n,j}^*\}_j)$ is w^*-sequentially dense in $Y_{n-1}^{\perp}|_{Y_n}$ we deduce that $\overline{T_n(Y_n)} = Y_n$ and $T_n^*(Y_{n-1}^{\perp}|_{Y_n})$ is w^*-sequentially dense (and hence w^*-dense) in $Y_{n-1}^{\perp}|_{Y_n}$. The latter yields $\ker T_n = Y_{n-1}$. Finally, if $(Y_n/Y_{n-1})^*$ is separable then $[\{s_{n,j}^*|_{Y_n}\}_j] = [\{v_{n,j}^*\}_j] = Y_{n-1}^{\perp}|_{Y_n}$, thus $\overline{T_n^*(Y_{n-1}^{\perp}|_{Y_n})} = Y_{n-1}^{\perp}|_{Y_n}$, and the claim is proved.

Notice that if $n \geq 1$ then

$$T_n(x) = \sum_{j\geq 1} 4^{-j} s_{n,j}^*(x) y_{n,j} = \sum_{j\geq 1} 2^{-j} s_{n,j}^*(x)(2^{-j} y_{n,j}) \in \|x\| K_n \quad \text{for all} \quad x \in E$$

and

$$T_n^*(f) = \sum_{j\geq 1} 2^{-j} f(y_{n,j})(2^{-j} s_{n,j}^*) \in \|f\| K_n' \quad \text{for all} \quad f \in E^*.$$

On the other hand, according to property (a_n) we get

$$R^+ \cap (\sum_{i=1}^{n} K_i) = \{0\} \text{ for all } n \geq 1 \quad \text{and} \quad K_m \cap (\sum_{i=m+1}^{n} K_i) = \{0\} \text{ for all } n > m \geq 1.$$

Thus, taking into account that R^+ is the union of a sequence of symmetric closed convex and bounded sets, the first equality and Lemma 5.2 ensure the existence of a sequence $\{\gamma_n\}_{n\geq 1} \subset (0,1]$ such that $\sum_{n\geq 1} \gamma_n < \infty$ and

$$R^+ \cap \operatorname{span}(\sum_{n\geq 1} \gamma_n K_n) = \{0\}. \tag{5.11}$$

Analogously, Lemma 5.3 yields a sequence $\{\gamma'_n\}_{n\geq 2} \subset (0,1]$ such that $\sum_{n\geq 2}\gamma'_n < \infty$ and

$$\operatorname{span}(K_n) \cap \operatorname{span}\big(\sum_{i\geq n+1} \gamma'_i K_i\big) = \{0\} \quad \text{for all} \quad n \geq 1. \tag{5.12}$$

Before performing the endomorphism T, we need to build one more symmetric compact convex set $K_\omega \subset B_E$ and one more nuclear endomorphism $T_\omega : E \to E$ with the following properties:

- (a_ω) $\big(R^+ + \operatorname{span}(\sum_{n\geq 1}\gamma_n K_n)\big) \cap \operatorname{span}(K_\omega) = \{0\}$.
- (b_ω) $T_\omega(E) + Y$ is dense in E and $T_\omega(x) \in \|x\| K_\omega$ for all $x \in E$.
- (c_ω) $T^*_\omega(Y^\perp)$ is a w^*-sequentially dense subspace of $Y^\perp$ (hence $\ker T_\omega = Y$), and if $Y^\perp$ is separable, $\overline{T^*_\omega(Y^\perp)} = Y^\perp$.
- (d_ω) If $\dim(E/Y) = \infty$ then $T^*_\omega(E^*) \cap W = \{0\}$.

To prove this assertion we assume first that $\dim(E/Y) = \infty$. As $\sum_{n\geq 1}\gamma_n K_n$ is a symmetric compact convex subset of Y and $R^+ \cap Y \in \mathcal{S}(Y)$, thanks to [13, Lemma 3.3] we have $R^+ + \operatorname{span}(\sum_{n\geq 1}\alpha_n K_n) \in \mathcal{S}(Y)$. Thus, proceeding as in the proof of the claim (with $n \geq 2$) we obtain a minimal sequence $\{x_j\}_j \subset B_E$ such that $[\{x_j\}_j] = E$ and the set $K_\omega = \overline{\operatorname{co}}(\{\pm 2^{-j} x_j\}_j)$ satisfies (a_ω). As in addition, $W \cap Y^\perp \in \mathcal{S}(Y^\perp)$, we can find an E-minimal sequence $\{s^*_j\}_j \subset B_{Y^\perp}$ such that $\operatorname{span}(\{s^*_j\}_j)$ is w^*-sequentially dense in $Y^\perp$ and $W \cap \operatorname{span}(\operatorname{co}\{\pm 2^{-j} s^*_j\}_j) = \{0\}$. Arguing as before we deduce that the operator $T_\omega : E \to E$ defined for each $x \in E$ as

$$T_\omega(x) = \sum_{j\geq 1} 4^{-j} s^*_j(x) x_j$$

satisfies (b_ω), (c_ω) and (d_ω).

On the other hand, if $\dim(E/Y) = d < \infty$ there exist linearly independent vectors $u_1, \ldots, u_d \in E$ such that $Y + \operatorname{span}(\{u_j\}_{j=1}^d) = E$. Lemma 2.1 (applied to any minimal sequence in E containing $\{u_j\}_{j=1}^d$) yields another linearly independent set $\{x_1, \ldots, x_d\} \subset B_E$ such that $\big(R^+ + \operatorname{span}(\sum_{n\geq 1}\gamma_n K_n)\big) \cap \operatorname{span}(\{x_j\}_{j=1}^d) = \{0\}$ and $Y + \operatorname{span}(\{x_j\}_{j=1}^d) = E$. In particular, the set $K_\omega = \operatorname{co}(\{\pm x_j\}_{j=1}^d)$ accomplishes (a_ω). Moreover, if $\{s^*_1, \ldots, s^*_d\}$ is any basis of $Y^\perp$ then the operator

$$T_\omega(x) = \sum_{j=1}^d 2^{-j} s^*_j(x) x_j, \quad x \in E$$

clearly satisfies (b_ω) and (c_ω), and the assertion is proved.

Now, we are able to construct the operator T. Let us write $\alpha_1 = \gamma_1$ and $\alpha_n = \gamma_n \gamma'_n$ if $n > 1$, where $\{\gamma_n\}_{n\geq 1}$ and $\{\gamma'_n\}_{n\geq 2}$ are the sequences in $(0,1]$ coming from equalities (5.11) and (5.12). Taking into account that $\{y_{n,j}\}_j \subset B_{Y_n}$ and $\{s^*_{n,j}\}_j \subset B_{E^*}$ for each $n \geq 1$, and $\sum_{n\geq 1}\alpha_n < \infty$, the formula

$$T(x) = T_\omega(x) + \sum_{n\geq 1} \alpha_n T_n(x) = T_\omega(x) + \sum_{n\geq 1}\sum_{j\geq 1} \alpha_n 4^{-j} s^*_{n,j}(x) y_{n,j}, \quad x \in E$$

defines a nuclear endomorphism on E. Moreover, if a vector $x \in E$ satisfies $T(x) = 0$, then $T_\omega(x) = -\sum_{n\geq 1} \alpha_n T_n(x)$. Thus,

$$T_\omega(x) \in \operatorname{span}(K_\omega) \cap \operatorname{span}\Big(\sum_{n\geq 1} \alpha_n K_n\Big) \subset \operatorname{span}(K_\omega) \cap \operatorname{span}\Big(\sum_{n\geq 1} \gamma_n K_n\Big),$$

and thanks to property (a_ω) we get $T_\omega(x) = 0$. Consequently $\sum_{n\geq 1} \alpha_n T_n(x) = 0$, thus $\alpha_1 T_1(x) = -\sum_{n\geq 1} \alpha_n T_n(x)$, and hence

$$T_1(x) \in \operatorname{span}(K_1) \cap \operatorname{span}\Big(\sum_{n\geq 2} \alpha_n K_n\Big) \subset \operatorname{span}(K_1) \cap \operatorname{span}\Big(\sum_{n\geq 2} \gamma'_n K_n\Big).$$

Using (5.12) we obtain $T_1(x) = 0$, and taking into account that $\ker T_1 = \{0\}$ we get $x = 0$, therefore T is injective.

Next, we shall prove properties (1) – (10).

(1) Fix $n \geq 1$. Since $\ker T_k = Y_{k-1} \supset Y_n$ whenever $k > n$ and $\ker T_\omega = Y \supset Y_n$, we have

$$T(y) = \sum_{k=1}^{n} \alpha_k T_k(y) \quad \text{for each} \quad y \in Y_n,$$

and bearing in mind that $T_k(E) \subset Y_n$ for $k \leq n$ we obtain $T(Y_n) \subset \sum_{k=1}^{n} T_k(Y_n) \subset Y_n$.

To show that $\overline{T(Y_n)} = Y_n$ we proceed inductively. As $Y_1 = \overline{T_1(Y_1)}$ and $T(Y_1) = T_1(Y_1)$ we get $Y_1 = \overline{T(Y_1)}$, and the assertion is proved if $n = 1$. Now, assume that $\overline{T(Y_{n-1})} = Y_{n-1}$ for some $n > 1$. We only need to check that $Y_n \subset \overline{T(Y_n)}$. For every $y \in Y_n$ we have $\alpha_n T_n(y) = T(y) - \sum_{k=1}^{n-1} \alpha_k T_k(y)$, therefore

$$T_n(Y_n) \subset T(Y_n) + \sum_{k=1}^{n-1} T_k(Y_n) \subset T(Y_n) + Y_{n-1} = T(Y_n) + \overline{T(Y_{n-1})} \subset \overline{T(Y_n)},$$

and taking into account that $\overline{T_n(Y_n)} = Y_n$ we obtain $Y_n \subset \overline{T(Y_n)}$.

(2) Since $\ker T_\omega = Y$ we have $T(y) = \sum_{n\geq 1} \alpha_n T_n(y)$ whenever $y \in Y$, and bearing in mind that $T_n(E) \subset Y_n \subset Y$ for all $n \geq 1$ we get $T(Y) \subset Y$. On the other hand, as by property (1), $Y_n = \overline{T(Y_n)} \subset \overline{T(Y)}$ for all $n \geq 1$, it follows that $Y \subset \overline{T(Y)}$, so $\overline{T(Y)} = Y$.

(3) For each $x \in E$ we have

$$T_\omega(x) = T(x) - \sum_{n\geq 1} \alpha_n T_n(x) \in T(E) + \overline{T(E)} = \overline{T(E)},$$

consequently $T_\omega(E) \subset \overline{T(E)}$. As by the previous property, $\overline{T(Y)} = Y$, we get

$$T_\omega(E) + Y \subset \overline{T(E)} + \overline{T(Y)} = \overline{T(E)},$$

and taking into account that $T_\omega(E) + Y$ is dense in E we obtain $E \subset \overline{T(E)}$.

(4) Pick $x \in E$ such that $T(x) \in R^+$. Then

$$T_\omega(x) = T(x) - \sum_{n\geq 1} \alpha_n T_n(x) \in R^+ + \operatorname{span}\Big(\sum_{n\geq 1} \alpha_n K_n\Big).$$

Thus

$$T_\omega(x) \in \operatorname{span}(K_\omega) \cap \big(R^+ + \operatorname{span}\big(\sum_{n\geq 1} \alpha_n K_n\big)\big),$$

therefore $T_\omega(x) = 0$, that is, $T(x) = \sum_{n\geq 1} \alpha_n T_n(x)$. Hence $T(x) \in \|x\| \sum_{n\geq 1} \gamma_n K_n$, so $T(x) \in R^+ \cap \operatorname{span}\big(\sum_{n\geq 1} \gamma_n K_n\big)$, and hence $T(x) = 0$.

(5) Since $\ker T_k \supset Y_n$ whenever $k > n \geq 1$ and $\ker T_\omega = Y$, we have

$$T^*(f)|_{Y_n} = \sum_{k=1}^{n} \alpha_k T_k^*(f)|_{Y_n} \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad f \in E^*.$$

Thus, if a functional $f \in E^*$ satisfies $T^*(f)|_{Y_1} \in W|_{Y_1}$ then $T_1^*(f)|_{Y_1} \in W|_{Y_1}$, therefore $T_1^*(f)|_{Y_1} \in W|_{Y_1} \cap \operatorname{span}(K_1'|_{Y_1})$, and hence $T_1^*(f)|_{Y_1} = 0$.

Now, fix $n \geq 2$ and choose $f \in E^*$ such that $T^*(f)|_{Y_n} \in W|_{Y_n}$. Then,

$$\alpha_n T_n^*(f)|_{Y_n} = T^*(f)|_{Y_n} - \sum_{j=1}^{n-1} \alpha_j T_j^*(f)|_{Y_n} \in \operatorname{span}\big(K_n'|_{Y_n}\big) \cap \left(W|_{Y_n} + \operatorname{span}\big(\sum_{i=1}^{n-1} K_i'|_{Y_n}\big) \right) = \{0\},$$

that is, $T_n^*(f)|_{Y_n} = 0$, hence $f|_{T_n(Y_n)} = 0$, and therefore $f|_{\overline{T_n(Y_n)}} = 0$. Since $\overline{T_n(Y_n)} = \overline{T(Y_n)}$ it follows that $f|_{T(Y_n)} = 0$, that is, $T^* f|_{Y_n} = 0$. Consequently $T^*(E^*)|_{Y_n} \cap W|_{Y_n} = 0$.

It remains to check that $T^*(E^*)|_Y \cap W|_Y = 0$. Choose $f \in E^*$ with $T^*(f)|_Y \in W|_Y$. Then, for each $n \geq 1$ we have $T^*(f)|_{Y_n} \in W|_{Y_n}$, and hence $T^*(f)|_{Y_n} = 0$. Since $\bigcup_n Y_n$ is dense in Y it follows that $T^*(f)|_Y = 0$.

(6) Firstly, we observe that if $f \in Y^\perp$ then $T_n^*(f)|_{Y_n} = 0$ for all $n \geq 1$, and therefore $T^*(f) = T_\omega^*(f)$. So $T^*(Y^\perp) = T_\omega^*(Y^\perp)$. Now, suppose that the space E/Y is infinite-dimensional, and take $f \in E^*$ such that $T^*(f) \in W$. Then $T^*(f)|_Y \in W|_Y$ for each $n \geq 1$, and because of the previous property we get $T^*(f)|_Y = 0$. Hence $f|_{\overline{T(Y)}} = 0$, so (by (2)) $f|_Y = 0$, consequently $T^*(f) = T_\omega^*(f) \in T_\omega^*(E^*)$, and bearing in mind that $T_\omega^*(E^*) \cap W = \{0\}$ we get $T^*(f) = 0$.

(7) Since $T^*(Y^\perp) = T_\omega^*(Y^\perp)$, this assertion is an immediate consequence of property (c_ω).

(8) We proceed inductively. If Y_1^* is separable then $\overline{T_1^*(E^*)|_{Y_1}} = Y_1^*$, and bearing in mind that $T^*(E^*)|_{Y_1} = T_1^*(E^*)|_{Y_1}$ we get $\overline{T^*(E^*)|_{Y_1}} = Y_1^*$.

Now, suppose that Y_n^* is separable for some $n \geq 2$, and assume that $\overline{T^*(E^*)|_{Y_{n-1}}} = Y_{n-1}^*$. Fix a functional $u^* \in Y_n^*$ and pick $\varepsilon > 0$. By the inductive hypothesis, there is $v^* \in E^*$ satisfying $\|u^*|_{Y_{n-1}} - T^*(v^*)|_{Y_{n-1}}\| < \varepsilon$. Since Y_{n-1}^* isometrically identifies with the quotient space $Y_n^*/(Y_{n-1}^\perp|_{Y_n})$, from the latter inequality we deduce the existence of a new functional $w^* \in Y_{n-1}^\perp$ such that

$$\|u^* - T^*(v^*)|_{Y_n} - w^*|_{Y_n}\| < \varepsilon. \tag{5.13}$$

On the other hand, as $\overline{T_n^*(Y_{n-1}^\perp)|_{Y_n}} = Y_{n-1}^\perp|_{Y_n}$, there is $z^* \in Y_{n-1}^\perp$ such that

$$\|T_n^*(z^*)|_{Y_n} - w^*|_{Y_n}\| < \varepsilon.$$

Observe that $T^*(z^*)|_{Y_n} = \alpha_n T_n^*(z^*)|_{Y_n}$. Indeed, for each $k \le n-1$ we have $T_k(E) \subset Y_{n-1}$, and taking into account that $z^* \in Y_{n-1}^{\perp}$ it follows that $T_k^*(z^*) = 0$. Hence

$$T^*(z^*)|_{Y_n} = \sum_{k=1}^{n-1} \alpha_k T_k^*(z^*)|_{Y_n} + \alpha_n T_n^*(z^*)|_{Y_n} = \alpha_n T_n^*(z^*)|_{Y_n}.$$

Therefore, $\|\alpha_n^{-1} T^*(z^*)|_{Y_n} - w^*|_{Y_n}\| < \varepsilon$. Combining this inequality with (5.13) we obtain

$$\|u^* - T^*(v^* + \alpha_n^{-1} z^*)|_{Y_n}\| \le \|u^* - T^*(v^*)|_{Y_n} - w^*|_{Y_n}\| + \|w^*|_{Y_n} - T^*(\alpha_n^{-1} z^*)|_{Y_n}\| < 2\varepsilon.$$

Thus, $u^* \in \overline{T^*(E^*)|_{Y_n}}$, and consequently $\overline{T^*(E^*)|_{Y_n}} = Y_n^*$.

(9) Assume that Y^* is separable, and let $q : E^* \to Y^*$ be the restriction map. Then, according to $(\star)$, the operator $(q \circ T_1^*)^* : Y^{**} \to E^{**}$ is one-to-one. We shall show that $(q \circ T^*)^* : Y^{**} \to E^{**}$ is one-to-one as well. Since $T_\omega^*(E^*) \subset Y^{\perp}$ we have $q \circ T_\omega^* = 0$, therefore

$$(q \circ T^*)^*(y^{**}) = \sum_{n \ge 1} \alpha_n (q \circ T_n^*)^*(y^{**}) \quad \text{for all} \quad y^{**} \in Y^{**}.$$

Thus, if a functional $y^{**} \in Y^{**}$ satisfies $(q \circ T^*)^*(y^{**}) = 0$ then

$$\alpha_1 (q \circ T_1^*)^*(y^{**}) = -\sum_{n \ge 2} \alpha_n (q \circ T_n^*)^*(y^{**}).$$

But for each $n \ge 1$,

$$(q \circ T_n^*)^*(y^{**}) = \sum_{j \ge 1} 4^{-j} y^{**}\left(s_{n,j}^*|_{Y_n}\right) y_{n,j}$$

(where the vectors $y_{n,j}$ are identified with elements in E^{**}), hence $(q \circ T_n^*)^*(y^{**}) \in \|y^{**}\| K_n$. Consequently

$$(q \circ T_1^*)^*(y^{**}) \in \operatorname{span}(K_1) \cap \operatorname{span}\Big(\sum_{n \ge 2} \alpha_n K_n\Big) \subset \operatorname{span}(K_1) \cap \operatorname{span}\Big(\sum_{n \ge 2} \gamma_n' K_n\Big).$$

So $(q \circ T_1^*)^*(y^{**}) = 0$, and thus $y^{**} = 0$. Consequently $(q \circ T^*)^*$ is an injective operator, and therefore $(q \circ T^*)(E^*)$ is a dense subspace of Y^*, that is, $\overline{T^*(E^*)|_Y} = Y^*$.

(10) The proof of this assertion can be achieved either arguing as in the previous one, by showing that the operator $T^{**} : E^{**} \to E^{**}$ is one-to-one via property $(\star\star)$, or combining assertions (7) and (9), since by the separability of E^*, $\overline{T^*(Y^{\perp})} = Y^{\perp}$ and $\overline{T^*(E^*)|_Y} = Y^*$.

Finally, if condition (a) is replaced with the weaker condition (a'): "$R \cap Y_n \in S(Y_n)$ for all $n \ge 1$", then every step in the proof holds true with the exception of property $R^+ \cap \operatorname{span}(\sum_{n \ge 1} \gamma_n K_n) = \{0\}$ (equality (5.11)), which has been needed only in the proof of assertion (4). Instead, if (a') holds, we can achive assertion (4'): "$T(Y_n) \cap R = \{0\}$ for all n and $T(E \setminus Y) \cap R = \{0\}$". Indeed, in this case and along the proof, the set R^+ is replaced with R (with the exception of equality (5.11)), and if $y \in Y_n$, then $T_k(y) = 0$ for $k > n$ and $T_\omega(y) = 0$. So $T(Y_n) \subset \operatorname{span}(\sum_{k=1}^n \alpha_n K_n)$. Since $R \cap (\sum_{i=1}^n K_i) = \{0\}$ we get that $T(Y_n) \cap R = \{0\}$ for all n. Also, if $z \in E \setminus Y$, then $T(z) = T_\omega(z) \in \operatorname{span} K_\omega$. Since $R \cap K_\omega = \{0\}$, we get $T(z) = 0$, which yields $T(E \setminus Y) = \{0\}$. $\square$

We end this paper with a pair of examples describing natural situations in which condition (a) in Theorem 5.1 is fulfilled.

Example 5.4. Let $\{Y_n\}_n$ be an increasing sequence of closed subspaces of a separable Banach space E satisfying the conditions of Theorem 5.1, and set $Y_0 = \{0\}$. For each $n \geq 1$, consider an arbitrary sequence $\{Y_{k,n}\}_k$ of closed infinite-codimensional subspaces of Y_n such that $Y_{k,n} \cap Y_{n-1} = \{0\}$ for all $k \geq 1$. As we already mentioned, $Y_{k,n} \in \mathcal{R}(Y_n)$ and $Y_{k,n}^+ = Y_{k,n}$ for every $k \geq 1$. Thus, the set $R = \bigcup_{n,k\geq 1} Y_{k,n}$ is an element of $\mathcal{S}(E)$ satisfying $R^+ = R$ and $R \cap Y_n = \bigcup_{k\geq 1} Y_{k,n} \in \mathcal{S}(Y_n)$ for each $n \geq 1$.

Example 5.5. Let $\{Y_n\}_n$ be an increasing sequence of closed subspaces of a separable Banach space E satisfying the conditions of Theorem 5.1, and set $Y_0 = \{0\}$. For each $n \geq 1$, we shall construct a proper dense operator range R_n in Y_n such that

$$R_n \cap Y_{n-1} = \{0\} \quad \text{and} \quad R_n^+ = R_n.$$

Since $Y_{n-1} \in \mathcal{R}(Y_n)$, there is a nuclear, one-to-one and dense-range operator $A_n : Y_n \to Y_n$ such that $A_n(Y_n) \cap Y_{n-1} = \{0\}$. On the other hand, as it is well-known, the separability of E yields a nuclear, one-to-one and dense-range operator $B_n : \ell^2 \to Y_n$. Consider the composition $T_n = A_n \circ B_n : \ell^2 \to Y_n$, and define

$$R_n = T_n(\ell^2).$$

Then, R_n is a proper dense operator range in Y_n, hence $R_n \in \mathcal{R}(E)$. Moreover, bearing in mind that $A_n(Y_n) \cap Y_{n-1} = \{0\}$ we get $R_n \cap Y_{n-1} = \{0\}$. On the other hand, because of the reflexivity of ℓ^2 we have $R_n^+ = R_n$ for each $n \geq 1$. In particular, the set $R = \bigcup_n R_n$ satisfies $R^+ \cap Y_n = R_n \in \mathcal{R}(Y_n) \subset \mathcal{S}(Y_n)$ for each $n \geq 1$.

More in general, if we consider for each $n \geq 1$ a sequence of proper and dense operator ranges $\{R_{k,n}\}_k \subset Y_n$ with the previous properties (that is, $R_{k,n} \cap Y_{n-1} = \{0\}$ and $R_{k,n}^+ = R_{k,n}$ for every $k \geq 1$), then the set $R := \bigcup_{n,k\geq 1} R_{k,n}$ lies in $\mathcal{S}(E)$ and satisfies $R^+ = R$ and $R \cap Y_n \in \mathcal{S}(Y_n)$ for all $n \geq 1$.

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References

[1] Y. A. Abramovich and C. D. Aliprantis, *An invitation to Operator Theory*. Graduate Studies in Mathematics vol. 50, Amer. Math. Soc., Providence, Rhode Island (2002).
[2] G. Bennet and N. Kalton, *Inclusion theorems for K–spaces*, Canadian J. of Math. **25** (1973), 511–524.
[3] J. M. Borwein and D. W. Tingley, *On supportless convex sets*, Proc. Amer. Math. Soc. **94** (1985), 471–476.
[4] J. B. Conway, *A Course in Functional Analysis*. Graduate Texts in Mathematics, Springer-Verlag, New York (1990).
[5] R. W. Cross, *On the continuous linear image of a Banach space*, J. Austral. Math. Soc. (Series A) **29** (1980), 219–234.
[6] R. W. Cross and V. Shevchik, *Disjointness of operator ranges*, Quaest. Math. **21** (1998), 247–260.
[7] I. Chalendar and J. Partington, *An image problem for compact operators*, Proc. Amer. Math. Soc. **134** (5) (2005), 1391–1396.

[8] L. Drewnowski, *A solution to a problem of De Wilde and Tsirulnikov*, Manuscripta Math. **37** (1) (1982), 61–64.
[9] A. F. M. ter Elst and M. Sauter, *Nonseparability and von Neumann's theorem for domains of unbounded operators*, J. Operator Theory **75** (2016), 367–386.
[10] M. Fabian, P. Habala, P. Hajek, V. Montesinos, V. Zizler, *Banach Space Theory: The basis for linear and nonlinear analysis.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011
[11] P. A. Fillmore and J. P. Williams, *On operator ranges*, Adv. Math. **7** (1971), 254–281.
[12] V. P. Fonf, *On a property of families of imbedded Banach spaces*, Teor. Funktsii Funktionsal Anal. i Priložen **55** (1991), 140–145.
[13] V. P. Fonf, S. Lajara, S. Troyanski and C. Zanco, *Operator ranges and quasicomplemented subspaces of Banach spaces*, Studia Math. **248** (2) (2019), 203–216.
[14] S. Grivaux, *Construction of operators with a prescribed behaviour*, Arch. Math. **81** (2003), 291–299.
[15] P. Hájek, V. Montesinos, J. Vanderwerff and V. Zizler, *Biorthogonal systems in Banach spaces.* CMS Books in Mathematics, Springer (2008).
[16] D. Kitson and R.M. Timoney, *Operator ranges and spaceability*, J. Math. Anal. Appl. **378** (2) (2011) 680–686.
[17] G. W. Mackey, *Note on a theorem of Murray*, Bull. Amer. Math. Soc. **52** (1946), 322–325.
[18] A. N. Plichko, *Some remarks on operator ranges* (Russian), Teor. Funktsii Funktsional. Anal. i Priložen. **53** (1990), 69–70. (English translation in J. Soviet Math. **58** (1992), 540–541).
[19] V. Shevchik, *On subspaces of a Banach space that coincide with the ranges of continuous linear operators* (Russian), Dokl. Akad. Nauk SSSR **263** (1982), 817–819.
[20] I. Singer, *On biorthogonal systems and total sequences of functionals*, Math. Ann. **193** (1971), 183–188.
[21] I. Singer, *On biorthogonal systems and total sequences of functionals II*, Math. Ann. **201** (1973), 1–8.
[22] I. Singer, Bases in Banach spaces II, Springer-Verlag, Berlin (1981).
[23] B. R. Yahaghi, *On injective or dense-range operators leaving a given chain of subspaces invariant*, Proc. Amer. Math. Soc. **132** (4) (2004), 1059–1066.

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