

## Comparison tests for improper integrals (of functions of one variable)

(1) Let  $f, g : [a, \infty) \rightarrow \mathbb{R}$  be integrable functions on  $[a, z]$  for all  $z > a$ .

- (a) If  $0 \leq f \leq g$  on  $[a, \infty)$  and the improper integral  $\int_a^\infty g$  converges, then the improper integral  $\int_a^\infty f$  converges.
- (b) If  $0 \leq f \leq g$  on  $[a, \infty)$  and the improper integral  $\int_a^\infty f$  diverges, then the improper integral  $\int_a^\infty g$  diverges.
- (c) If  $f$  and  $g$  are non-negative and continuous functions on  $[a, \infty)$ ,  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$  and  $\int_a^\infty g$  converges, then  $\int_a^\infty f$  converges.
- (d) If  $f$  and  $g$  are non-negative and continuous functions on  $[a, \infty)$ ,  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$  and  $\int_a^\infty g$  diverges, then  $\int_a^\infty f$  diverges.
- (e) If  $f$  and  $g$  are nonnegative and continuous functions on  $[a, \infty)$  and  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \neq 0$  (and it is finite), then  $\int_a^\infty f$  converges if and only if  $\int_a^\infty g$  converges.

(2) Let  $f, g : (a, b] \rightarrow \mathbb{R}$  be integrable functions on  $(z, b]$  for all  $a < z < b$ .

- (a) If  $0 \leq f \leq g$  on  $(a, b]$  and the improper integral  $\int_a^b g$  converges, then the improper integral  $\int_a^b f$  converges.
  - (b) If  $0 \leq f \leq g$  on  $(a, b]$  and the improper integral  $\int_a^b f$  diverges, then the improper integral  $\int_a^b g$  diverges.
  - (c) If  $f$  and  $g$  are non-negative functions on  $(a, b]$ ,  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = 0$  and  $\int_a^b g$  converges, then  $\int_a^b f$  converges.
  - (d) If  $f$  and  $g$  are non-negative functions on  $(a, b]$ ,  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \infty$  and  $\int_a^b g$  diverges, then  $\int_a^b f$  diverges.
  - (e) If  $f$  and  $g$  are non-negative functions on  $(a, b]$  and  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} \neq 0$  (and it is finite), then  $\int_a^b f$  converges if and only if  $\int_a^b g$  converges.
- (3) (a) Let  $f : [a, \infty) \rightarrow \mathbb{R}$  be an integrable function on  $[a, z]$  for all  $a < z$ . If the improper integral  $\int_a^\infty |f|$  converges, then the improper integral  $\int_a^\infty f$  converges.
- (b) Let  $f : (a, b] \rightarrow \mathbb{R}$  be an integrable function on  $[z, b]$  for all  $a < z < b$ . If the improper integral  $\int_a^b |f|$  converges, then the improper integral  $\int_a^b f$  converges.
- (c) **Remark.** The converse implications of (a) and (b) do not hold: The function  $f(x) = \frac{\sin x}{x}$ , for  $x \in (0, \infty)$ , is integrable on  $(0, z)$  for all  $z > 0$  and  $\int_0^\infty \frac{\sin x}{x} dx$  converges. However,  $\int_0^\infty \frac{|\sin x|}{x} dx$  diverges.

## Criterios de comparación para integrales impropias (de funciones de una variable)

(1) Sean  $f, g : [a, \infty) \rightarrow \mathbb{R}$  funciones integrables en  $[a, z]$  para todo  $z > a$ .

- (a) Si  $0 \leq f \leq g$  en  $[a, \infty)$  y la integral impropia  $\int_a^\infty g$  converge, entonces la integral impropia  $\int_a^\infty f$  converge.
- (b) Si  $0 \leq f \leq g$  en  $[a, \infty)$  y la integral impropia  $\int_a^\infty f$  diverge, entonces la integral impropia  $\int_a^\infty g$  diverge.
- (c) Si  $f$  y  $g$  son funciones continuas y no negativas en  $[a, \infty)$ ,  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$  y  $\int_a^\infty g$  converge, entonces  $\int_a^\infty f$  converge.
- (d) Si  $f$  y  $g$  son funciones continuas y no negativas en  $[a, \infty)$ ,  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$  y  $\int_a^\infty g$  diverge, entonces  $\int_a^\infty f$  diverge.
- (e) Si  $f$  y  $g$  son funciones continuas y no negativas en  $[a, \infty)$  y  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \neq 0$  (y es finito), entonces  $\int_a^\infty f$  converge si y solo si  $\int_a^\infty g$  converge.

(2) Sean  $f, g : (a, b] \rightarrow \mathbb{R}$  funciones integrables en  $(z, b]$  para todo  $a < z < b$ .

- (a) Si  $0 \leq f \leq g$  en  $(a, b]$  y la integral impropia  $\int_a^b g$  converge, entonces la integral impropia  $\int_a^b f$  converge.
- (b) Si  $0 \leq f \leq g$  en  $(a, b]$  y la integral impropia  $\int_a^b f$  diverge, entonces la integral impropia  $\int_a^b g$  diverge.
- (c) Si  $f$  y  $g$  son funciones no negativas en  $(a, b]$ ,  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = 0$  y  $\int_a^b g$  converge entonces  $\int_a^b f$  converge.
- (d) Si  $f$  y  $g$  son funciones no negativas en  $(a, b]$ ,  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \infty$  y  $\int_a^b g$  diverge entonces  $\int_a^b f$  diverge.
- (e) Si  $f$  y  $g$  son funciones no negativas en  $(a, b]$  y  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} \neq 0$  (y es finito), entonces  $\int_a^b f$  converge si y solo si  $\int_a^b g$  converge.

- (3) (a) Sea  $f : [a, \infty) \rightarrow \mathbb{R}$  una función integrable en  $[a, z]$  para todo  $a < z$ . Si la integral impropia  $\int_a^\infty |f|$  converge, entonces la integral impropia  $\int_a^\infty f$  converge.
- (b) Sea  $f : (a, b] \rightarrow \mathbb{R}$  una función integrable en  $[z, b]$  para todo  $a < z < b$ . Si la integral impropia  $\int_a^b |f|$  converge, entonces la integral impropia  $\int_a^b f$  converge.
- (c) **Observación.** Los recíprocos de (a) y (b) no son ciertos: La función  $f(x) = \frac{\sin x}{x}$ , para  $x \in (0, \infty)$ , es integrable en  $(0, z)$  para todo  $z > 0$  y  $\int_0^\infty \frac{\sin x}{x} dx$  converge. Sin embargo,  $\int_0^\infty \frac{|\sin x|}{x} dx$  diverge.