

Integral Calculus. Problem Set 1

1. Draw the following sets in \mathbb{R}^2 :

- (a) $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 2; y \geq 0\}$
- (b) $B = \{(x, y) : 4x^2 + 9y^2 \leq 1\}$
- (c) $C = \{(x, y); x^2 - y^2 \leq 1\}$
- (d) $D = \{(x, y) : 0 \leq x \leq 1; 0 \leq y \leq \sqrt{1 - x^2}\}$
- (e) $E = \{(x, y) : 1 \leq x \leq 2; 2x \leq y \leq 3x + 1\}$
- (f) $F = \{(x, y) : -1 \leq x \leq 1; -2|x| \leq y \leq |x|\}$
- (g) $G = \{(x, y) : 0 \leq y \leq 1; y^3 \leq x \leq y^2\}$
- (h) $H = \{(x, y) : -3 \leq y \leq 1; -\sqrt{9 - y^2} \leq x \leq \sqrt{9 - y^2}\}$

2. Draw the following surfaces in \mathbb{R}^3 :

- (a) $z = x^2 + y^2$
- (b) $z = \sqrt{x^2 + y^2}$
- (c) $z = x^2 - y^2$
- (d) $z^2 = y^2 + 4$
- (e) $4x^2 + 9y^2 = 1$
- (f) $4x^2 + 9y^2 + z^2 = 1$
- (g) $x^2 + y^2 - 2x = 0$
- (h) $y^2 = x^2 - z^2$
- (i) $\frac{y^2}{9} + \frac{z^2}{4} = 1 + \frac{x^2}{16}$
- (j) $\frac{x}{4} = \frac{y^2}{4} + \frac{z^2}{9}$

3. Draw the following sets in \mathbb{R}^3 :

- (a) $A = \{(x, y, z) : x^2 + y^2 \leq z \leq 1\}$
- (b) $B = \{(x, y, z) : 0 \leq z^2 \leq x^2 + y^2\}$
- (c) $C = \{(x, y, z); \sqrt{x^2 + y^2} \leq z \leq 1\}$
- (d) $D = \{(x, y, z) : x + y + z = 1; x \geq 0; y \geq 0; z \geq 0\}$
- (e) $E = \{(x, y, z) : 4x^2 + 9y^2 \leq 1; 0 \leq z \leq 4\}$
- (f) $F = \{(x, y, z) : z^2 \leq x^2 + y^2 \leq 4\}$
- (g) $G = \{(x, y, z) : x^2 + y^2 \leq 1 + z^2; -1 \leq z \leq 1\}$
- (h) $H = \{(x, y, z) : x^2 + y^2 \leq 1; z \geq 0; x^2 + y^2 + z^2 \leq 4\}$
- (i) $I = \{(x, y, z) : -1 \leq x \leq 0; 0 \leq y \leq 1 + x; 0 \leq z \leq 1 - y + x\}$
- (j) $J = \{(x, y, z) : -1 \leq x \leq 1; -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}; -\sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2}\}$

4. (a) Let us consider the function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $\varphi(x, y) = (x + y, x - y)$, the square $C = [0, 1] \times [0, 1]$ and the triangle T with vertexes $(0, 0)$, $(0, 1)$ and $(1, 0)$. Draw the sets $\varphi(C)$ and $\varphi(T)$.

- (b) Find $\varphi(C)$ and $\varphi(T)$ for $\varphi(x, y) = (x^2 - y^2, 2xy)$.

Cálculo Integral. Hoja 1

1. Dibujar aproximadamente los conjuntos siguientes en \mathbb{R}^2 :

- (a) $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 2; y \geq 0\}$
- (b) $B = \{(x, y) : 4x^2 + 9y^2 \leq 1\}$
- (c) $C = \{(x, y); x^2 - y^2 \leq 1\}$
- (d) $D = \{(x, y) : 0 \leq x \leq 1; 0 \leq y \leq \sqrt{1 - x^2}\}$
- (e) $E = \{(x, y) : 1 \leq x \leq 2; 2x \leq y \leq 3x + 1\}$
- (f) $F = \{(x, y) : -1 \leq x \leq 1; -2|x| \leq y \leq |x|\}$
- (g) $G = \{(x, y) : 0 \leq y \leq 1; y^3 \leq x \leq y^2\}$
- (h) $H = \{(x, y) : -3 \leq y \leq 1; -\sqrt{9 - y^2} \leq x \leq \sqrt{9 - y^2}\}$

2. Dibujar aproximadamente las superficies siguientes en \mathbb{R}^3 :

- (a) $z = x^2 + y^2$
- (b) $z = \sqrt{x^2 + y^2}$
- (c) $z = x^2 - y^2$
- (d) $z^2 = y^2 + 4$
- (e) $4x^2 + 9y^2 = 1$
- (f) $4x^2 + 9y^2 + z^2 = 1$
- (g) $x^2 + y^2 - 2x = 0$
- (h) $y^2 = x^2 - z^2$
- (i) $\frac{y^2}{9} + \frac{z^2}{4} = 1 + \frac{x^2}{16}$
- (j) $\frac{x}{4} = \frac{y^2}{4} + \frac{z^2}{9}$

3. Dibujar aproximadamente los conjuntos siguientes en \mathbb{R}^3 :

- (a) $A = \{(x, y, z) : x^2 + y^2 \leq z \leq 1\}$
- (b) $B = \{(x, y, z) : 0 \leq z^2 \leq x^2 + y^2\}$
- (c) $C = \{(x, y, z); \sqrt{x^2 + y^2} \leq z \leq 1\}$
- (d) $D = \{(x, y, z) : x + y + z = 1; x \geq 0; y \geq 0; z \geq 0\}$
- (e) $E = \{(x, y, z) : 4x^2 + 9y^2 \leq 1; 0 \leq z \leq 4\}$
- (f) $F = \{(x, y, z) : z^2 \leq x^2 + y^2 \leq 4\}$
- (g) $G = \{(x, y, z) : x^2 + y^2 \leq 1 + z^2; -1 \leq z \leq 1\}$
- (h) $H = \{(x, y, z) : x^2 + y^2 \leq 1; z \geq 0; x^2 + y^2 + z^2 \leq 4\}$
- (i) $I = \{(x, y, z) : -1 \leq x \leq 0; 0 \leq y \leq 1 + x; 0 \leq z \leq 1 - y + x\}$
- (j) $J = \{(x, y, z) : -1 \leq x \leq 1; -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}; -\sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2}\}$

4. (a) Definimos la aplicación $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ por $\varphi(x, y) = (x + y, x - y)$. Consideramos el cuadrado $C = [0, 1] \times [0, 1]$ y el triángulo T con vértices $(0, 0)$, $(0, 1)$ y $(1, 0)$. Dibujar los conjuntos $\varphi(C)$ y $\varphi(T)$.

- (b) Haz lo mismo para $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ por $\varphi(x, y) = (x^2 - y^2, 2xy)$.

Integral Calculus. Problem Set 2

1. Consider the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{for } x \in \mathbb{Q} \\ 1 - x & \text{for } x \notin \mathbb{Q} \end{cases}$$

Evaluate the upper and lower integral of f over $[0, 1]$.

2. Consider a subset $A \subset \mathbb{R}^n$ and a bounded function $f : A \rightarrow \mathbb{R}$. Show that:

- (a) $\sup\{\lambda f(x) : x \in A\} = \lambda \sup\{f(x) : x \in A\}$, whenever $\lambda > 0$.
- (b) $\sup\{\lambda f(x) : x \in A\} = \lambda \inf\{f(x) : x \in A\}$, whenever $\lambda < 0$.
- (c) $\sup\{f(x) + f(y) : x, y \in A\} = \sup\{f(x) : x \in A\} + \sup\{f(x) : x \in A\}$.
- (d) $\sup\{f(x) - f(y) : x, y \in A\} = \sup\{f(x) : x \in A\} - \inf\{f(x) : x \in A\}$.

3. Consider a rectangle $R \subset \mathbb{R}^n$, an integrable function $f : R \rightarrow \mathbb{R}$ and a constant $\lambda \in \mathbb{R}$.

Show that the function λf is integrable and

$$\int_R \lambda f = \lambda \int_R f.$$

4. Consider a rectangle $R \subset \mathbb{R}^n$, a bounded function $f : R \rightarrow \mathbb{R}$ and a partition P of R .

Show that f is integrable over R if and only if f is integrable over every subrectangle S of P . Moreover, in this case, show the following identity

$$\int_R f = \sum_{S \in P} \int_S f.$$

5. Consider a rectangle $R \subset \mathbb{R}^n$, a bounded and integrable function $f : R \rightarrow \mathbb{R}$ and constants m, M such that $m \leq f(x) \leq M$ for all $x \in R$. Show that

$$m \cdot v(R) \leq \int_R f \leq M \cdot v(R).$$

6. Show the following **Mean Value Theorem for Integrals**: Let us consider a rectangle $R \subset \mathbb{R}^n$ and a continuous function $f : R \rightarrow \mathbb{R}$. Then, there is a point $x_0 \in R$ satisfying

$$\int_R f = f(x_0) \cdot v(R).$$

7. Consider a n -rectangle (with positive n -dimensional volume) $R \subset \mathbb{R}^n$ and an integrable function $f : R \rightarrow [0, \infty)$ such that $\int_R f = 0$. Assume that f is continuous at x_0 . Show that $f(x_0) = 0$.

8. Consider the square $I = [0, 1] \times [0, 1]$ and the function $f : I \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 0 & \text{for } x \notin \mathbb{Q} \\ 0 & \text{for } x \in \mathbb{Q} \text{ and } y \notin \mathbb{Q} \\ \frac{1}{q} & \text{for } x \in \mathbb{Q} \text{ and } y = \frac{p}{q} \text{ lowest terms.} \end{cases}$$

Show that f is integrable over I and compute the integral.

9. Show that for E_1, \dots, E_m with content zero in \mathbb{R}^n , the union $\bigcup_{j=1}^m E_j$ has content zero as well.

10. (a) Show that the set $E_1 = \{(x, y) : x = y; 0 \leq x \leq 1\}$ has content zero in \mathbb{R}^2 .
 (b) Show that the set $E = \{(x, y) : x = y\}$ has measure zero in \mathbb{R}^2 .

11. (a) Consider a n -rectangle $R \subset \mathbb{R}^n$. Show that for any given $\varepsilon > 0$ there are n -cubes Q_1, \dots, Q_m such that $R \subset \bigcup_{i=1}^m Q_i$ and $\sum_{i=1}^m v(Q_i) < v(R) + \varepsilon$.
 (b) Show that in the definitions of measure zero and content zero the rectangles can be replaced with cubes.

- 12.** (a) Consider a rectangle $R \subset \mathbb{R}^n$ and a bounded function $f : R \rightarrow \mathbb{R}$. Assume that f vanishes outside a subset of content zero. Show that f is integrable over R and

$$\int_R f = 0.$$

- (b) Consider a rectangle $R \subset \mathbb{R}^n$ and bounded functions $f, g : R \rightarrow \mathbb{R}^n$. Assume that f is integrable and $f = g$ outside a subset of content zero. Show that g is integrable and

$$\int_R f = \int_R g.$$

- 13.** (a) Consider an integrable function $f : [a, b] \rightarrow \mathbb{R}$, where $[a, b] \subset \mathbb{R}$ is an interval. Show that the graph of f , $G_f = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], y = f(x)\}$, has content zero in \mathbb{R}^2 .
(b) Let us consider a rectangle R in \mathbb{R}^n and an integrable function $f : R \rightarrow \mathbb{R}$. Show that the graph of f , $G_f = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in R, t = f(x)\}$, has content zero in \mathbb{R}^{n+1} .

- 14.** Consider a Lipschitz function $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

- (a) Show that there is $K > 0$ such that for any n -cube $Q \subset \mathbb{R}^n$ of edge length ℓ the image $L(Q)$ is contained in a n -cube of edge length $K \cdot \ell$.
(b) Show that for any $E \subset \mathbb{R}^n$ with content zero (measure zero) the image $L(E)$ has content zero (measure zero).

- 15.** (a) Consider a Lipschitz function $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$, where $m < n$. Show that the image $L(\mathbb{R}^m)$ has measure zero in \mathbb{R}^n .

- (b) Deduce that every line in \mathbb{R}^2 has measure zero and every plane in \mathbb{R}^3 has measure zero.

- 16.** Consider a C^1 smooth path $\gamma : [a, b] \rightarrow \mathbb{R}^2$. Show that the image $\text{Im}(\gamma) := \gamma([a, b])$ has measure zero in \mathbb{R}^2 .

- 17.** Let A denote the set of points of \mathbb{R}^2 with at least one rational coordinate. Determine whether the set A has measure zero.

- 18.** Let B denote the set of points of the square $[0, 1] \times [0, 1]$ in \mathbb{R}^2 with at least one irrational coordinate. Determine whether the set B has area.

Cálculo Integral. Hoja 2

1. Consideramos la función $f : [0, 1] \rightarrow \mathbb{R}$ definida por

$$f(x) = \begin{cases} x & \text{si } x \in \mathbb{Q} \\ 1 - x & \text{si } x \notin \mathbb{Q} \end{cases}$$

Calcular la integral superior y la integral inferior de f en $[0, 1]$.

2. Sean $A \subset \mathbb{R}^n$ y $f : A \rightarrow \mathbb{R}$ una función acotada. Demostrar que:

- (a) Si $\lambda > 0$, entonces $\sup\{\lambda f(x) : x \in A\} = \lambda \sup\{f(x) : x \in A\}$.
- (b) Si $\lambda < 0$, entonces $\sup\{\lambda f(x) : x \in A\} = \lambda \inf\{f(x) : x \in A\}$.
- (c) $\sup\{f(x) + f(y) : x, y \in A\} = \sup\{f(x) : x \in A\} + \sup\{f(x) : x \in A\}$.
- (d) $\sup\{f(x) - f(y) : x, y \in A\} = \sup\{f(x) : x \in A\} - \inf\{f(x) : x \in A\}$.

3. Sean $R \subset \mathbb{R}^n$ un rectángulo, $f : R \rightarrow \mathbb{R}$ una función integrable y $\lambda \in \mathbb{R}$. Demostrar que la función λf es integrable y además

$$\int_R \lambda f = \lambda \int_R f.$$

4. Sean $R \subset \mathbb{R}^n$ un rectángulo, P una partición de R y $f : R \rightarrow \mathbb{R}$ una función acotada. Demostrar que f es integrable en R si, y sólo si, f es integrable en cada subrectángulo S de P . Además, en este caso,

$$\int_R f = \sum_{S \in P} \int_S f.$$

5. Sea $R \subset \mathbb{R}^n$ un rectángulo. Demostrar que si $f : R \rightarrow \mathbb{R}$ es una función integrable y $m \leq f(x) \leq M$ para todo $x \in R$, entonces

$$m \cdot v(R) \leq \int_R f \leq M \cdot v(R).$$

6. Demostrar el siguiente **teorema del valor medio para la integral**: Si R es un rectángulo de \mathbb{R}^n y $f : R \rightarrow \mathbb{R}$ es una función continua, entonces existe un punto $x_0 \in R$ tal que

$$\int_R f = f(x_0) \cdot v(R).$$

7. Sea $R \subset \mathbb{R}^n$ un rectángulo, $f : R \rightarrow \mathbb{R}$ integrable con $f \geq 0$ y supongamos que $\int_R f = 0$. Si f es continua en un punto $x_0 \in R$, demostrar que $f(x_0) = 0$.

8. Sea $I = [0, 1] \times [0, 1]$. Sea $f : I \rightarrow \mathbb{R}$ la función definida por

$$f(x, y) = \begin{cases} 0 & \text{si } x \notin \mathbb{Q} \\ 0 & \text{si } x \in \mathbb{Q} \text{ e } y \notin \mathbb{Q} \\ \frac{1}{q} & \text{si } x \in \mathbb{Q} \text{ e } y = \frac{p}{q} \text{ irreducible.} \end{cases}$$

Demostrar que f es integrable en I y calcular su integral.

9. Demostrar que si E_1, \dots, E_m tienen contenido cero en \mathbb{R}^n entonces la unión $\cup_{j=1}^m E_j$ también tiene contenido cero.

10. (a) Demostrar que el conjunto $E_1 = \{(x, y) : x = y; 0 \leq x \leq 1\}$ tiene contenido cero en \mathbb{R}^2 .

- (b) Demostrar que el conjunto $E = \{(x, y) : x = y\}$ tiene medida cero en \mathbb{R}^2 .

11. (a) Sea $R \subset \mathbb{R}^n$ un n -rectángulo. Demostrar que dado $\varepsilon > 0$ existen n -cubos Q_1, \dots, Q_m tales que $R \subset \cup_{i=1}^m Q_i$ y $\sum_{i=1}^m v(Q_i) < v(R) + \varepsilon$.

- (b) Demostrar que en las definiciones de contenido cero y medida cero pueden sustituirse los rectángulos por cubos.

- 12.** (a) Sean $R \subset \mathbb{R}^n$ un rectángulo y $f : R \rightarrow \mathbb{R}$ acotada. Supongamos que f se anula excepto en un conjunto de contenido nulo. Demostrar que f es integrable en R , y además

$$\int_R f = 0.$$

- (b) Sean $R \subset \mathbb{R}^n$ un rectángulo y $f, g : R \rightarrow \mathbb{R}^n$ acotadas. Supongamos que f es integrable y que $f = g$ excepto en un conjunto de contenido nulo. Demostrar que g también es integrable, y además

$$\int_R f = \int_R g.$$

- 13.** (a) Sea $f : [a, b] \rightarrow \mathbb{R}$ una función integrable en el intervalo $[a, b]$. Demostrar que su gráfica $G_f = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], y = f(x)\}$ tiene contenido nulo en \mathbb{R}^2 .

- (b) Sea $R \subset \mathbb{R}^n$ un rectángulo, y sea $f : R \rightarrow \mathbb{R}$ integrable. Demostrar que la gráfica $G_f = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in R, t = f(x)\}$ tiene contenido nulo en \mathbb{R}^{n+1} .

- 14.** Sea $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ una función Lipschitz.

- (a) Demostrar que existe $K > 0$ tal que, si $Q \subset \mathbb{R}^n$ es un n -cubo de lado ℓ , entonces $L(Q)$ está contenido en un n -cubo de lado $K \cdot \ell$.

- (b) Demostrar que si $E \subset \mathbb{R}^n$ tiene contenido cero (respectivamente, medida cero), entonces $L(E)$ también tiene contenido cero (respectivamente, medida cero).

- 15.** (a) Sea $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ una función Lipschitz, donde $m < n$. Demostrar que la imagen $L(\mathbb{R}^m)$ tiene medida cero en \mathbb{R}^n .

- (b) Deducir que toda recta en \mathbb{R}^2 tiene medida cero, y todo plano en \mathbb{R}^3 tiene medida cero.

- 16.** Sea $\gamma : [a, b] \rightarrow \mathbb{R}^2$ una curva paramétrica de clase C^1 . Demostrar que su imagen $\text{Im}(\gamma) := \gamma([a, b])$ tiene medida nula en \mathbb{R}^2 .

- 17.** Sea A el conjunto de puntos del plano \mathbb{R}^2 que tienen al menos una coordenada racional. Estudiar si A tiene medida nula.

- 18.** Sea B el conjunto de puntos del cuadrado $[0, 1] \times [0, 1]$ que tienen al menos una coordenada irracional. Estudiar si B tiene área.

Integral Calculus. Problem Set 3

1. Let us consider the set $A = \{(x, y) : x^2 + y^2 \leq 4\}$. Determine whether the following functions are integrable over A :

$$f(x, y) = \begin{cases} \sin \frac{1}{x+y} & \text{for } x+y \notin Q \\ 2 & \text{for } x+y \in Q \end{cases}$$

$$g(x, y) = \begin{cases} \sin \frac{1}{x+y} & \text{for } x+y \neq 0 \\ 2 & \text{for } x+y = 0 \end{cases}$$

2. Rewrite the following iterated integrals as double integrals over a region, sketch the region and interchange the order of integration. Finally, evaluate the integrals:

(a) $\int_{-1}^1 \int_0^1 (x^4 y + y^2) dy dx$

(b) $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x+y) dy dx$

(c) $\int_{-3}^2 \int_0^{y^2} (x^2 + y) dx dy$

(d) $\int_0^1 \int_{e^x}^{e^{2x}} x \log y dy dx$

(e) $\int_{-1}^1 \int_0^{|x|} 1 dy dx$

(f) $\int_{-1}^1 \int_{x^2}^1 x^2 y^3 dy dx$

3. Rewrite the following iterated integrals as triple integrals over a region, and interchange the order of integration.

(a) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dy dx$

(b) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^1 dz dy dx$

(c) $\int_0^1 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} dy dx dz$

(d) $\int_0^1 \int_0^x \int_0^{x^2} (x - 2y - 3z) dz dy dx$

(e) $\int_0^1 \int_0^z \int_0^{\sqrt{z^2-x^2}} dy dx dz$

(f) $\int_0^1 \int_0^1 \int_0^{\sqrt{1-x^2}} (x + 2y + 3z) dy dx dz$

4. Interchange the order of integration in the following iterated integrals:

(a) $\int_0^1 \int_{-\sqrt{1-y^2}}^{1-y} f(x, y) dx dy$

(b) $\int_0^a \int_{\frac{b}{a}\sqrt{a^2-x^2}}^b f(x, y) dy dx$, where $a, b > 0$

(c) $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 f(x, y, z) dz dy dx$

(d) $\int_0^1 \int_0^1 \int_0^{x^2+y^2} f(x, y, z) dz dx dy$

5. Evaluate the following integrals:

(a) $\int_D x^2 y dx dy$, where D is the triangle with vertexes $(0, 0)$, $(0, 1)$ and $(1, 0)$.

(b) $\int_D x^2 y dx dy$, where D is the triangle with vertexes $(0, 0)$, $(1, 1)$ and $(2, 4)$.

(c) $\int_D x dx dy$, where $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \sqrt{\pi}, 0 \leq y \leq \sin x^2\}$.

(d) $\int_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dx dy$ where D is the interior of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(e) $\int_D |\max\{x, y\}| dx dy$, where $D = [-2, 2] \times [-1, 1]$.

(f) $\int_V (x^2 + y^2) dx dy dz$, where $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 2z \leq 4\}$.

(g) $\int_V z dx dy dz$, where $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$.

(h) $\int_V z dx dy dz$, where $V = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq (R^2 - x^2 - y^2)^{1/2}\}$, $(R > 0)$.

6. Assume that R is a rectangle in \mathbb{R}^4 . Evaluate the volume of the image $L(R)$, in the following cases:

(a) $L : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is the linear function $L(x_1, x_2, x_3, x_4) = (x_1, x_1 + x_2, x_3, x_4)$.

(b) $L : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is the linear function $L(x_1, x_2, x_3, x_4) = (x_1, x_2, x_1 + 5x_3, x_4)$.

Cálculo Integral. Hoja 3

1. Consideramos el conjunto $A = \{(x, y) : x^2 + y^2 \leq 4\}$. Estudiar si las funciones siguientes son integrables en A :

$$f(x) = \begin{cases} \sin \frac{1}{x+y} & \text{si } x+y \notin Q \\ 2 & \text{si } x+y \in Q \end{cases}$$

$$g(x) = \begin{cases} \sin \frac{1}{x+y} & \text{si } x+y \neq 0 \\ 2 & \text{si } x+y = 0 \end{cases}$$

2. Expresar las integrales iteradas siguientes como integrales dobles sobre un recinto, dibujar el recinto y cambiar el orden de integración; finalmente, calcular:

$$(a) \int_{-1}^1 \int_0^1 (x^4 y + y^2) dy dx$$

$$(b) \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x+y) dy dx$$

$$(c) \int_{-3}^2 \int_0^{y^2} (x^2 + y) dx dy$$

$$(d) \int_0^1 \int_{e^x}^{e^{2x}} x \log y dy dx$$

$$(e) \int_{-1}^1 \int_0^{|x|} 1 dy dx$$

$$(f) \int_{-1}^1 \int_{x^2}^1 x^2 y^3 dy dx$$

3. Expresar la integrales iteradas siguientes como integrales triples sobre un recinto, y cambiar el orden de integración:

$$(a) \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dy dx$$

$$(b) \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^1 dz dy dx$$

$$(c) \int_0^1 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} dy dx dz$$

$$(d) \int_0^1 \int_0^x \int_0^{x^2} (x - 2y - 3z) dz dy dx$$

$$(e) \int_0^1 \int_0^z \int_0^{\sqrt{z^2-x^2}} dy dx dz$$

$$(f) \int_0^1 \int_0^1 \int_0^{\sqrt{1-x^2}} (x + 2y + 3z) dy dx dz$$

4. Cambiar el orden de integración en las siguientes integrales iteradas:

$$(a) \int_0^1 \int_{-\sqrt{1-y^2}}^{1-y} f(x, y) dx dy$$

$$(b) \int_0^a \int_{\frac{b}{a}\sqrt{a^2-x^2}}^b f(x, y) dy dx, \text{ siendo } a, b > 0.$$

$$(c) \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 f(x, y, z) dz dy dx$$

$$(d) \int_0^1 \int_0^1 \int_0^{x^2+y^2} f(x, y, z) dz dx dy$$

5. Calcular las integrales siguientes:

$$(a) \int_D x^2 y dx dy, \text{ siendo } D \text{ el triángulo de vértices } (0, 0), (0, 1) \text{ y } (1, 0).$$

$$(b) \int_D x^2 y dx dy, \text{ siendo } D \text{ el triángulo de vértices } (0, 0), (1, 1) \text{ y } (2, 4).$$

$$(c) \int_D x dx dy, \text{ siendo } D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \sqrt{\pi}, 0 \leq y \leq \sin x^2\}.$$

$$(d) \int_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dx dy \text{ siendo } D \text{ el interior de la elipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$(e) \int_D |\max\{x, y\}| dx dy, \text{ siendo } D = [-2, 2] \times [-1, 1].$$

$$(f) \int_V (x^2 + y^2) dx dy dz, \text{ donde } V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 2z \leq 4\}.$$

$$(g) \int_V z dx dy dz, \text{ donde } V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}.$$

$$(h) \int_V z dx dy dz, \text{ donde } V = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq (R^2 - x^2 - y^2)^{1/2}\}, (R > 0).$$

6. Sea R un rectángulo in \mathbb{R}^4 . Calcular el volumen del conjunto $L(R)$ en los siguientes casos:

$$(a) L : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \text{ es la aplicación lineal } L(x_1, x_2, x_3, x_4) = (x_1, x_1 + x_2, x_3, x_4).$$

$$(b) L : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \text{ es la aplicación lineal } L(x_1, x_2, x_3, x_4) = (x_1, x_2, x_1 + 5x_3, x_4).$$

Integral Calculus. Problem Set 4

- 1.** Find the area of the region bounded by the lines $x+y = a$, $x+y = b$, $2x-3y = c$, $2x-3y = d$ (where $a < b$ and $c < d$).

- 2.** Let E be the region of the first quadrant bounded by the lines $y = \frac{x}{2}$, $y = 3x$ and the hyperbolas $xy = 1$, $xy = 2$. Evaluate the integral

$$\int_E x^2 y^2 dx dy$$

by making the change of variables $u = xy$, $v = \frac{y}{x}$.

- 3.** Evaluate the following integrals by making a change of variables to polar coordinates:

- (a) $\int_D x^2 dx dy$, where $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2, y \geq 0\}$.
- (b) $\int_D |x+y| dx dy$, where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.
- (c) $\int_D (x^2 + y^2)^{-3} dx dy$, where $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$.

- 4.** Calculate the following areas:

- (a) The area of the region bounded by the curve with polar coordinates $\rho = a(1 + \cos \theta)$; ($a > 0$). (Cardioid)
- (b) The area of the region bounded by the curve with polar coordinates $\rho = a|\sin 3\theta|$; ($a > 0$). (Rose curve)
- (c) The area of the region bounded by the lemniscate $(x^2 + y^2)^2 = 2a(x^2 - y^2)$; ($a > 0$).

- 5.** Let us consider the mapping $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $\phi(u, v) = (x = u + v, y = v - u^2)$. Let D denote the triangle with vertexes $(0, 0)$, $(2, 0)$ and $(0, 2)$ in the plane (u, v) . Show that ϕ is a change of variables in a neighborhood of D . Find the set $\phi(D)$ and calculate its area.

- 6.** Let us consider the mapping $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $\phi(u, v) = (u^2 - v^2, 2uv)$.

- (a) Assume that C_R is the circle with center at the origin and radius R . Find $\phi(C_R)$.
- (b) Assume that S_θ is the line extending from the origin with (counterclockwise) angle θ from the positive x -axis. Find $\phi(S_\theta)$.
- (c) Let us consider the set $D = \{(u, v) \in \mathbb{R}^2 : 1 \leq u^2 + v^2 \leq 9; u \geq 0, v \geq 0\}$. Find $\phi(D)$.
- (d) Let us consider the set $E = \{(u, v) \in \mathbb{R}^2 : 1 \leq u^2 - v^2 \leq 9; 2 \leq uv \leq 4, u \geq 0\}$. Find $\phi(E)$.
- (e) Prove that $\phi : \{(u, v) : v > 0\} \rightarrow \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$ is a change of variables.
- (f) Evaluate $\int_E (u^2 + v^2)uv du dv$.

- 7.** Compute the following integrals:

- (a) $\int_V (x^2 + y^2) dx dy dz$, where $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 2z \leq 4\}$.
- (b) $\int_V z dx dy dz$, where $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$.
- (c) $\int_V z(x+y) dx dy dz$, where V is the solid bounded by $z = 0$, $z = a$, $xy = a^2$, $2(x+y) = 5a$, ($a > 0$).
- (d) $\int_V e^z dx dy dz$, where $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$.
- (e) $\int_V (x^2 + y^2 + z^2)^{-2} dx dy dz$, where $V = \{(x, y, z) \in \mathbb{R}^3 : a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$ ($a > 0$).
- (f) $\int_D \frac{x}{4x^2 + y^2} dx dy$, where $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq 4x^2 + y^2 \leq 16, x \geq 0, y \geq 0\}$.

- 8.** Compute the volume of the following solids:

- (a) The solid bounded by the cylinder $x^2 + y^2 = 1$, the plane $z = 0$ and the surface $z = x^2 + y^2$.
- (b) The solid bounded by the elliptic paraboloids $\frac{x^2}{3} + y^2 = 2z - 4$ and $\frac{2x^2}{3} + 3y^2 = 2z$.
- (c) The solid bounded by the surfaces $x + y = z$, $xy = 1$, $y = x$, $y = 2x$, $z = 0$ and lying in the first octant.
- (d) The solid bounded by the cylinder $x^2 + y^2 = ay$ and the sphere $x^2 + y^2 + z^2 = a^2$, ($a > 0$).

- (e) The solid bounded by the hyperboloid of one sheet $x^2 + y^2 = 1 + z^2$ and the planes $z = 0, z = 1$.
 - (f) The solid bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.
 - (g) The solid bounded by the paraboloid $z = x^2 + y^2$ and the planes $z = 2$ and $z = 4$.
9. Calculate the volume of the 4-dimensional euclidean ball of radius R :

$$B_R = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq R^2\}.$$

Cálculo Integral. Hoja 4

1. Calcular el área del recinto limitado por las rectas: $x + y = a$, $x + y = b$, $2x - 3y = c$, $2x - 3y = d$ (siendo $a < b$ y $c < d$).

2. Sea E el recinto del primer cuadrante limitado por las rectas $y = \frac{x}{2}$, $y = 3x$ y las hipérbolas $xy = 1$, $xy = 2$. Utilizar el cambio de variables $u = xy$, $v = \frac{y}{x}$ para calcular

$$\int_E x^2 y^2 dx dy.$$

3. Utilizando coordenadas polares, calcular las integrales:

- (a) $\int_D x^2 dx dy$, siendo $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2, y \geq 0\}$.
- (b) $\int_D |x + y| dx dy$, siendo $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.
- (c) $\int_D (x^2 + y^2)^{-3} dx dy$, siendo $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$.

4. Calcular las áreas siguientes:

- (a) El área limitada por la curva en polares: $\rho = a(1 + \cos \theta)$; ($a > 0$). (Cardioide)
- (b) El área limitada por la curva en polares: $\rho = a|\sin 3\theta|$; ($a > 0$). (Rosa)
- (c) El área limitada por la lemniscata $(x^2 + y^2)^2 = 2a(x^2 - y^2)$; ($a > 0$).

5. Se considera la transformación $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ dada por $\phi(u, v) = (x = u + v, y = v - u^2)$. Sea D el triángulo de vértices $(0, 0)$, $(2, 0)$ y $(0, 2)$ en el plano (u, v) . Comprobar que ϕ es un cambio de variables alrededor de D . Determinar el conjunto $\phi(D)$ y calcular su área.

6. Se considera la transformación $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ definida por $\phi(u, v) = (u^2 - v^2, 2uv)$.
- (a) Si C_R es la circunferencia centrada en el origen y con radio R , determinar $\phi(C_R)$.
 - (b) Si S_θ es la semirecta que parte del origen y forma un ángulo θ con el semi-eje positivo x , determinar $\phi(S_\theta)$.
 - (c) Si $D = \{(u, v) \in \mathbb{R}^2 : 1 \leq u^2 + v^2 \leq 9; u \geq 0, v \geq 0\}$, determinar $\phi(D)$.
 - (d) Si $E = \{(u, v) \in \mathbb{R}^2 : 1 \leq u^2 - v^2 \leq 9; 2 \leq uv \leq 4, u \geq 0\}$, determinar $\phi(E)$.
 - (e) Prueba que $\phi : \{(x, y) : y > 0\} \rightarrow \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$ es un cambio de variables.
 - (f) Calcula $\int_E (u^2 + v^2)uv du dv$.

7. Calcular:

- (a) $\int_V (x^2 + y^2) dx dy dz$, donde $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 2z \leq 4\}$.
- (b) $\int_V z dx dy dz$, donde $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$.
- (c) $\int_V z(x + y) dx dy dz$, donde V está limitado por: $z = 0$, $z = a$, $xy = a^2$, $2(x + y) = 5a$, ($a > 0$).
- (d) $\int_V e^z dx dy dz$, donde $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$.
- (e) $\int_V (x^2 + y^2 + z^2)^{-2} dx dy dz$, donde $V = \{(x, y, z) \in \mathbb{R}^3 : a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$.
- (f) $\int_D \frac{x}{4x^2 + y^2} dx dy$, siendo $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq 4x^2 + y^2 \leq 16, x \geq 0, y \geq 0\}$.

8. Halla el volumen de los cuerpos siguientes:

- (a) El cuerpo limitado por el cilindro $x^2 + y^2 = 1$, el plano $z = 0$ y la superficie $z = x^2 + y^2$.
- (b) El cuerpo limitado por los paraboloides elípticos $\frac{x^2}{3} + y^2 = 2z - 4$ y $\frac{2x^2}{3} + 3y^2 = 2z$.
- (c) El cuerpo limitado por las superficies $x + y = z$, $xy = 1$, $y = x$, $y = 2x$, $z = 0$ y está en el primer octante.
- (d) El cuerpo limitado por el cilindro $x^2 + y^2 = ay$ y la esfera $x^2 + y^2 + z^2 = a^2$, ($a > 0$).
- (e) El cuerpo limitado por el hiperbolóide de una hoja $x^2 + y^2 = 1 + z^2$ y los planos $z = 0$, $z = 1$.
- (f) El cuerpo limitado por los cilindros $x^2 + y^2 = 1$ y $x^2 + z^2 = 1$.
- (g) El cuerpo limitado por el parabolóide elíptico $z = x^2 + y^2$ y los planos $z = 2$ y $z = 4$.

9. Calcular el volumen de la bola euclídea 4-dimensional de radio R :

$$B_R = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq R^2\}.$$

Integral Calculus. Problem Set 5

- 1.** Use the comparison tests to determine whether or not the following improper integrals are convergent:

(a) $\int_0^\infty \frac{x^3 - x + 1}{3x^5 + x^2 + 5} dx.$

(b) $\int_0^1 \frac{x + e^x}{\sqrt{x}} dx.$

(c) $\int_1^\infty \frac{\cos x}{x^2} dx.$

(d) $\int_0^\infty x^5 e^{-x/2} dx.$

(e) $\int_0^\infty \frac{e^{-2x}}{\sqrt[3]{x}} dx.$

- 2.** Evaluate the following improper integrals:

(a) $\int_A (1+y)^{-2} dxdy,$ where $A = \{(x,y) : 0 \leq x \leq y \leq 2x\}.$

(b) $\int_B (1+y)^{-2} dxdy,$ where $B = \{(x,y) : x^2 \leq y \leq 2x^2\}.$

- 3.** Evaluate the improper integrals:

(a) $\int_A \frac{1}{x+y} dxdy,$ where $A = (0,1] \times [0,1].$

(b) $\int_B xe^{-y^3} dxdy,$ where $B = \{(x,y) : 0 \leq x \leq y\}.$

(c) $\int_C e^{-(x^2+y^2)} dxdy,$ where $C = \{(x,y) : x \geq 0; y \geq 0\}.$

(d) $\int_D \frac{3x^2 - 2y^2}{\sqrt{4-x^2-y^2}} dxdy,$ where $D = \{(x,y) : x^2 + y^2 < 4\}.$

(d) $\int_D \frac{3x^2 - 2y^2}{(x^2 + y^2)^{5/3}} dxdy,$ where $D = \{(x,y) : x^2 + y^2 < 1\}.$

- 4.** Let us consider the function $f(x,y,z) = (x^2 + y^2 + z^2)^{-p},$ where $p \in \mathbb{R},$ and a constant $R > 0.$ Determine the values of p for which the improper integral of f over A_R converges and compute it:

(a) $A_R = \{(x,y,z) : x^2 + y^2 + z^2 \geq R^2\}$

(b) $A_R = \{(x,y,z) : 0 < x^2 + y^2 + z^2 \leq R^2\}$

- 5.** Let us consider the rectangle $R = [a,b] \times [c,d].$ Calculate

$$\lim_{n \rightarrow \infty} \int_R e^{\frac{x^2+y^2}{n}} dx dy.$$

- 6.** Evaluate the following integrals by differentiating under the integral sign:

- (a) For $t \in \mathbb{R},$ calculate

$$\int_0^\infty \frac{\sin(tx)}{x} e^{-x} dx$$

- (b) For $t > 0,$ calculate

$$\int_0^\infty \frac{\sin x}{x} e^{-tx} dx$$

(c) For $p, q > 0$, calculate

$$\int_0^\infty \frac{e^{-px} - e^{-qx}}{x} dx$$

Cálculo Integral. Hoja 5

1. Usa los criterios de comparación para determinar si las siguientes integrales impropias existen:

(a) $\int_0^\infty \frac{x^3 - x + 1}{3x^5 + x^2 + 5} dx.$

(b) $\int_0^1 \frac{x + e^x}{\sqrt{x}} dx.$

(c) $\int_1^\infty \frac{\cos x}{x^2} dx.$

(d) $\int_0^\infty x^5 e^{-x/2} dx.$

(e) $\int_0^\infty \frac{e^{-2x}}{\sqrt[3]{x}} dx.$

2. Calcula, si existen, las siguientes integrales impropias:

(a) $\int_A (1+y)^{-2} dxdy,$ siendo $A = \{(x,y) : 0 \leq x \leq y \leq 2x\}.$

(b) $\int_B (1+y)^{-2} dxdy,$ siendo $B = \{(x,y) : x^2 \leq y \leq 2x^2\}.$

3. Calcula, si existen, las siguientes integrales impropias:

(a) $\int_A \frac{1}{x+y} dxdy,$ siendo $A = (0,1] \times [0,1].$

(b) $\int_B xe^{-y^3} dxdy,$ siendo $B = \{(x,y) : 0 \leq x \leq y\}.$

(c) $\int_C e^{-(x^2+y^2)} dxdy,$ siendo $C = \{(x,y) : x \geq 0; y \geq 0\}.$

(d) $\int_D \frac{3x^2 - 2y^2}{\sqrt{4-x^2-y^2}} dxdy,$ siendo $D = \{(x,y) : x^2 + y^2 < 4\}.$

(e) $\int_D \frac{3x^2 - 2y^2}{(x^2 + y^2)^{5/3}} dxdy,$ siendo $D = \{(x,y) : x^2 + y^2 < 1\}.$

4. Considera la función $f(x,y,z) = (x^2 + y^2 + z^2)^{-p}$, donde $p \in \mathbb{R}$ y $R > 0$. Halla los valores de p para los cuales la integral impropia de f en A_R existe y calcular en su caso la integral:

(a) $A_R = \{(x,y,z) : x^2 + y^2 + z^2 \geq R^2\}$

(b) $A_R = \{(x,y,z) : 0 < x^2 + y^2 + z^2 \leq R^2\}$

5. Consideramos el rectángulo $R = [a,b] \times [c,d]$. Calcula

$$\lim_{n \rightarrow \infty} \int_R e^{\frac{x^2+y^2}{n}} dx dy.$$

6. Evalua las siguientes integrales, derivando con respecto a los parámetros:

- (a) Para $t \in \mathbb{R}$, calcula

$$\int_0^\infty \frac{\sin(tx)}{x} e^{-x} dx$$

- (b) Para $t > 0$, calcula

$$\int_0^\infty \frac{\sin x}{x} e^{-tx} dx$$

- (c) Para $p, q > 0$, calcula

$$\int_0^\infty \frac{e^{-px} - e^{-qx}}{x} dx$$

Integral Calculus. Problem Set 6

1. Sketch the following paths and calculate their length:
 - (a) $\gamma(t) = (R \cos t, -R \sin t)$, $0 \leq t \leq 2\pi$.
 - (b) $\gamma(t) = (R \cos(t^2), R \sin(t^2))$, $0 \leq t \leq \sqrt{\pi}$.
 - (c) $\gamma(t) = (t^4, t^4)$, $-1 \leq t \leq 1$.
 - (d) $\gamma(t) = (t^3, |t|^3)$, $-1 \leq t \leq 1$.
 - (e) $\gamma(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 4\pi$.
2. Let us consider the planar paths $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$ for $0 \leq t \leq 1$ and $\sigma(s) = (\cos(2s), -\sin(2s))$, for $0 \leq s \leq \pi$. Are they equivalent paths?
3. Sketch the path γ and evaluate the integral of the scalar field f along γ , where
 - (a) $f(x, y) = 1 + y$; $\gamma(t) = (\cos^3 t, \sin^3 t)$, $0 \leq t \leq 3\pi/2$.
 - (b) $f(x, y, z) = xyz$; $\gamma(t) = (\cos t, \sin t, 3)$; $0 \leq t \leq 2\pi$.
 - (c) $f(x, y, z) = x + y + z$; $\gamma(t) = (\sin t, \cos t, t)$, $0 \leq t \leq 2\pi$.
4. Let C denote the boundary of the square $|x| + |y| \leq 2$ and define $f(x, y) = e^{x+y}$. Evaluate $\int_C f$.
5. (a) Calculate the area of a vertical fence built over the plane curve $y^2 = x$ for $0 \leq x \leq 1$ and variable height $h(x, y) = |y|$.

 (b) The average value of a scalar function f along a path γ is given by $\frac{\int_{\gamma} f}{\ell(\gamma)}$. Find the average value of the function $f(x, y, z) = x + y + z$ along the semicircle parametrized by $\tau(t) = (0, \sin \theta \cos \theta)$, $\theta \in [0, \pi]$.

 (c) Suppose the helix $\gamma(t) = (\cos t, \sin t, t)$, $t \in [0, 2\pi]$, is made of a wire with density of $\rho(x, y, z) = z$ grams per unit length or wire. What is the total mass of the wire?
6. Sketch the path γ and evaluate the integral of the vector field F along γ , where
 - (a) $F(x, y) = (-x^2 y, x y^2)$; $\gamma(t) = (\sin t, \cos t)$, $0 \leq t \leq 2\pi$.
 - (b) $F(x, y, z) = (x, y, xz)$; $\gamma(t) = (\sin t, \cos t, t)$, $0 \leq t \leq 2\pi$.
7. Find:
 - (a) $\int_{\gamma} y dx - x dy$; where $\gamma(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 2\pi$.
 - (b) $\int_{\gamma} \sin z dx + \cos z dy - (xy)^{1/3} dz$; where $\gamma(t) = (\cos^3 t, \sin^3 t, t)$, $0 \leq t \leq 7\pi/2$.
 - (c) $\int_{\gamma} y dx + (3y^2 - x) dy + z dz$; where $\gamma(t) = (t, t^n, 0)$, $0 \leq t \leq 1$; $n \in \mathbb{N}$.
 - (d) $\int_C x^2 dx + xy dy$; where C is the boundary of the square with vertexes $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$ oriented counterclockwise.
 - (e) $\int_C 2xy dx + (x^2 + z) dy + yz dz$; where C is the line segment from $(1, 0, 2)$ to $(3, 4, 1)$.
8. Consider the force field $F(x, y, z) = (xy, y, z)$. Compute the work done on a particle moving along the parabola $y = x^2$, $z = 0$, from $x = -1$ to $x = 2$.
9. Determine whether the following vector fields F are conservative or not. If so, find a potential function for F :
 - (a) $F(x, y) = (3x^2 + y, e^y + x)$ in \mathbb{R}^2 .
 - (b) $F(x, y) = (x + 4y, ax + y)$ in \mathbb{R}^2 , where $a \in \mathbb{R}$.
 - (c) $F(x, y, z) = (3y^2 z + ye^x, 6xyz + e^x, 3xy^2)$ in \mathbb{R}^3 .
10. Evaluate:
 - (a) $\int_{\gamma} x dy + y dx$, where γ is a path joining $(-1, 2)$ to $(2, 3)$.
 - (b) $\int_{\gamma} yz dx + zx dy + xy dz$, where γ is a path joining $(1, 1, 1)$ to $(1, 2, 3)$.
11. Let us consider an open and convex set $A \subset \mathbb{R}^n$ and a C^1 vector field $F : A \rightarrow \mathbb{R}^n$. Let us assume that F is conservative. Show that, for every $a \in A$, the function

$$\varphi(x) = \int_0^1 \langle F(a + t(x - a)), x - a \rangle dt$$

is a potential function for F . (Hint: Notice that the above integral is the line integral of F along the line segment connecting a and x .)

- 12.** Show that the vector field $F : \mathbb{R}^3 \setminus \{(0, 0, 0)\} \rightarrow \mathbb{R}^3$ defined by

$$F(x, y, z) = -\frac{K}{(x^2 + y^2 + z^2)^{3/2}}(x, y, z)$$

is conservative and the function

$$\phi(x, y, z) = \frac{K}{(x^2 + y^2 + z^2)^{1/2}}$$

is a potential function for F .

- 13.** Using Green's Theorem, evaluate

$$\int_{C^+} (y^2 + x^3)dx + x^4dy,$$

where C^+ is the perimeter of $[0, 1] \times [0, 1]$ oriented positively (i.e. counterclockwise).

- 14.** Verify Green's theorem for the vector field $F(x, y) = (2xy - x^2, x^2 + y^2)$ and the triangular region D with vertexes $(0, 0)$, $(0, 2)$ and $(1, 0)$.
- 15.** Verify Green's theorem for the vector field $F(x, y) = (-y, x)$ and the multiply connected region $D = \{(x, y) : x^2 + y^2 \leq 16; (x - 2)^2 + y^2 \geq 1; (x + 2)^2 + y^2 \geq 1\}$.
- 16.** Let $C \subset \mathbb{R}^2$ be a piecewise regular Jordan curve and let D denote the “inner” region of C . Use Green's theorem to prove that the area of D is

$$\text{Area}(D) = \int_{C^+} xdy = \int_{C^+} (-y)dx = \frac{1}{2} \int_{C^+} (xdy - ydx).$$

- 17.** Calculate the area enclosed by:

- (a) The ellipse $\gamma(t) = (a \cos t, b \sin t)$, $0 \leq t \leq 2\pi$.
 (b) The astroid $\gamma(t) = (a \cos^3 t, b \sin^3 t)$, $0 \leq t \leq 2\pi$.

- 18.** Let $F : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2$ be the vector field defined by $F(x, y) = (\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2})$ and let $C \subset \mathbb{R}^2$ be a piecewise regular Jordan curve such that the origin lies in the “inner” region of C .

- (a) Evaluate $\int_{\gamma} F$, where $\gamma(t) = (R \cos t, R \sin t)$, $0 \leq t \leq 2\pi$. Deduce that F is not a conservative vector field.
 (b) Use Green's theorem to evaluate $\int_{C^+} F$.
 (c) Consider a piecewise regular Jordan curve A lying in $R^2 \setminus \{(0, 0)\}$. Find the possible values of the integral of F along A^+ .

Cálculo Integral. Hoja 6

1. Dibujar los caminos siguientes y calcular su longitud:
 - (a) $\gamma(t) = (R \cos t, -R \sin t)$, $0 \leq t \leq 2\pi$.
 - (b) $\gamma(t) = (R \cos(t^2), R \sin(t^2))$, $0 \leq t \leq \sqrt{\pi}$.
 - (c) $\gamma(t) = (t^4, t^4)$, $-1 \leq t \leq 1$.
 - (d) $\gamma(t) = (t^3, |t|^3)$, $-1 \leq t \leq 1$.
 - (e) $\gamma(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 4\pi$.
2. Consideramos los caminos en el plano $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$, definido para $0 \leq t \leq 1$ y $\sigma(s) = (\cos(2s), -\sin(2s))$, definido para $0 \leq s \leq \pi$. ¿Son caminos equivalentes?
3. En los siguientes casos, calcular la integral de f a lo largo de γ y dibujar el camino γ :
 - (a) $f(x, y) = 1 + y$; $\gamma(t) = (\cos^3 t, \sin^3 t)$, $0 \leq t \leq 3\pi/2$.
 - (b) $f(x, y, z) = xyz$; $\gamma(t) = (\cos t, \sin t, 3)$; $0 \leq t \leq 2\pi$.
 - (c) $f(x, y, z) = x + y + z$; $\gamma(t) = (\sin t, \cos t, t)$, $0 \leq t \leq 2\pi$.
4. Si C es el borde del cuadrado $|x| + |y| \leq 2$ y tomamos $f(x, y) = e^{x+y}$, calcular $\int_C f$.
5. Calcular el área de la valla vertical construida sobre la curva plana $y^2 = x$ siendo $0 \leq x \leq 1$, con una altura variable $h(x, y) = |y|$.
6. En los siguientes casos, calcular la integral del campo vectorial F a lo largo de γ y dibujar γ :
 - (a) $F(x, y) = (-x^2 y, x y^2)$; $\gamma(t) = (\sin t, \cos t)$, $0 \leq t \leq 2\pi$.
 - (b) $F(x, y, z) = (x, y, zx)$; $\gamma(t) = (\sin t, \cos t, t)$, $0 \leq t \leq 2\pi$.
7. Calcular:
 - (a) $\int_{\gamma} y dx - x dy$; donde $\gamma(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 2\pi$.
 - (b) $\int_{\gamma} \sin z dx + \cos z dy - (xy)^{1/3} dz$; donde $\gamma(t) = (\cos^3 t, \sin^3 t, t)$, $0 \leq t \leq 7\pi/2$.
 - (c) $\int_{\gamma} y dx + (3y^2 - x) dy + z dz$; donde $\gamma(t) = (t, t^n, 0)$, $0 \leq t \leq 1$; $n \in \mathbb{N}$.
 - (d) $\int_C x^2 dx + xy dy$; donde C es el cuadrado de vértices $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$ que se recorre en sentido positivo.
 - (e) $\int_C 2xy dx + (x^2 + z) dy + z dz$; donde C es el segmento de $(1, 0, 2)$ a $(3, 4, 1)$.
8. Consideramos la fuerza $F(x, y, z) = (xy, y, z)$. Calcular el trabajo realizado sobre una partícula la cual se mueve sobre la parábola $y = x^2$, $z = 0$, desde $x = -1$ hasta $x = 2$.
9. Estudiar si los campos siguientes son conservativos, y calcular en su caso una función de potencial:
 - (a) $F(x, y) = (3x^2 + y, e^y + x)$ en \mathbb{R}^2 .
 - (b) $F(x, y) = (x + 4y, ax + y)$ en \mathbb{R}^2 , donde $a \in \mathbb{R}$.
 - (c) $F(x, y, z) = (3y^2 z + ye^x, 6xyz + e^x, 3xy^2)$, en \mathbb{R}^3 .
10. Calcular:
 - (a) $\int_{\gamma} x dy + y dx$, si γ es un camino de $(-1, 2)$ a $(2, 3)$.
 - (b) $\int_{\gamma} y z dx + z x dy + x y dz$, si γ es un camino de $(1, 1, 1)$ a $(1, 2, 3)$.
11. Sea $A \subset \mathbb{R}^n$ abierto convexo y sea $F : A \rightarrow \mathbb{R}^n$ un campo vectorial C^1 . Supongamos que F es conservativo. Demostrar que para cada $a \in A$ la función

$$\varphi(x) = \int_0^1 \langle F(a + t(x - a)), x - a \rangle dt$$
 es una función de potencial para F . (Indicación: Observa que la anterior integral es la integral de F a lo largo del segmento que une a con x .)
12. Comprobar que el campo $F : \mathbb{R}^3 \setminus \{(0, 0, 0)\} \rightarrow \mathbb{R}^3$ definido por

$$F(x, y, z) = -\frac{K}{(x^2 + y^2 + z^2)^{3/2}}(x, y, z)$$

es conservativo, y tiene como función de potencial:

$$\phi(x, y, z) = \frac{K}{(x^2 + y^2 + z^2)^{1/2}}.$$

- 13.** Utilizar el teorema de Green para calcular

$$\int_{C^+} (y^2 + x^3)dx + x^4dy,$$

donde C^+ es el perímetro de $[0, 1] \times [0, 1]$, orientado positivamente.

- 14.** Verificar el teorema de Green para el campo $F(x, y) = (2xy - x^2, x^2 + y^2)$ y el dominio triangular D cuyos vértices son los puntos $(0, 0)$, $(0, 2)$ y $(1, 0)$.

- 15.** Verificar el teorema de Green general para el campo $F(x, y) = (-y, x)$ y el dominio múltiplemente conexo $D = \{(x, y) : x^2 + y^2 \leq 16; (x - 2)^2 + y^2 \geq 1; (x + 2)^2 + y^2 \geq 1\}$.

- 16.** Sea $C \subset \mathbb{R}^2$ una curva de Jordan, regular a trozos, y sea D el recinto acotado y limitado por C (la parte “interna” de C). Utilizar el teorema de Green para demostrar que el área de D es

$$\text{Area}(D) = \int_{C^+} xdy = \int_{C^+} (-y)dx = \frac{1}{2} \int_{C^+} (xdy - ydx).$$

- 17.** Calcular el área encerrada por:

- (a) La elipse $\gamma(t) = (a \cos t, b \sin t)$, $0 \leq t \leq 2\pi$.
- (b) La astroide $\gamma(t) = (a \cos^3 t, b \sin^3 t)$, $0 \leq t \leq 2\pi$.

- 18.** Sea $F : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2$ el campo definido por $F(x, y) = (\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2})$, y sea $C \subset \mathbb{R}^2$ una curva de Jordan, regular a trozos, tal que el origen pertenece a la parte interna de C .

- (a) Calcular $\int_{\gamma} F$, siendo $\gamma(t) = (R \cos t, R \sin t)$, $0 \leq t \leq 2\pi$. Deducir que F no es conservativo.
- (b) Utilizar el teorema de Green para calcular $\int_{C^+} F$.
- (c) Sea A una curva de Jordan regular a trozos, contenida en $\mathbb{R}^2 \setminus \{(0, 0)\}$. Hallar los posibles valores de la integral de F a lo largo de A^+

Integral Calculus. Problem Set 7

1. Let us consider the open set $U = \{(x, y, z) : (x, y) \neq (0, 0)\}$ and let $\mathbf{F} : U \rightarrow \mathbb{R}^3$ be a vector field defined by

$$\mathbf{F}(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right).$$

- (a) Calculate $\operatorname{curl} \mathbf{F}$.
- (b) Is \mathbf{F} a conservative field?

2. Find parametrizations of the following surfaces:

- (a) $S = \{(x, y, z) : x^2 + y^2 = z; 1 \leq z \leq 2\}$.
- (b) $S = \{(x, y, z) : x^2 + y^2 = 4; -1 \leq z \leq 1\}$.
- (c) $S = \{(x, y, z) : x^2 + y^2 = z^2; 1 \leq z \leq 2\}$.
- (d) $S = \{(x, y, z) : x^2 + y^2 = 1 + z^2; -1 \leq z \leq 1\}$.
- (e) $S = \{(x, y, z) : x^2 + y^2 + z^2 = R^2; z \geq 0\}$.

3. Find the area of the following surfaces:

- (a) The portion of the unit sphere lying in the region $x^2 + y^2 \leq z^2, z \geq 0$.
- (b) The portion of the sphere $x^2 + y^2 + z^2 = R^2$ lying in the region $x^2 + y^2 \leq Ry$.
- (c) The portion of the cone $z^2 = 3(x^2 + y^2)$ lying in the region $z \geq x^2 + y^2$.

4. Assume $0 < r < R$. Consider the circle with center $(R, 0, 0)$ and radius r in the xz -plane. Find the area of the torus generated by revolving this circle in three-dimensional space about the z -axis. The parametric equations of the torus are

$$\begin{aligned} x &= (R + r \cos \psi) \cos \theta \\ y &= (R + r \cos \psi) \sin \theta \\ z &= r \sin \psi \end{aligned}$$

for $0 \leq \theta \leq 2\pi$ and $0 \leq \psi \leq 2\pi$.

5. Suppose $f : [a, b] \rightarrow [0, \infty)$ is a C^1 smooth function.

- (a) Find a parametrization of the surface of revolution S generated by revolving the graph of the function f about the x -axis.
- (b) Show that the area of S is

$$a(S) = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx.$$

6. For each of the following, evaluate the integral of f over the surface S :

- (a) $f(x, y, z) = x^2 + y^2; S = \{(x, y, z) : x^2 + y^2 + z^2 = R^2\}$.
- (b) $f(x, y, z) = xyz; S$ is the triangle with vertexes $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 1, 1)$.
- (c) $f(x, y, z) = z; S = \{(x, y, z) : z = x^2 + y^2 \leq 1\}$.

7. For each of the following, evaluate the integral of the vector field \mathbf{F} over the surface S :

- (a) $\mathbf{F}(x, y, z) = (x, y, -y); S = \{(x, y, z) : x^2 + y^2 = 1; 0 \leq z \leq 1\}$ oriented by the outward normal.
- (b) $\mathbf{F}(x, y, z) = (yz, xz, xy); S$ is the surface of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0, x + y + z = 1$, oriented by the outward normal.
- (c) $\mathbf{F}(x, y, z) = (x^2, y^2, z^2); S = \{(x, y, z) : z^2 = x^2 + y^2, 1 \leq z \leq 2\}$ oriented by the outward normal.
- (d) $\mathbf{F}(x, y, z) = (x, y, z); S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ oriented by the outward normal.

8. Consider the following surfaces: $S_1 = \{(x, y, z) : x^2 + y^2 = 1; 0 \leq z \leq 1\}$, $S_2 = \{(x, y, z) : x^2 + y^2 + (z - 1)^2 = 1; z \geq 1\}$ and $S = S_1 \cup S_2$ oriented by the outward normal. Consider the vector field \mathbf{F} defined by $\mathbf{F}(x, y, z) = (zx + z^2y + x, z^3xy + y, z^2x^2)$. Evaluate

$$\int_S \operatorname{curl} \mathbf{F}.$$

- 9.** Use Gauss' divergence theorem to evaluate $\int_S \mathbf{F}$, where $\mathbf{F}(x, y, z) = (xy^2, x^2y, y)$ and $S = \{x^2 + y^2 = 1, -1 < z < 1\} \cup \{x^2 + y^2 \leq 1, z = 1\} \cup \{x^2 + y^2 \leq 1, z = -1\}$.
- 10.** Consider $f(x, y, z) = x^2 + 2xy + z^2 - 3x + 1$, $\mathbf{F}(x, y, z) = (e^{-xy} + z, z \sin y, x^2 - z^2 + y^2)$ and $V = \{(x, y, z) : 0 \leq z \leq 3 - x^2 - y^2, x^2 + y^2 + z^2 \geq 4z - 3\}$. Evaluate
- $$\int_{\partial V} \nabla f + \text{curl } \mathbf{F},$$
- where ∂V is oriented by the outward normal.
- 11.** Consider the sets $V = \{(x, y, z) : 0 \leq z \leq 1 - x^2 - y^2, x \geq 0, y \geq 0\}$, $S = \{(x, y, z) : z = 1 - x^2 - y^2, x \geq 0, y \geq 0, z \geq 0\}$ (oriented by the outward normal) and let C denote the boundary of S .
- (a) Evaluate the area of S .
 - (b) Evaluate the volume of V .
 - (c) Evaluate $\int_C \mathbf{F}$, where $\mathbf{F}(x, y, z) = (1 - 2z, 0, 2y)$,
- 12.** Consider the following surfaces: $S_1 = \{(x, y, z) : x^2 + y^2 = z^2; 1 \leq z \leq 2\}$, $S_2 = \{(x, y, z) : x^2 + y^2 = 4; 2 \leq z \leq 3\}$ and $S = S_1 \cup S_2$. Consider the vector field $\mathbf{F}(x, y, z) = (y, -x, x + y + z)$.
- (a) Find parametrizations of S_1 and S_2 with compatible orientations. Describe the induced orientation on ∂S_1 , ∂S_2 and ∂S .
 - (b) Evaluate $\int_S \text{curl } \mathbf{F}$, directly by using the parametrizations obtained in (a).
 - (c) Evaluate $\int_S \text{curl } \mathbf{F}$, by using Stokes' theorem.

Cálculo Integral. Hoja 7

1. Consideremos el conjunto abierto $U = \{(x, y, z) : (x, y) \neq (0, 0)\}$ de \mathbb{R}^3 y el campo vectorial $\mathbf{F} : U \rightarrow \mathbb{R}^3$ definido por

$$\mathbf{F}(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right).$$

- (a) Calcular $\text{rot } \mathbf{F}$.
- (b) Estudiar si \mathbf{F} es conservativo en U .

2. Encontrar parametrizaciones de las superficies siguientes:

- (a) $S = \{(x, y, z) : x^2 + y^2 = z; 1 \leq z \leq 2\}$.
- (b) $S = \{(x, y, z) : x^2 + y^2 = 4; -1 \leq z \leq 1\}$.
- (c) $S = \{(x, y, z) : x^2 + y^2 = z^2; 1 \leq z \leq 2\}$.
- (d) $S = \{(x, y, z) : x^2 + y^2 = 1 + z^2; -1 \leq z \leq 1\}$.
- (e) $S = \{(x, y, z) : x^2 + y^2 + z^2 = R^2; z \geq 0\}$.

3. Calcular el área de las superficies siguientes:

- (a) La parte de la esfera unitaria dentro de la región $x^2 + y^2 \leq z^2, z \geq 0$.
- (b) La parte de la esfera $x^2 + y^2 + z^2 = R^2$ dentro de la región $x^2 + y^2 \leq Ry$
- (c) La parte del cono $z^2 = 3(x^2 + y^2)$ dentro de la región $z \geq x^2 + y^2$.

4. Sean $0 < r < R$. Calcular el área del toro obtenido al girar la circunferencia de centro $(R, 0, 0)$ y radio r del plano xz alrededor del eje z . Las ecuaciones paramétricas del toro son

$$\begin{aligned} x &= (R + r \cos \psi) \cos \theta \\ y &= (R + r \cos \psi) \sin \theta \\ z &= r \sin \psi \end{aligned}$$

con $0 \leq \theta \leq 2\pi$ y $0 \leq \psi \leq 2\pi$.

5. Sea $f : [a, b] \rightarrow [0, \infty)$ una función de clase C^1 .

- (a) Obtener una parametrización de la superficie de revolución S obtenida al girar la gráfica de f alrededor del eje x .
- (b) Demostrar que el área de S es:

$$a(S) = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx.$$

6. En los siguientes casos, calcular la integral de f sobre la superficie S :

- (a) $f(x, y, z) = x^2 + y^2; S = \{(x, y, z) : x^2 + y^2 + z^2 = R^2\}$.
- (b) $f(x, y, z) = xyz; S$ es el triángulo de vértices $(1, 0, 0)$, $(0, 2, 0)$ y $(0, 1, 1)$.
- (c) $f(x, y, z) = z; S = \{(x, y, z) : z = x^2 + y^2 \leq 1\}$.

7. En los siguientes casos, calcular la integral del campo \mathbf{F} sobre la superficie S :

- (a) $\mathbf{F}(x, y, z) = (x, y, -y); S = \{(x, y, z) : x^2 + y^2 = 1; 0 \leq z \leq 1\}$ orientada con la normal exterior.
- (b) $\mathbf{F}(x, y, z) = (yz, xz, xy); S$ es la superficie del tetraedro limitado por los planos $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$ orientada con la normal exterior.
- (c) $\mathbf{F}(x, y, z) = (x^2, y^2, z^2); S = \{(x, y, z) : z^2 = x^2 + y^2, 1 \leq z \leq 2\}$ orientada con la normal exterior.
- (d) $\mathbf{F}(x, y, z) = (x, y, z); S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ orientada con la normal exterior.

8. Consideremos las superficies siguientes: $S_1 = \{(x, y, z) : x^2 + y^2 = 1; 0 \leq z \leq 1\}$, $S_2 = \{(x, y, z) : x^2 + y^2 + (z - 1)^2 = 1; z \geq 1\}$ y $S = S_1 \cup S_2$. Sea el campo $\mathbf{F}(x, y, z) = (zx + z^2y + x, z^3xy + y, z^2x^2)$. Calcular

$$\int_{\partial S} \text{rot } \mathbf{F}.$$

- 9.** Utilizar el teorema de la divergencia para calcular $\int_S \mathbf{F}$, donde $\mathbf{F}(x, y, z) = (xy^2, x^2y, y)$ y $S = \{x^2 + y^2 = 1, -1 < z < 1\} \cup \{x^2 + y^2 \leq 1, z = 1\} \cup \{x^2 + y^2 \leq 1, z = -1\}$.
- 10.** Consideramos $f(x, y, z) = x^2 + 2xy + z^2 - 3x + 1$, $\mathbf{F}(x, y, z) = (e^{-xy} + z, z \sin y, x^2 - z^2 + y^2)$ y sea $V = \{(x, y, z) : 0 \leq z \leq 3 - x^2 - y^2, x^2 + y^2 + z^2 \geq 4z - 3\}$. Calcular
- $$\int_{\partial V} \nabla f + \text{rot } \mathbf{F}.$$
- 11.** Sean $V = \{(x, y, z) : 0 \leq z \leq 1 - x^2 - y^2, x \geq 0, y \geq 0\}$, $S = \{(x, y, z) : z = 1 - x^2 - y^2, x \geq 0, y \geq 0, z \geq 0\}$ y sea C el borde de S .
- Calcular el área de S .
 - Calcular el volumen de V .
 - Calcular $\int_C \mathbf{F}$, donde $\mathbf{F}(x, y, z) = (1 - 2z, 0, 2y)$.
- 12.** Consideramos las superficies siguientes: $S_1 = \{(x, y, z) : x^2 + y^2 = z^2; 1 \leq z \leq 2\}$, $S_2 = \{(x, y, z) : x^2 + y^2 = 4; 2 \leq z \leq 3\}$ y $S = S_1 \cup S_2$. Consideramos también el campo $\mathbf{F}(x, y, z) = (y, -x, x + y + z)$.
- Obtener parametrizaciones de S_1 y de S_2 que determinen orientaciones compatibles. Describir la orientación inducida en ∂S .
 - Calcular $\int_S \text{rot } \mathbf{F}$ directamente, utilizando las parametrizaciones obtenidas (a).
 - Calcular $\int_S \text{rot } \mathbf{F}$ utilizando el teorema de Stokes.