OPERATOR RANGES IN BANACH SPACES WITH WEAK STAR SEPARABLE DUAL

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Abstract. We provide several extensions for Banach spaces with weak*-separable dual of a theorem of Schevchik ensuring that for every proper dense operator range R in a separable Banach space E, there exists a one-to-one and dense-range operator such that $T(E) \cap R = \{0\}$. These results lead to several characterizations of Banach spaces with weak*-separable dual in terms of disjointness properties of operator ranges, which yield a refinement of a theorem of Plichko concerning the spaceability of the complementary set of a proper dense operator range, and an affirmative solution to a problem of Borwein and Tingley for the class of Banach spaces with a separable quotient and weak*-separable dual. We also provide an extension to these spaces of a theorem of Cross and Shevchik, which guarantees that for every proper dense operator range R in a separable Banach space E there exist two closed quasicomplementary subspaces X and Y of E such that $R \cap (X+Y) = \{0\}$. Finally, we prove that some weak forms of the theorems of Shevchik and Cross and Shevchik do not hold in any nonseparable weakly Lindelöf determined Banach space.

1. Introduction

A linear subspace R of a Banach space E is called an **operator range** if R is the image of a bounded linear operator $T : X \to E$, for some Banach space X. If $X = E$, then we say that R is an **endomorphism range** in E. The class of operator ranges in a Banach space contains the family of its closed subspaces, but it has a much more flexible structure. This is due in part to the fact that the sum of two operator ranges in a Banach space E is again an operator range in E (cf. [7, Proposition 2.2]). These subspaces play an important role in several areas of Functional Analysis, in particular in Banach space Theory. In this respect, it is worth to mention the characterization of Saxon and Wilansky of Banach spaces with an infinite-dimensional separable quotient as those spaces which contain a proper dense operator range [24], and the result of Bennet and Kalton in [5] ensuring that a dense linear subspace V of a Banach space E is non-barrelled if, and only if, there exists a proper dense operator range R in E such that $V \subset R$. For detailed accounts on operator ranges in Hilbert and Banach spaces, we refer respectively to the works [13] and [7].

In this paper, we are concerned with some disjointness properties of operator ranges. A classical result of von Neumann, restated by Dixmier in terms of operator ranges (cf. [13, Theorem 3.6]) asserts that, if H is a separable Hilbert space, then for every non-closed endomorphism range $R \subset H$ there exists a unitary operator $T : H \to H$ such that $T(R)$ and

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R are **essentially disjoint**, that is, $T(R) \cap R = \{0\}$. As it has been recently shown by ter Elst and Sauter [12], this result does not hold true if H is any nonseparable Hilbert space. In the Banach space setting, Shevchik [25] proved that if E is a separable Banach space, then for every proper dense operator range R in E there exists a nuclear, one-to-one and dense-range endomorphism $T : E \to E$ such that $T(E) \cap R = \{0\}$. (Recall that an operator $T : E \to Y$ between Banach spaces is said to be **nuclear** whenever there exist sequences $\{y_n\}_n$ in Y and $\{f_n\}_n$ in E^* (the dual space of E) such that $\sum_{n\geq 1} \|f_n\|\|y_n\| < \infty$ and $T(x) = \sum_{n\geq 1} f_n(x)y_n$ for all $x \in E$). In particular, for every proper dense operator range $R \subset E$ there is a dense operator range $V \subset E$ which is essentially disjoint with respect to R. In [8, Theorem 6.2], Cross and Shevchik obtained a strengthening of the latter, by proving that if R is a proper dense operator range in a separable Banach space E, then R is essentially disjoint with respect to an operator range of the form $V = X + Y$, for some **proper quasicomplementary subspaces** X and Y of E (that is, such that $X \cap Y = \{0\}$, $X + Y$ is dense in E and $X + Y \neq E$). This statement constitutes as well an extension in the separable case of a result of Plichko [21] (see also [11] and [18]), who showed that if R is a proper dense operator range in any Banach space E, then the set $E \setminus R$ is **spaceable** (that is, there exists a closed infinite-dimensional subspace $X \subset E$ such that $R \cap X = \{0\}$). We point out that, according to the referred result in [5], the proper dense operator range R in these statements can be replaced with any dense non-barrelled subspace. However, as Drewnowski showed in [10], every Banach space E contains an infinite-codimensional linear subspace V such that $V \cap T(X) \neq \{0\}$ for any one-to-one operator $T : X \to E$ defined on an infinite-dimensional Banach space X.

Shevchik's theorem was used by Plichko [22] to prove that a Banach space E contains a couple of dense essentially disjoint operator ranges whenever E has a fundamental biorthogonal system (that is, a biorthogonal system $\{e_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset E \times E^*$ such that $\{e_\gamma\}_{\gamma\in\Gamma}$ is linearly dense in E). This result solves, for those spaces, a problem posed by Borwein and Tingley [6, Problem 8], which asks if for any given Banach space E (with a separable quotient, in particular if $E = \ell_\infty$), there exists a pair of dense essentially disjoint operator ranges in E.

In the recent paper [16], we have obtained several generalizations of Shevchik's theorem, which entail in particular that, if E is a separable Banach space, and R and V are respectively (countable unions of) infinite-codimensional operator ranges in E and E^*, then for any $0 < \lambda < 1$ there exists a nuclear, one-to-one and dense-range endomorphism $T : E \to E$ such that $T(E) \cap R = \{0\}$, $T^*(E^*) \cap V = \{0\}$ (where T^* stands for the adjoint operator of T) and the linear subspace $F = T^*(E^*) \subset E^*$ is **λ-norming** (that is, $\sup\{f(x) : f \in B_F\} \geq \lambda\|x\|$ for all $x \in E$, being B_F the closed unit ball of F).

In this work, we provide extensions of some of the previous statements in the setting of Banach spaces with weak*-separable dual. We also show that some weak forms of the theorems of Shevchik and Cross and Shevchik characterize separable Banach spaces among weakly Lindelöf determined spaces. All the considered Banach spaces are real. Given such a space E, we denote by $\mathcal{R}(E)$ the family of infinite-codimensional operator ranges in E, and by $\mathcal{S}(E)$ the class of countable unions of elements from $\mathcal{R}(E)$. It is well-known (see e.g. [1, Corollary 2.17]) that $\mathcal{R}(E)$ contains the family of proper dense operator ranges in E, which will be denoted by $\mathcal{R}_d(E)$. In the sequel, otherwise stated, the terms **"closed subspace"** and

"separable quotient" refer to a **"closed infinite-dimensional subspace"** of a Banach space and **"separable infinite-dimensional quotient"**, respectively.

The paper consists of four sections, apart from this Introduction. In the next one, we obtain, for a Banach space E with weak*-separable dual, several results concerning the existence of endomorphisms on E having a prescribed behaviour with respect to given closed subspaces $Y \subset E$ and/or $Z \subset E^*$ and countable unions of infinite-codimensional operator ranges in Y and/or Z, which yield generalizations of some statements in [16] and Shevchik's theorem. These results lead to various characterizations of the classes of Banach spaces with weak*-separable dual, Banach spaces with a separable quotient that have weak*-separable dual and separable spaces, to which the third section is devoted. In particular, we prove that if a Banach space E has a separable quotient, then E^* is weak*-separable if and only if for every $R \in \mathcal{R}_d(E)$ the set $E \setminus R$ is spaceable through a closed subspace $Y \subset E$ such that E/Y is separable, if and only if for every $R \in \mathcal{R}_d(E)$ there exists an isomorphism $\varphi : E \to E$ such that $\varphi(R) \cap R = \{0\}$. The latter yields an affirmative solution to the aforementioned problem of Borwein and Tingley for Banach spaces with weak*-separable dual and a separable quotient. As regards Plichko's solution to this problem in [22], we stress that under some set-theoretical assumptions, there exist Banach spaces with weak*-separable dual and a separable quotient lacking of fundamental biorthogonal systems (cf. [15, p. 143]).

In Section 4, we provide several strengthenings of the former results, which involve the existence of quasicomplementary subspaces of Banach spaces with some special features with respect to operator ranges in those spaces. The first result of that section yields that, if E is a Banach space with weak*-separable dual and X is a closed subspace of E such that E/X is separable, then for every $R \in \mathcal{R}_d(E)$ containing X there exists an isomorphism $\varphi : E \to E$ such that $R \cap \varphi(R) = \{0\}$ while the sum $X + \varphi(X)$ still is dense in E (in particular, the closed subspaces X and $\varphi(X)$ are quasicomplementary). We also show that if E is a Banach space with weak*-separable dual then for any couple of proper quasicomplementary subspaces $X, Y \subset E$ there exist $R, V \in \mathcal{R}_d(E)$ such that $X \subset R$, $Y \subset V$ and $R \cap V = \{0\}$. Next, we obtain a generalization of the aforementioned result of Cross and Shevchik in [8] by showing that if E is a Banach space with weak*-separable dual and a separable quotient, then for every $R \in \mathcal{R}_d(E)$ there exist two closed isomorphic quasicomplementary subspaces $X, Y \subset E$ such that E/X and E/Y are separable and $R \cap (X + Y) = \{0\}$. Finally, we characterize the closed subspaces X of a Banach space E with weak*-separable dual for which there exists a couple of quasicomplementary subspaces $X, Y \subset E$ satisfying $X \cap (Y + Z) = \{0\}$.

In the last section, we show that some of the former results do not hold in a large class of Banach spaces with no weak*-separable dual. In particular, we show that if E is a non-separable weakly Lindelöf determined space, then E contains a proper dense endomorphism range R such that $R \cap V \neq \{0\}$ whenever V is either a dense endomorphism range in E, or a dense operator range of the form $V = X + Y$, for some closed subspaces $X, Y \subset E$.

The notation we use is standard. The norm of a normed space E is usually denoted by $\|\cdot\|$, and the symbols S_E, B_E and I_E stand for the unit sphere, the closed unit ball and the identity operator of that space, respectively. Given a set $A \subset E$, we denote by $\operatorname{span}(A)$ (or $\operatorname{span} A$) the linear span of A, and by $A^\perp$ its annihilator subspace, that is,

$A^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in A\}$. If $\{x_n\}_n$ is a sequence in E, we write $[\{x_n\}_n]$ for the closed linear span of $\{x_n\}_n$.

2. Endomorphisms with a prescribed behaviour

In this section we establish several generalizations for Banach spaces with weak*-separable dual of some results from [16] and Shevchik's theorem. One of the main ingredients in their proofs is the next lemma, which is proven in [16], and guarantees the existence of minimal sequences in a Banach space with a "disjoint behaviour" with respect to countable unions of infinite-codimensional operator ranges in that space. In order to establish this result, we fix some terminology. Recall that a sequence $\{x_n\}_n$ in a Banach space E is said to be **minimal** whenever there exists a sequence $\{f_n\}_n \subset E^*$ such that $\{x_n, f_n\}_n$ is a biorthogonal system in E. If all the functionals f_n lie in a given subspace $F \subset E^*$, we say that $\{x_n\}_n$ is an F-**minimal** sequence. According to the terminology from [16], we say that a bounded sequence $\{x_n\}_n$ in a Banach space E has **property** $(*)$ **with respect to a subset** $V \subset E$ if the conditions $\{a_n\}_n \in \ell_1$ and $\sum_n a_n x_n \in V$ imply that $a_n = 0$ for all n.

Lemma 2.1. [16, Lemma 2.1] *Let E be a Banach space, let $X \subset E$ and $F \subset E^*$ be closed subspaces and $\varepsilon \in (0, 1)$. If $\{x_n\}_n$ is an F-minimal sequence in X then, for every $S \in \mathcal{S}(X)$ there exist an isomorphism $\varphi : E \longrightarrow E$ and an F-minimal sequence $\{y_n\}_n \subset B_X$ such that:*

(1) $\|\varphi - I_E\| \le \varepsilon$.
(2) $\varphi(X) = X$ and $\varphi^*(F) = F$.
(3) $\varphi(x_n)$ and y_n are collinear for each n.
(4) $\{y_n\}_n$ satisfies property $(*)$ with respect to S.

It is worth to mention that the isomorphism $\varphi : E \to E$ in the former lemma can be constructed to satisfy the following refinement of property (2): *for any two linear subspaces $Y \subset E$ and $G \subset E^*$ such that $X \subset Y$ and $F \subset G$ we have $\varphi(Y) = Y$ and $\varphi^*(G) = G$.* Indeed, such isomorphism is of the form $\varphi = I_E + T$ where $T : E \to E$ is a nuclear operator such that $T(E) \subset X$ and $T^*(E^*) \subset F$. From the first inclusion we get $\varphi(Y) \subset Y + X = Y$. On the other hand, for each $y \in Y$ there exists a (unique) vector $u \in E$ such that $y = \varphi(u) = u + T(u)$, so $u = y - T(u) \in Y + X = Y$. Consequently $\varphi(Y) = Y$. Analogously, since $T^*(E^*) \subset F$ we have $\varphi^*(G) = G$. This fact yields to the following refinement of Lemma 2.1 in the dual case.

Lemma 2.2. *Let E be a Banach space, let $X \subset E$ and $F \subset E^*$ be closed subspaces and $V \in \mathcal{S}(F)$. If $\{f_n\}_n$ is an X-minimal sequence in F then, for every $\varepsilon \in (0, 1)$ there exist an isomorphism $\varphi : E \longrightarrow E$ and an X-minimal sequence $\{g_n\}_n \subset B_F$ such that:*

(1) $\|\varphi - I_E\| \le \varepsilon$.
(2) $\varphi(X) = X$ and $\varphi^*(F) = F$.
(3) $\varphi^*(f_n)$ and g_n are collinear for each n.
(4) $\{g_n\}_n$ satisfies property $(*)$ with respect to V.

Proof. The previous lemma and the subsequent observation, applied in E^* (with $\{f_n\}_n$ instead of $\{x_n\}_n$, and F instead of X), entail the existence of an isomorphism $\psi : E^* \to E^*$ and an X-minimal sequence $\{g_n\}_n \subset B_F$ such that $\|\psi - I_{E^*}\| < \varepsilon$, $\psi(F) = F$, $\psi^*(Z) = Z$ for each subspace $Z \subset E^{**}$ containing X, $\psi(f_n)$ and $\{g_n\}_n$ are collinear for each n, and $\{g_n\}_n$

satisfies property $(*)$ with respect to V. Let φ be the restriction of ψ^* to E. Then $\varphi(E) = E$, therefore φ is an isomorphism on E. It is easily checked that $\varphi^* = \psi$, so properties (1), (2) and (3) are proved. □

The next result is an analogue of Theorem 3.1 in [16] for the class of Banach spaces with weak*-separable dual.

Theorem 2.3. *Let E be a Banach space with weak*-separable dual, let Y be a closed subspace of E and Z be a closed, weak*-separable and total subspace of E^*. Then, for any $V \in \mathcal{S}(Y)$ and any $W \in \mathcal{S}(Z)$ there exists a one-to-one nuclear operator $T : E \to E$ such that*

(1) $T(E) \subset Y$.
(2) If Y is separable, then $\overline{T(E)} = Y$.
(3) $T(E) \cap V = \{0\}$.
(4) $T^*(E^*) \subset Z$.
(5) If Z is separable, $\overline{T^*(E^*)} = Z$.
(6) $T^*(E^*) \cap W = \{0\}$.

Proof. First, let us assume that the subspaces Y and Z are separable. According to Markushevich theorem (see e.g. [15, Theorem 1.22]), Y has an M-basis, that is, there is a fundamental minimal sequence $\{x_n\}_n \subset Y$ whose sequence $\{y_n^*\}_n \subset Y^*$ of biorthogonal functionals is total over Y. Lemma 2.1 (applied to the subspaces $X = Y \subset E$, $F = E^*$ and the M-basis $\{x_n\}_n$) yields a sequence $\{y_n\}_n \subset B_Y$ which is an M-basis of Y satisfying property $(*)$ with respect to the set V. On the other hand, because of the weak*-density of the separable subspace Z in E^*, thanks again to [15, Theorem 1.22] we can find an E-minimal sequence $\{f_n\}_n \subset Z$ such that

$$[\{f_n\}_n] = Z \quad \text{and} \quad \overline{[\{f_n\}_n]}^{w^*} = E^*, \tag{2.1}$$

Now, an appeal to Lemma 2.2 (applied to the E-minimal sequence $\{f_n\}_n$) yields a bounded E-minimal sequence $\{g_n\}_n \subset Z$ enjoying property $(*)$ with respect to W, and an isomorphism $\varphi : E \to E$ such that $\varphi^*([\{f_n\}_n]) = [\{g_n\}_n]$ and $\varphi^*(Z) = Z$. Thanks to (2.1) we obtain

$$\overline{[\{g_n\}_n]}^{w^*} = E^*. \tag{2.2}$$

Now, consider the operator $T : E \to E$ given by the formula

$$T(x) = \sum_{n \geq 1} 2^{-n} g_n(x) y_n, \quad x \in E,$$

which is well-defined and continuous because the sequences $\{g_n\}_n$ and $\{y_n\}_n$ are bounded. Clearly, T is nuclear and satisfies (1). Moreover, the E-minimality of $\{g_n\}_n$ provides a sequence $\{u_n\}_n \subset E$ such that $g_n(u_m) = \delta_{n,m}$ for all n, m. In particular, $T(u_n) = 2^{-n} y_n$, which yields $\operatorname{span}(\{y_n\}_n) \subset T(E)$, thus $Y = \overline{T(E)}$ and assertion (2) is proved. Now, pick a vector $x \in E$ such that $T(x) \in V$. Since $\{2^{-n} g_n(x)\}_n \in \ell_1$ and the sequence $\{y_n\}_n$ satisfies property $(*)$ with respect to V we have $g_n(x) = 0$ for all n, hence $T(x) = 0$, and property (3) is fulfilled. Since the sequence $\{g_n\}_n$ is total over E, this argument guarantees as well the injectivity of T. Notice also that

$$T^*(f) = \sum_{n \geq 1} 2^{-n} f(y_n) g_n, \quad \text{for all} \quad f \in E^*.$$

Taking into account that $\{g_n\}_n \subset Z$ we get $T^*(E^*) \subset Z$. Moreover, the minimality of $\{y_n\}_n$ yields a sequence $\{h_n\}_n \subset E^*$ such that $\{y_n, h_n\}_n$ is a biorthogonal system. In particular, $T^*(h_n) = 2^{-n}g_n$, and so $g_n \in T^*(E^*)$ for all n. Consequently $[\{g_n\}_n] = \overline{T^*(E^*)}$, and bearing in mind that $\{g_n\}_n$ is linearly dense in Z we deduce (5). To prove (6), take a functional $f \in E^*$ such that $T^*(f) \in W$. Since $\{g_n\}_n$ enjoys property $(*)$ with respect to W, we have $f(y_n) = 0$ for all n, therefore $T^*(f) = 0$, and hence $T^*(f) = 0$. Thus, $T^*(E^*) \cap W = \{0\}$.

If Y is not separable, the only modification in the above proof is that the sequence $\{x_n\}_n$ is chosen to be a minimal sequence in Y. In the case that Z is not separable, the only modification in the above proof is that the sequence $\{f_n\}_n$ is chosen to satisfy only the second condition $\overline{[\{f_n\}_n]}^{w^*} = E^*$ in (2.1). □

As a consequence of Theorem 2.3 it follows that *if E is a Banach space with weak*-separable dual then, for any $W \in \mathcal{S}(E^*)$ there exists a total linear subspace $N \subset E^*$ such that $N \cap W = \{0\}$*. Another application of that result yields to the following corollary.

Corollary 2.4. *Let E be a Banach space with weak*-separable dual. Then, for every couple of elements $R \in \mathcal{S}(E)$ and $V \in \mathcal{S}(E^*)$ there exists a one-to-one nuclear operator $T : E \to E$ such that $T(E) \cap R = \{0\}$ and $T^*(E^*) \cap V = \{0\}$.*

The next result provides another extension of Shevchik's theorem and [16, Theorem 3.3]. As regards assertion (6) in that result, we stress that the dual of a Banach space E has a norming sequence whenever B_{E^*} is weak*-separable (cf. [9]).

Theorem 2.5. *Let E be a Banach space with weak*-separable dual, let Y be a closed subspace of E and let $q : E^* \to Y^*$ the canonical restriction mapping. Then, for any $V \in \mathcal{S}(Y)$ and any $M \in \mathcal{S}(Y^*)$ there exists a one-to-one nuclear operator $T : E \to E$ such that*

(1) $T(E) \subset Y$.
(2) If Y is separable, $\overline{T(E)} = Y$.
(3) $T(E) \cap V = \{0\}$.
(4) $T^*(E^*) \cap q^{-1}(M) = \{0\}$.
(5) $T^*(E^*)|_Y \cap M = \{0\}$.
(6) If $(E/Y)^*$ is weak*-separable and Y^* has a λ-norming sequence for Y, for some $\lambda \in (0, 1]$, then the subspace $\overline{T^*(E^*)|_Y} \subset Y^*$ is λ-norming for Y.

In order to prove assertion (6) in this theorem, we will need the following result of Singer (cf. [27, Lemma 7.5, p. 207]). We give the proof for the sake of completeness.

Lemma 2.6. *Let E be a Banach space with weak*-separable dual, let Y be a closed subspace of E such that $(E/Y)^*$ is weak*-separable. If $\{y_n^*\}_n \subset Y^*$ is a total sequence over Y then there is a sequence $\{f_n\}_n$ in E^* such that $f_n|_Y = y_n^*$ for each n and $\{f_n\}_n$ is total over E.*

Proof. Since $(E/Y)^*$ identifies with $Y^\perp \subset E^*$, by assumption there is a sequence $\{z_n^*\}_n \subset Y^\perp$ which is weak* dense in $Y^\perp$. For every n, let us consider a linear continuous extension to E of the functional y_n^*, and denote it by $h_n^* \in E^*$. For convenience, let us relabel the sequence $\{h_n^*\}_n$ in the form $\{h_{m,j}^*\}_{m,j\in\mathbb{N}}$. Since $z_m^* + \frac{h_{m,j}^*}{j\|h_{m,j}^*\|} \xrightarrow{j\to\infty} z_m^*$ for every fixed m, we have $z_m^* \in [\{z_k^* + \frac{h_{k,j}^*}{j\|h_{k,j}^*\|} : k, j \in \mathbb{N}\}]$. Thus for every m, j we get $h_{m,j}^* \in [\{z_k^* + \frac{h_{k,i}^*}{i\|h_{k,i}^*\|} : k, i \in \mathbb{N}\}]$.

Let us define $f_{m,j} = j\|h^*_{m,j}\|z^*_m + h^*_{m,j}$ for every m, j. Clearly, $f_{m,j}|_Y = h^*_{m,j}|_Y$. and $[\{f_{k,j} : k, j \in \mathbb{N}\}] = [\{z^*_k + \frac{h^*_{k,j}}{j\|h^*_{k,j}\|} : k, j \in \mathbb{N}\}]$. Let us check that $\{f_{m,j}\}_{m,j}$ is total over E. Let us fix $x \in E$. If $x \in Y$, since $\{h^*_{m,j}\}_{m,j}$ is total over Y we get that there are m_0, j_0 satisfying $h^*_{m_0,j_0}(x) \neq 0$ so $f_{m_0,j_0}(x) = h^*_{m_0,j_0}(x) \neq 0$. Now, if $x \notin Y$, let us consider the non-zero coset $\widehat{x} \in E/Y$. Since $\{z^*_m\}_m$ is total over E/Y, there is m_0 such that $z^*_{m_0}(x) \neq 0$. So for j_0 large enough $|z^*_{m_0}(x)| > \frac{1}{j_0}\frac{h^*_{m_0,j_0}(x)}{\|h^*_{m_0,j_0}\|}$, which yields $f_{m_0,j_0}(x) \neq 0$. Finally, we rewrite $\{f_{m,j}\}_{m,j}$ as $\{f_n\}_n$ by reverting the first relabeling, which yields $f_n|_Y = h^*_n|_Y = y^*_n$ for each n, and the proof is finished. □

Proof of Theorem 2.5. In the case that $(E/Y)^*$ is weak*-separable and there is a sequence $\{y^*_n\}_n \subset Y^*$ such that $[\{y^*_n\}_n]$ is λ-norming for Y, thanks to Lemma 2.6 we can consider a sequence $\{f_n\}_n \subset E^*$ total over E such that $[\{f_n|_Y\}]$ is λ-norming for Y and define $\widetilde{Z} = [\{f_n\}_n]$. Otherwise we set $\widetilde{Z} = E^*$.

Now, let us consider a sequence $\{M_n\}_n \subset \mathcal{R}(Y^*)$ such that $M = \bigcup_n M_n$. Since, for each n, M_n has infinite codimension in Y^* we have that $q^{-1}(M_n)$ has infinite codimension in E^*, thus there is a sequence of linearly independent elements $\{h_{n,k}\}_k \subset E^*$ such that

$$\operatorname{span}(\{h_{n,k}\}_k) \cap q^{-1}(M_n) = \{0\}.$$

Define

$$Z = [\widetilde{Z} \cup \{h_{n,k}\}_{n,k}] \quad \text{and} \quad W = q^{-1}(M) \cap Z.$$

We shall prove that $W \in \mathcal{S}(Z)$. Since $W = \bigcup_n q^{-1}(M_n) \cap Z$, it is enough to show that $q^{-1}(M_n) \cap Z \in \mathcal{R}(Z)$ for all n. Firstly, because of the definition of Z, the subspace $q^{-1}(M_n) \cap Z$ has infinite codimension in Z. Secondly, since for every n, M_n is an operator range in Y^*, thanks to [7, Proposition 2.1], there is a complete norm $\|\cdot\|_{M_n}$ on M_n such that

$$\|y^*\|_{Y^*} \leq \|y^*\|_{M_n} \quad \text{for all} \quad y^* \in M_n,$$

where $\|\cdot\|_{Y^*}$ denotes the norm on M_n inherited from Y^*. Let us define a new norm on $q^{-1}(M_n)$ by the following expression

$$\|w\|_n = \|w\|_{E^*} + \|q(w)\|_{M_n}, \quad w \in q^{-1}(M_n),$$

being $\|\cdot\|_{E^*}$ the norm on $q^{-1}(M_n)$ inherited from E^*. It is easy to check that $\|\cdot\|_n$ is complete on $q^{-1}(M_n)$, thus using again [7, Proposition 2.1] it follows that $q^{-1}(M)$ is an operator range in E^*. Therefore $q^{-1}(M_n) \in \mathcal{R}(E^*)$ for every n, and thus $q^{-1}(M) \in \mathcal{S}(E^*)$. Since the intersection of two operator ranges is an operator range ([7, Proposition 2.2]), we have $q^{-1}(M_n) \cap Z \in \mathcal{R}(Z)$, and thus $W = q^{-1}(M) \cap Z \in \mathcal{S}(Z)$, as we wanted.

Since Z is a closed, weak*-separable and total subspace of E^*, thanks to Theorem 2.3 we can deduce the existence of a one-to-one endomorphism $T : E \to E$ satisfying assertions (1), (2) and (3), $T^*(E^*) \subset Z$ and $T^*(E^*) \cap W = \{0\}$. Consequently,

$$T^*(E^*) \cap q^{-1}(M) = T^*(E^*) \cap \big(q^{-1}(M) \cap Z\big) = T^*(E^*) \cap W = \{0\},$$

thus property (4) is also fulfilled. To prove (5) take a functional $f \in E^*$ such that $T^*(f)|_Y \in M$. Then $T^*(f) \in q^{-1}(M) \cap Z = W$. So $T^*(f) \in T^*(E^*) \cap W$, hence $T^*(f) = 0$ and in particular $T^*(f)|_Y = 0$. Therefore, $T^*(E^*)|_Y \cap M = \{0\}$. Finally, assume that $(E/Y)^*$ is weak*-separable and Y^* has a λ-norming sequence for Y. In this case the subspace Z is separable and Theorem 2.3 provides the additional property $\overline{T^*(E^*)} = Z$. Thus, for every n, k there is $g_{n,k} \in E^*$ such that $\|T^*(g_{n,k}) - f_n\| < \frac{1}{k}$, and hence $\|T^*(g_{n,k})|_Y - f_n|_Y\| < \frac{1}{k}$. Since the subspace $[\{f_n|_Y\}] \subset Y^*$ is λ-norming for Y it follows that $T^*(E^*)|_Y$ is also λ-norming for Y, and the proof is finished. □

As a consequence of the former theorem we obtain the following result.

Corollary 2.7. *Let E be a Banach space with weak*-separable dual and let Y be a closed subspace of E. If $R \in \mathcal{S}(E)$ and $W \in \mathcal{S}(E^*)$ satisfy $R \cap Y \in \mathcal{S}(Y)$ and $W|_Y \in \mathcal{S}(Y^*)$ then there exists a nuclear one-to-one operator $T : E \to E$ such that*

(1) $T(E) \subset Y$.
(2) If Y is separable, $\overline{T(E)} = Y$.
(3) $T(E) \cap R = \{0\}$.
(4) $T^*(E^*) \cap W = \{0\}$.
(5) $T^*(E^*)|_Y \cap W|_Y = \{0\}$.
(6) If $(E/Y)^*$ is weak-separable and Y has a sequence in Y^* λ-norming for Y for some $\lambda \in (0, 1]$, then $T^*(E^*)|_Y$ is λ-norming for Y.*

Proof. Let us denote by $q : E^* \to Y^*$ the canonical restriction mapping. Applying Theorem 2.5 to the subspace $Y \subset E$ and the elements $V = R \cap Y \in \mathcal{S}(Y)$ and $M = W|_Y \in \mathcal{S}(Y^*)$, we deduce the existence of a one-to-one nuclear operator $T : E \to E$ satisfying assertions (1) and (2),

$$T(E) \cap V = \{0\}, \quad T^*(E^*) \cap q^{-1}(M) = \{0\} \quad \text{and} \quad T^*(E^*)|_Y \cap q^{-1}(M)|_Y = \{0\},$$

and (6) (the last one, assuming that $(E/Y)^*$ is weak*-separable and Y^* has a λ-norming sequence for Y). Clearly, $T(E) \cap R = T(E) \cap R \cap Y = T(E) \cap V = \{0\}$, which proves (3). As $q^{-1}(M) = q^{-1}(q(W)) \supset W$ we obtain $T^*(E^*) \cap W = \{0\}$. Moreover, since $q^{-1}(M)|_Y = q(q^{-1}(q(W))) = W|_Y$ we get $T^*(E^*)|_Y \cap W|_Y = \{0\}$, so properties (4) and (5) are also satisfied. □

3. Characterizations of weak star separability

In this section, we obtain several characterizations of Banach spaces with weak*-separable dual and separable spaces in terms of disjointness properties of operator ranges. The first result in this direction reads as follows.

Theorem 3.1. *If E is an infinite-dimensional Banach space then the following assertions are equivalent:*

(1) E^* is weak-separable.*
(2) For each $R \in \mathcal{R}(E)$ and each $\varepsilon > 0$ there exists an isomorphism $\varphi : E \to E$ such that $\|\varphi - I_E\| < \varepsilon$ and $\varphi(R) \cap R = \{0\}$.
(3) For each $R \in \mathcal{R}(E)$ there is a one-to-one operator $T : R \to E$ with $T(R) \cap R = \{0\}$.
(4) For each closed infinite-codimensional subspace $X \subset E$ there exists a one-to-one operator $T : X \to E$ such that $T(X) \cap X = \{0\}$.
(5) There exist two closed infinite-codimensional subspaces $X, Y \subset E$ such that $(E/X)^*$ and $(E/Y)^*$ are weak-separable and $X \cap Y = \{0\}$.*

Proof. First, we establish the equivalence between assertions $(1)-(4)$. The implication $(2) \Rightarrow (3)$ is obvious, and $(3) \Rightarrow (4)$ follows from the fact that closed subspaces of a Banach space are operator ranges. Let us check that $(1) \Rightarrow (2)$ and $(4) \Rightarrow (1)$. Suppose that E^* is weak*-separable, and fix $R \in \mathcal{R}(E)$ and $\varepsilon > 0$. Theorem 2.3 guarantees the existence of a one-to-one operator $T : E \to E$ such that $T(E) \cap R = \{0\}$. We can assume without loss of generality that $\varepsilon \in (0, 1)$ and $\|T\| < \varepsilon$, therefore the operator $\varphi = I_E - T$ is an isomorphism on E satisfying $\|\varphi - I_E\| < \varepsilon$. Pick $x \in R \cap \varphi(R)$. Then $x = x' - T(x')$ for some $x' \in R$. Consequently, $T(x') = x' - x \in R \cap T(E)$, hence $T(x') = 0$, and the injectivity of T entails that $x' = x = 0$. Thus, $R \cap \varphi(R) = \{0\}$, and the implication $(1) \Rightarrow (2)$ is checked.

Now, we shall prove that $(4) \Rightarrow (1)$. Let Z be a closed separable subspace of E^*, and set $X = Z_\perp$ (the annihilator of Z in E). Since $(E/X)^*$ identifies with the weak*-closure of Z, we have that E/X is infinite-dimensional, that is, $\operatorname{codim}_E(X) = \infty$. By the hypothesis, there exists a one-to-one operator $T : X \to E$ satisfying $T(X) \cap X = \{0\}$. Consider the operator $S : Q \circ T : X \to E/X$, where $Q : E \to E/X$ denotes the canonical quotient map. Bearing in mind that T is one-to-one and $T(X) \cap X = \{0\}$ it follows that S is one-to-one. Therefore, the range of S^* is weak*-dense in X^*. Since Z is separable, we have that $(E/X)^*$ is weak*-separable, thus X^* is weak*-separable as well. On the other hand, thanks to Lemma 2.6 it follows that the fact of having a weak*-separable dual is a three space property. Therefore, E^* is weak*-separable, so $(4) \Rightarrow (1)$.

Next, we will check the implication $(1) \Rightarrow (5)$. Let X be any closed infinite-codimensional subspace of E such that $(E/X)^*$ is weak*-separable. Since X is an infinite-codimensional operator range in E, according to Lemma 2.1 (applied to any minimal sequence in E), there exists a minimal sequence $\{e_n\}_n \subset B_E$ with property $(*)$ with respect to X. On the other hand, because of the weak*-separability of E^*, we can find a total sequence $\{f_n\}_n \subset E^*$. We can assume that $\|f_n\| \leq 4^{-n}$ for all n. Let us write, for each $u \in E$,

$$\varphi(u) = u + \sum_{n \geq 1} f_n(u) e_n.$$

Since $\sum_{n \geq 1} \|f_n\| \|e_n\| < 1$, we have that φ is an isomorphism on E. Set $Y = \varphi(X)$. Then, the formula $\Phi(u + X) = \varphi(u) + Y$ defines an isomorphism $\Phi : E/X \to E/Y$, thus $\Phi^* : (E/Y)^* \to (E/X)^*$ is a (weak*, weak*)-isomorphism, and taking into account that $(E/X)^*$ is weak*-separable, we deduce that $(E/Y)^*$ is weak*-separable as well. Now, pick a vector $x \in X$ such that $\varphi(x) \in X$. Then $\varphi(x) - x \in X$, and therefore

$$\sum_{n \geq 1} f_n(x) e_n \in X.$$

Since $\{f_n(x)\}_n \in \ell_1$ and $\{e_n\}_n$ has property $(*)$ with respect to X we get $f_n(x) = 0$ for all n, and bearing in mind that $\{f_n\}_n$ is total over E we get $x = 0$. Thus, $X \cap Y = \{0\}$.

It remains to show that $(5) \Rightarrow (1)$. Let X and Y be closed subspaces of E satisfying property (5), and $T : X \to E/Y$ be the restriction to X of the quotient map $Q : E \to E/Y$. Since $X \cap Y = \{0\}$ we have that T is one-to-one. Thus, its adjoint $T^* : (E/Y)^* \to X^*$ has weak*-dense range. Because of the weak*-separability of $(E/Y)^*$ we deduce that X^* is weak*-separable. Since $(E/X)^*$ is also weak*-separable it follows that E^* is weak*-separable. □

It is not known if a Banach space with weak*-separable dual has a separable quotient. In fact, the separable quotient problem reduces to the case of Banach spaces with weak*-separable dual. The next result provides a strengthening of the former theorem in the class of Banach spaces E which have a separable quotient (or equivalently, a proper dense operator range). The equivalence between properties (1) and (5) in this result refines the aforementioned theorem in [21] ensuring that the complementary set of a proper dense operator range in every Banach space is spaceable. On the other hand, the implication (1) ⇒ (4) yields a strong answer to the problem of Borwein and Tingley in the class of Banach spaces with weak*-separable dual and a separable quotient. As regards the solution to this problem in [22] for Banach spaces with a fundamental biorthogonal system, we notice that, under some set-theoretical assumptions there exist compact sets K such that the space $\mathcal{C}(K)$ lacks of any fundamental biorthogonal system while its dual space $\mathcal{C}(K)^*$ is weak*-separable. Moreover, as every space of continuous functions, $\mathcal{C}(K)$ has a separable quotient (cf. [19, 23]).

Theorem 3.2. *If E is a Banach space with a separable quotient, then the following assertions are equivalent:*

(1) *E^* is weak*-separable.*
(2) *For each (some) closed subspace $X \subset E$ such that E/X is separable and infinite-dimensional there exists a one-to-one operator $T : X \to E$ such that $T(X) \cap X = \{0\}$.*
(3) *For each $R \in \mathcal{R}_d(E)$ there is a one-to-one operator $T : R \to E$ with $T(R) \cap R = \{0\}$.*
(4) *For each $R \in \mathcal{R}_d(E)$ and each $\varepsilon > 0$ there exists an isomorphism $\varphi : E \to E$ such that $\|\varphi - I_E\| < \varepsilon$ and $\varphi(R) \cap R = \{0\}$.*
(5) *For each $R \in \mathcal{R}_d(E)$ there exists a closed subspace $Y \subset E$ such that E/Y is separable and $R \cap Y = \{0\}$.*

In the proof of Theorem 3.2 we shall use the following refinement of the aforementioned result of Saxon and Wilansky in [24].

Proposition 3.3. *Let E be a Banach space and $R \in \mathcal{R}_d(E)$. Then, there exists a closed (infinite-dimensional) subspace $X \subset E$ such that E/X is separable and $R + X$ has infinite codimension in E.*

Proof. Firstly, let us assume that E is separable. According to the result of Plichko in [21], there is a closed infinite-dimensional subspace $X \subset E$ with $R \cap X = \{0\}$. Then, $R + X$ is an operator range in E, and taking into account that R is not closed and $R \cap X = \{0\}$, according to [7, Theorem 2.4] it follows that $R + X$ is not closed. Moreover, thanks to [1, Corollary 2.17] we have $\operatorname{codim}_E(R + X) = \infty$. Now, suppose that E is not separable. Let Z be a Banach space and $T : Z \to E$ be a bounded operator such that $T(Z) = R$. Define $B_0 = \overline{T(B_Z)} \subset E$. The construction given in [24, §1.9 - §1.10] (see also [20, proof of Theorem 3.2]) yields the existence of sequences $\{x_n\}_n \subset S_E$ and $\{f_n\}_n \subset E^*$, and a strictly increasing sequence of convex bounded and symmetric sets $B_n \subset E$ such that

(i) $f_n(x_n) = 1$,
(ii) $f_n(x_k) = 0$ for all $k > n$,
(iii) $B_n = B_{n-1} + \{\sum_{i=1}^n \alpha_i x_i : |\alpha_i| \leq 1\}$,
(iv) $\sup_{B_{n-1}} |f_n| \leq 2^{-n}$,

for every n, and the closed subspace $X = \bigcap_{n=1}^{\infty} \ker f_n$ satisfies that E/X is infinite-dimensional and separable. Since E is nonseparable we have that X is infinite-dimensional.

Now, let us write

$$R_1 = \{x \in E : \text{ the sequence } \left(nf_n(x)\right)_n \text{ converges}\},$$

and let $\|\cdot\|$ be the norm of E. It is easy to see that the formula

$$|||x||| = \|x\| + \sup_n n|f_n(x)|, \quad x \in R_1,$$

defines a complete norm on R_1, stronger than $\|\cdot\|$. Therefore, thanks to [7, Proposition 2.1] we have that R_1 is an operator range in E. It is clear that $X \subset R_1$, and because of inequality (iv) it also follows that $R \subset R_1$, thus $R + X \subset R_1$. Moreover, $R_1 \neq E$. Indeed, if $R_1 = E$ then the sequence $(nf_n)_n$ would be bounded, by the uniform boundedness principle. But bearing in mind that $(x_n)_n \subset S_E$ and property (i) we have $\|nf_n\| \geq n|f_n(x_n)| = n$ for all n, a contradiction. Thus, $R_1 \in \mathcal{R}_d(E)$, and hence $\operatorname{codim}_E(R_1) = \infty$. Consequently $\operatorname{codim}_E(R + X) = \infty$. □

We shall also need the following result, essentially proven in [17].

Lemma 3.4. *If X is a closed infinite-codimensional subspace of a Banach space E, then the following assertions are equivalent.*

(1) E/X has a separable quotient.
(2) There exists a closed subspace $Z \subset E$ such that E/Z is separable, $X + Z$ is dense in E and $X + Z \neq E$.
(3) There exists $R \in \mathcal{R}_d(E)$ such that $X \subset R$.

Proof. The implication (1) ⇒ (2) is given by [17, Proposition 3.5], and (2) ⇒ (3) follows from the fact that the sum of two operator ranges is again an operator range. Assume that (3) is satisfied, let $Q : E \to E/X$ be the quotient map, and set $R_1 = Q(R)$. Then R_1 is a proper dense operator range in E/X (notice that, since $X \subset R$, $Q(x) \notin Q(R)$ whenever $x \notin R$), and using the aforementioned result in [24] we deduce that E/X has a separable quotient, hence (3) ⇒ (1). □

Proof of Theorem 3.2. First, we establish the equivalence between assertions (1) – (4). Since $\mathcal{R}_d(E) \subset \mathcal{R}(E)$ the implication (1) ⇒ (4) follows from Theorem 3.1, and (4) ⇒ (3) is trivial.

Assume that property (3) holds. Let X be a closed subspace of E such that E/X is separable and infinite-dimensional. According to Lemma 3.4 there is an operator range $R \in \mathcal{R}_d(E)$ such that $X \subset R$. By our assumption, there exists one-to-one operator $T : R \to E$ such that $T(R) \cap R = \{0\}$, in particular $T(X) \cap X = \{0\}$, so (3) ⇒ (2).

Now, suppose that the parenthetic part of property (2) is satisfied. Let X be a closed subspace of E such that E/X is separable, and let $T : X \to E$ be a one-to-one operator such that $T(X) \cap X = \{0\}$. Then, the operator $S = Q \circ T : X \to E/X$ is one-to-one, hence its adjoint $S^* : (E/X)^* \to X^*$ has weak*-dense range, and taking into account that $(E/X)^*$ is weak*-separable (being the dual of a separable space), it follows that X^* is weak*-separable. Since having a weak*-separable dual is a three-space property, we have that E^* is weak*-separable, and thus (2) ⇒ (1).

Next, we shall prove that (4) $\Rightarrow$ (5). Let $R \in \mathcal{R}_d(E)$. Because of Proposition 3.3 there exists a closed subspace $X \subset E$ such that E/X is separable and $\operatorname{codim}_E(R+X) = \infty$. Then, the subspace $V = R+X$ is a proper dense operator range in E. Using the hypothesis, we deduce the existence of an isomorphism $\varphi : E \to E$ such that $\varphi(V) \cap V = \{0\}$. Let $Y = \varphi(X)$. Then Y is a closed subspace of E such that E/Y is separable (notice that the formula $\Phi(u+X) = \varphi(u) + Y,\ \ u \in E$, defines an isomorphism from E/X onto E/Y), and $R \cap Y \subset V \cap \varphi(V)$, thus $Y \cap R = \{0\}$.

To finish, we shall show that (5) $\Rightarrow$ (1). Consider a closed subspace $X \subset E$ such that E/X is separable and infinite-dimensional. Thanks to Lemma 3.4 there exists an operator range $R \in \mathcal{R}_d(E)$ such that $X \subset R$. Because of our assumption we can find a closed subspace $Y \subset E$ such that E/Y is separable and $R \cap Y = \{0\}$, in particular $X \cap Y = \{0\}$, and using Theorem 3.1 (5) we deduce that E^* is weak*-separable. □

We end this section with the next characterization of separable Banach spaces in terms of essentially disjoint operator ranges. As regards assertion (4) in this result, we notice that, according to a well-known theorem of Argyros, Dodos and Kanellopoulos [2], the dual of every Banach space E has a separable quotient, and hence $R_d(E^*) \neq \emptyset$.

Theorem 3.5. *If E is a Banach space then the following assertions are equivalent:*

(1) E is separable.
(2) For every $R \in \mathcal{R}(E)$, every $V \in \mathcal{R}(E^*)$ and every $\varepsilon > 0$ there exists an isomorphism $\varphi : E \to E$ such that $\|\varphi - I_E\| < \varepsilon$, $R \cap \varphi(R) = \{0\}$ and $V \cap \varphi^*(V) = \{0\}$.
(3) For every $V \in \mathcal{R}(E^*)$ and every $\varepsilon > 0$ there exists an isomorphism $\varphi : E \to E$ such that $\|\varphi - I_E\| < \varepsilon$ and $V \cap \varphi^*(V) = \{0\}$.
(4) For every $V \in \mathcal{R}_d(E^*)$ there exists an isomorphism $\varphi : E \to E$ with $V \cap \varphi^*(V) = \{0\}$.
(5) For every weak*-closed infinite-codimensional subspace $Z \subset E^*$ there exists an isomorphism $\varphi : E \to E$ such that $\varphi^*(Z) \cap Z = \{0\}$.

Proof. The implication (1) $\Rightarrow$ (2) is proved in [16, Corollary 3.8], and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. Let us show that (4) $\Rightarrow$ (5) $\Rightarrow$ (1). Let Z be a weak*-closed infinite-codimensional subspace of E^*. Since E^*/Z is isomorphic to the dual space of $Z_\perp$, according to the aforementioned result in [2] it follows that E^*/Z has a separable quotient, and Lemma 3.4 yields an operator range $V \in \mathcal{R}_d(E^*)$ such that $Z \subset V$. Using now the hypothesis, we deduce the existence of an isomorphism $\varphi : E \to E$ satisfying $V \cap \varphi^*(V) = \{0\}$. In particular $Z \cap \varphi^*(Z) = \{0\}$, and the implication (4) $\Rightarrow$ (5) is proved.

Now, assume that assertion (5) is satisfied, consider a separable infinite-codimensional subspace $Y \subset E$ and set $Z = Y^\perp$. Then, Z is a weak*-closed infinite-codimensional (and infinite-dimensional) subspace of E^*. By our assumption, there is an isomorphism $\varphi : E \to E$ such that $\varphi^*(Z) \cap Z = \{0\}$. Set $X = \varphi^{-1}(Y)$. Since Y is separable, so is X. Moreover, as

$$(X+Y)^\perp = \left[\varphi^{-1}(Y)\right]^\perp \cap Y^\perp = \varphi^*(Y^\perp) \cap Y^\perp = \varphi^*(Z) \cap Z$$

we get $(X+Y)^\perp = \{0\}$. Consequently, $X+Y$ is dense in E. Since the subspaces X and Y are separable, it follows that E is separable, thus (5) $\Rightarrow$ (1). □

Remark 3.6. Assertion (4) from Theorem 3.5 cannot be replaced with the weaker one *"for each $V \in \mathcal{R}_d(E^*)$ there exists an isomorphism $\psi : E^* \to E^*$ with $V \cap \psi(V) = \{0\}$."* Indeed, let E be a nonseparable Banach space such that E^{**} is weak*-separable (an example is $E = c_0(\omega_1)$, where ω_1 denotes the first uncountable ordinal). According to Theorem 3.2, for every $V \in \mathcal{R}_d(E^*)$ there exists an isomorphism $\psi : E^* \to E^*$ such that $\psi(V) \cap V = \{0\}$. On the other hand, if for every $V \in \mathcal{R}_d(E^*)$ such isomorphism were a dual operator, then E would be separable by Theorem 3.5.

4. Essentially disjoint operator ranges and quasicomplements

In this section, we establish some results which involve the existence of quasicomplementary subspaces of a Banach space with a special behaviour with respect to operator ranges in that space, which enhance the characterizations of Banach spaces with weak*-separable dual and separable spaces given in the previous section. Before stating them, we need to introduce some terminology from the recent paper [17]. We say that a closed subspace Y of a Banach space E is a **nuclearly adjacent quasicomplement** of a closed subspace $X \subset E$ provided that X and Y are quasicomplementary and the restriction to Y of the canonical quotient map $Q : E \to E/X$ is nuclear (that is, $Q|_Y : Y \to E/X$ is one-to-one, dense-range and nuclear). Two closed quasicomplementary subspaces X and Y are called **mutually nuclearly adjacent** if X is nuclearly adjacent to X and Y is nuclearly adjacent to X. Observe that, since every nuclear operator is compact, we have that if a subspace $X \subset E$ has a nuclearly adjacent quasicomplementary subspace, then E/X is separable, and that if a subspace $Y \subset E$ is a nuclearly adjacent quasicomplement of X, then the sum $X + Y_1$ is not closed for any closed infinite-dimensional subspace $Y_1 \subset Y$.

In [17, Corollary 2.12] it is shown that, if E is a Banach space with weak*-separable dual, then for any closed subspace $X \subset E$ such that E/X is separable, each sequence $\{R_k\}_k \subset \mathcal{R}(E)$ with $X \subset \bigcap_{k\geq 1} R_k$ and each $\varepsilon > 0$ there exists an isomorphism $\varphi : E \to E$ with $\|\varphi - I_E\| < \varepsilon$ such that:

(1) $\varphi(X) \cap (\bigcup_{k\geq 1} R_k) = \{0\}$ and
(2) the subspaces X and $\varphi(X)$ are mutually nuclearly adjacent quasicomplementary.

In the case that $\{R_k\}_k$ reduces to only one operator range R, the construction of the isomorphism φ can be carried out so that, in addition to these properties, the operator range $\varphi(R)$ is essentially disjoint with respect to R. More precisely, we have the following result, which enhances as well assertion (4) from Theorem 3.2.

Theorem 4.1. *Let E be a Banach space with weak*-separable dual. If E has a separable quotient then, for each closed subspace $X \subset E$ such that E/X is separable, each $R \in \mathcal{R}(E)$ with $X \subset R$ and each $\varepsilon > 0$ there exists an isomorphism φ with $\|\varphi - I_E\| < \varepsilon$ such that:*

(a) $\varphi(R) \cap R = \{0\}$, and
(b) the subspaces X and $\varphi(X)$ are mutually nuclearly adjacent quasicomplementary.

Proof. Let Q be the canonical quotient map $E \to E/X$. Since E/X is separable, there is a sequence $\{u_n\}_n \subset E$ such that $\{Q(u_n)\}_n$ is an M-basis of E/X. Using Lemma 2.1 (applied to the sequence $\{u_n\}_n$ and the operator range R), we deduce the existence of a minimal sequence $\{e_n\}_n \subset B_E$, an isomorphism $\psi : E \to E$ and a sequence of (nonzero) scalars $\{\lambda_n\}_n$ such that $\{e_n\}_n$ has property $(*)$ with respect to R and $\psi(u_n) = \lambda_n e_n$ all n. From this, it easily follows that $\{Q(e_n)\}_n$ is an M-basis of E/X. On the other hand, as E^* is weak*-separable, so is X^*, and [15, Theorem 1.22] yields a total X-minimal sequence $\{x_n^*\}_n \subset X^*$. Since $(E/X)^*$ is weak*-separable (being the dual of a separable space), thanks to Lemma 2.6 there is a sequence $\{f_n\}_n \subset E^*$ which is total over E and satisfies $f_n|_X = x_n^*$ for all n.

Choose constants $c_n > 0$ such that $\sum_{n\geq 1} c_n \|f_n\| < \varepsilon/(1+\varepsilon)$. Then, the formula

$$\varphi(u) = u + \sum_{n\geq 1} c_n f_n(u) e_n, \quad u \in E$$

defines an isomorphism on E satisfying $\|\varphi - I_E\| < \varepsilon$.

Pick a vector $v \in R \cap \varphi(R)$. Then, $v = \varphi(u)$ for some $u \in R$, hence $v - u \in R$, and thus

$$\sum_{n\geq 1} c_n f_n(u) e_n \in R.$$

Since $\{c_n f_n(u)\}_n \in \ell_1$ and the sequence $\{e_n\}_n$ satisfies property $(*)$ with respect to R it follows that $f_n(u) = 0$ for all n, and taking into account that $\{f_n\}_n$ is total over E we get $u = 0$. Therefore, $v = 0$, and assertion (a) is proved.

To check (b), we can follow some arguments from the proof of [17, Theorem 2.1]. We give the details for the sake of completeness. Set $Y = \varphi(X)$, let $\{x_n\}_n$ be any sequence in X such that $\{x_n, f_n\}_n$ is a biorthogonal system, and define, for each $k \geq 1$, $y_k = \varphi(x_k)$. Then

$$Q(y_k) = Q\left(x_k + \sum_{n\geq 1} c_n f_n(x_k) e_n\right) = \sum_{n\geq 1} c_n f_n(x_k) Q(e_n) = c_k Q(e_k).$$

Since $\{Q(e_k)\}_k$ is linearly dense in E/X we get $\overline{Q(Y)} = E/X$. Therefore, $X + Y$ is dense in E. Moreover, as $X = \varphi^{-1}(Y)$ and φ is the sum of the identity operator on E and a nuclear endomorphism, thanks to [17, Lemma 2.3] we have that $Q|_Y$ is nuclear. Notice also that

$$\varphi^{-1}(u) = u - \sum_{n\geq 1} c_n f_n(u) \varphi^{-1}(e_n) \quad \text{for all} \quad u \in E,$$

that is, φ^{-1} is the sum of the identity on E plus a nuclear endomorphism, and a new appeal to [17, Lemma 2.3] entails that the restriction to X of the quotient map $E \to E/Y$ is nuclear. Therefore, the subspaces X and Y are mutually nuclearly adjacent quasicomplements. $\square$

In the case that E is separable, we have the following analogue of the previous result, which yields a strengthening of Theorem 3.5.

Theorem 4.2. *If E is a separable Banach space then, for each closed subspace $X \subset E$, each $R \in \mathcal{R}_d(E)$ containing X, each $V \in \mathcal{R}_d(E^*)$ and each $\varepsilon > 0$ there exists an isomorphism $\varphi : E \to E$ with $\|\varphi - I_E\| < \varepsilon$ such that:*

- *(a) $\varphi(R) \cap R = \{0\}$,*
- *(b) $\varphi^*(V) \cap V = \{0\}$, and*
- *(c) the subspaces X and $\varphi(X)$ are mutually nuclearly adjacent quasicomplementary.*

Proof. Let $Q : E \to E/X$ be the quotient map, and set $R_1 = Q(R)$. Since R is a proper dense operator range in E and $X \subset R$ we have that R_1 is a proper dense operator range in E/X, and because of Lemma 2.1 (applied to any M-basis of E/X), there exists a sequence $\{u_n\}_n \subset E$ such that $\{Q(u_n)\}_n$ is a bounded M-basis of E/X satisfying property $(*)$ with respect to R_1. Using now [26, Theorem 3], we can find an M-basis $\{e_n\}_n$ of E (not necessarily bounded) such that $Q(e_n) = Q(u_n)$ for all n.

On the other hand, according to Lemma 2.6 there is an X-minimal sequence $\{g_n\}_n \subset E^*$ which is total over E. Thus, thanks to Lemma 2.2 there exists an X-minimal sequence $\{f_n\}_n \subset B_{E^*}$ satisfying property $(*)$ with respect to V, an isomorphism $\phi : E \to E$ and scalars $\lambda_n \neq 0$ such that $f_n = \lambda_n \phi^*(g_n)$ for all n. Notice that if $f_n(u) = 0$ for some $u \in E$ and all n, then $g_n(\psi(u)) = 0$ for all n, and taking into account that $\{g_n\}_n$ is total over E we get $\psi(u) = 0$. Therefore $u = 0$, that is, $\{f_n\}_n$ is total over E.

Now, choose scalars $c_n > 0$ such that $\sum_{n\geq 1} c_n \|e_n\| < \varepsilon/(1+\varepsilon)$. Then, the formula

$$\varphi(u) = u + \sum_{n\geq 1} c_n f_n(u) e_n, \quad u \in E$$

defines an isomorphism on E satisfying $\|\varphi - I_E\| < \varepsilon$. Take $u \in R$ such that $\varphi(u) \in R$. Then $Q(\varphi(u)) \in R_1$, thus $Q(\varphi(u)) - Q(u) \in R_1$, and therefore

$$\sum_{n\geq 1} c_n f_n(u) Q(e_n) \in R_1.$$

Since the sequence $\{Q(e_n)\}_n$ satisfies property $(*)$ with respect to R_1 we obtain $f_n(u) = 0$ for all n, and bearing in mind that $\{f_n\}_n$ is total over E we get $u = 0$, thus assertion (a) is proved. Analogously, if a functional $f \in V$ satisfies $\varphi^*(f) \in V$, then

$$\sum_{n\geq 1} c_n f(e_n) f_n = \varphi^*(f) - f \in V.$$

As $\{f_n\}_n$ has property $(*)$ with respect to V and $[\{e_n\}_n] = E$ we get $f = 0$, so assertion (b) is also fulfilled. Assertion (c) can be achieved arguing as in the proof of Theorem 4.1. □

The next result ensures the posibility of separating two proper quasicomplementary subspaces through proper dense operator ranges.

Proposition 4.3. *If E is a Banach space with weak*-separable dual and a separable quotient, then the following properties hold:*

(1) For any $R \in \mathcal{R}_d(E)$ and any closed subspace $Y \subset E$ such that $R \cap Y = \{0\}$ there exists $V \in \mathcal{R}_d(E)$ satisfying $Y \subset V$ and $R \cap V = \{0\}$.
(2) For any two closed proper quasicomplementary subspaces X and Y of E there exist $R, V \in \mathcal{R}_d(E)$ such that $X \subset R$, $Y \subset V$ and $V \cap R = \{0\}$.

Proof. (1) Let us write $R_1 = R + Y$. Then, R_1 is a dense operator range in E. As $R \cap Y = \{0\}$ and R is not closed, thanks to [7, Theorem 2.4] we have that $R_1 \neq E$, hence $R_1 \in \mathcal{R}_d(E)$. Since E^* is weak*-separable, using Theorem 3.2 we deduce the existence of an isomorphism $\varphi : E \to E$ such that $R_1 \cap \varphi(R_1) = \{0\}$. It is clear that $\varphi(R_1) \in \mathcal{R}_d(E)$.

Now, define $V = \varphi(R_1) + Y$. Since $\varphi(R_1) \cap Y = \{0\}$, again by [7, Theorem 2.4] we get $V \in R_d(E)$. Pick $v \in V \cap R$. Then there exist $x \in R_1$ and $y \in Y$ such that $\varphi(x) = y - v \in Y + R = R_1$, hence $\varphi(x) \in \varphi(R_1) \cap R_1$, so $\varphi(x) = 0$, thus $v \in Y \cap R$ and therefore $v = 0$. Consequently, $V \cap R = \{0\}$.

(2) Let us write $R_0 = X + Y$. Since X and Y are proper quasicomplementary subspaces of E we get $R_0 \in \mathcal{R}_d(E)$, and an appeal to Theorem 3.2 yields the existence of a proper dense operator range $R_1 \subset E$ such that $R_1 \cap R_0 = \{0\}$.

Define $R = X + R_1$. Since $R_1 \in \mathcal{R}_d(E)$ and $X \cap R_1 = \{0\}$ it follows that $R \in \mathcal{R}_d(E)$, and taking into account that $R_1 \cap R_0 = \{0\}$ we get $Y \cap R = \{0\}$. Thus, thanks to the first assertion we deduce the existence of an operator range $V \in \mathcal{R}_d(E)$ such that $Y \subset V$ and $V \cap R = \{0\}$. □

Remark 4.4. In assertion (2) from Proposition 4.3, it is essential to assume that the quasicomplementary subspaces X and Y are proper. Indeed, let $R, V \in \mathcal{R}_d(E)$ such that $X \subset R$ and $Y \subset V$. If $X + Y = E$ then $R + V = E$, and taking into account that R and V are not closed it follows that $R \cap V \neq \{0\}$. The latter assertion is a consequence of [7, Theorem 2.4].

As we mentioned, Cross and Shevchik [8, Theorem 6.2] proved that if E is a separable Banach space, then for every $R \in \mathcal{R}_d(E)$ there exists a couple of closed quasicomplementary subspaces $X, Y \subset E$ such that $R \cap (X + Y) = \{0\}$. Next, we provide an extension of this theorem for the class of Banach spaces with weak*-separable dual and a separable quotient, which yields as well another solution to the problem of Borwein and Tingley for this class of spaces.

Theorem 4.5. *Let E be a Banach space with weak*-separable dual and a separable quotient. If $R \in \mathcal{R}_d(E)$ then, for any $\varepsilon > 0$ there exist a closed subspace $X \subset E$ and an isomorphism $\varphi : E \to E$ such that $\|\varphi - I_E\| < \varepsilon$, and if $Y = \varphi(X)$ then*

(1) X and Y are mutually nuclearly adjacent quasicomplementary subspaces, and
(2) $R \cap (X + Y) = \{0\}$.

Proof. Let R be a proper dense operator range in E. According to Proposition 3.3, there exists a closed subspace $Z \subset E$ such that E/Z is separable and $R+Z$ has infinite codimension in E.

Set $R_1 = R + Z$. Then, R_1 is an operator range in E, and thus $R_1 \in \mathcal{R}_d(E)$. Since E^* is weak*-separable so is Z^*. Therefore, thanks to Theorem 4.1 there exists an isomorphism $\psi : E \to E$ such that $\|I_E - \psi\| < \varepsilon/(1 + \varepsilon)$ and if $F = \psi(Z)$, then $R_1 \cap F = \{0\}$ and the subspaces F and Z are mutually nuclearly adjacent quasicomplementary.

Set $R_2 = R_1 + F$. Then, R_2 is an operator range in E. Taking into account that $R_1 \cap F = \{0\}$, F is closed in E and R_1 is not closed in E, according to [7, Theorem 2.4] it follows that R_2 is not closed in E, and thus $R_2 \in \mathcal{R}_d(E)$. Now, an appeal to Theorem 3.2 ensures the existence of an isomorphism $S : E \to E$ such that $\|S\|\|S^{-1}\| < 1 + \varepsilon$ and $S(R_2) \cap R_2 = \{0\}$. Consider the isomorphism $\varphi = S \circ \psi \circ S^{-1}$, and the subspaces

$$X = S(Z) \quad \text{and} \quad Y = S(F).$$

Then $\|\varphi - I_E\| < \varepsilon$ and $Y = \varphi(X)$. Moreover, since F is a quasicomplement of Z, it follows that Y is a quasicomplement of X. We claim that Y is nuclearly adjacent to X. Indeed, let $Q_1 : E \to E/Z$ and $Q_2 : E \to E/X$ denote the canonical quotient maps from E onto E/Z and E/X respectively, and let $\Psi : E/Z \to E/X$ be the map defined as

$$\Psi(Q_1(u)) = Q_2(S(u)), \quad u \in E.$$

It can be verified that Ψ is an isomorphism between E/Z and E/X. For every $u \in E$ we have $\Psi\big(Q_1\big(S^{-1}(u)\big)\big) = Q_2(u)$, hence $Q_2|_Y = \Psi \circ Q_1|_F \circ S^{-1}|_Y$. As F is a nuclearly adjacent quasicomplement of Z we have that $Q_1|_F$ is a nuclear operator, thus, $Q_2|_Y$ is nuclear as well. Therefore, Y is nuclearly adjacent to X. The same argument yields that X is nuclearly adjacent to Y.

To finish, notice that, since $Z \subset R_1$ we have $X + Y = S(Z + F) \subset S(R_1 + F) = S(R_2)$. Therefore, if $w \in R \cap (X + Y)$ then $w \in R_2 \cap S(R_2)$, and hence $w = 0$. Consequently $R \cap (X + Y) = \{0\}$, as we wanted. □

In view of Theorem 4.5, it is natural to investigate the possibility of finding, for a closed infinite-codimensional subspace X of a Banach space E with weak*-separable dual E^*, a pair of quasicomplementary subspaces $Y, Z \subset E$ such that $X \cap (Y + Z) = \{0\}$, or merely a proper dense operator range $R \subset E$ such that $X \cap R = \{0\}$. The next result guarantees that both properties are actually equivalent.

Theorem 4.6. *Let E be a Banach space with weak*-separable dual. If X is a closed subspace of E with $\operatorname{codim}_E(X) = \infty$, then the following assertions are equivalent:*

(1) E/X has a separable quotient.
(2) There exists an operator range $R \in \mathcal{R}_d(E)$ such that $R \cap X = \{0\}$.
(3) There exists a pair of proper quasicomplementary subspaces $Y, Z \subset E$ such that $X \cap (Y + Z) = \{0\}$.
(4) There exists a pair of isomorphic mutually nuclearly adjacent quasicomplementary subspaces $Y, Z \subset E$ such that $X \cap (Y + Z) = \{0\}$.
(5) There exists a closed subspace $Y \subset E$ such that $X \cap Y = \{0\}$, E/Y is separable and the sum $X + Y$ is not closed.

Proof. The implications (4) ⇒ (3) ⇒ (2) are trivial. We shall show that (1) ⇒ (5) ⇒ (4) and (2) ⇒ (1). Assume that property (1) is satisfied. Since X^* is weak*-separable (as E^* is) and E/X has a separable quotient, thanks to Theorem [17, Corollary 2.9] X has a proper quasicomplement Y such that E/Y is separable. Thus (1) ⇒ (5).

Now, let $Y_0 \subset E$ be a closed subspace verifying property (5), and define $R = X + Y_0$. Then R is an operator range in E. Bearing in mind that R is not closed in E, according to [1, Corollary 2.17] we get $\operatorname{codim}_E(R) = \infty$. Since E/Y_0 is separable and $Y_0 \subset R$, an appeal to [17, Corollary 2.12] (or Theorem 4.1) entails the existence of a closed subspace $Y_1 \subset E$, isomorphic to Y_0, such that Y_1 is a nuclearly adjacent quasicomplement of Y_0 and $Y_1 \cap R = \{0\}$.

Set $R_1 = R + Y_1$. Since $Y_0 + Y_1$ is dense in E we have that R_1 is a dense operator range in E, and taking into account that, by hypothesis, R is not closed in E, thanks to [7, Theorem 2.4] it follows that $R_1 \neq E$. Consequently, $R_1 \in \mathcal{R}_d(E)$. Using now Theorem 4.5 we deduce the existence of a closed subspace $Y \subset E$ and an isomorphism $\varphi : E \to E$ such that Y and $Z = \varphi(Y)$ are mutually nuclearly adjacent quasicomplementary subspaces and $R_1 \cap (Y + Z) = \{0\}$. In particular, as $R_1 \supset X$ we obtain $X \cap (Y + Z) = \{0\}$. Therefore (5) ⇒ (4). Finally, assume there exists $R \in \mathcal{R}_d(E)$ such that $R \cap X = \{0\}$, and set $V = R + X$. Since $V \in \mathcal{R}_d(E)$, Lemma 3.4 yields that E/X has a separable quotient, thus (2) ⇒ (1). □

Remark 4.7. Rosenthal [23] (see e.g. [15, Theorem 5.83]) proved that if X is a closed infinite-codimensional subspace of ℓ_∞, then the subspace $X^\perp \subset (\ell_\infty)^*$ contains a reflexive subspace, in particular ℓ_∞/X has a separable quotient. Therefore, thanks to Theorem 4.6 there exist closed quasicomplementary subspaces $Y, Z \subset \ell_\infty$ such that $X \cap (Y + Z) = \{0\}$.

In the case that the space E is hereditarily indecomposable, we have the following result, which simplifies the spaceability condition (5) of the set $E \setminus X$ in Theorem 4.6.

Corollary 4.8. *If X is a closed infinite-codimensional subspace of an hereditarily indecomposable Banach space E, then the following assertions are equivalent:*

(1) X is quasicomplemented in E.
(2) There exists a closed subspace $Y \subset E$ such that E/Y is separable and $X \cap Y = \{0\}$.
(3) There exist two closed mutually nuclearly adjacent quasicomplementary isomorphic subspaces $Y, Z \subset E$ such that $X \cap (Y + Z) = \{0\}$.

Proof. Being hereditarily indecomposable, the space E has weak*-separable dual (cf. [4, Theorem 1.3]). Moreover, if X satisfies (1) and X_1 is a quasicomplement of X, then (as X is infinite-codimensional and E is hereditarily indecomposable) we have that $X + X_1$ is not closed, in particular X_1 is a proper quasicomplement of E, thus $R = X + X_1$ is a proper dense operator range in E. By Theorem 4.6, there exist isomorphic mutually nuclearly adjacent quasicomplementary subspaces $Y, Z \subset E$ such that $R \cap (Y + Z) = \{0\}$, therefore $X \cap (Y + Z) = \{0\}$, and so (1) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) is trivial. On the other hand, if a subspace $Y \subset E$ satisfies (2) then, as E is hereditarily indecomposable the sum $X + Y$ is not closed. Thus, again by Theorem 4.6 we have that E/X has a separable quotient, and an appeal to the Theorem of Lindenstrauss and Rosenthal (cf. [15, Theorem 5.79], see also [17, Theorem 3.1]) yields that X is quasicomplemented in E, therefore (2) $\Rightarrow$ (1). □

Another application of Theorem 4.6 yields the following result.

Corollary 4.9. *Let E be a Banach space with weak*-separable dual, and let X be a closed infinite-codimensional subspace of E. Assume there exists a closed subspace $Y_0 \subset E$ such that:*

(1) $X \cap Y_0 = \{0\}$,
(2) E/Y_0 is separable, and
(3) Y_0 has a separable quotient.

Then, there exist two isomorphic mutually nuclearly adjacent quasicomplementary subspaces $Y, Z \subset E$ such that $X \cap (Y + Z) = \{0\}$.

Proof. Let us write $R = X + Y_0$. Then, R is an operator range in E. Assume first that $\operatorname{codim}_E(R) = \infty$. Since E/Y_0 is separable and Y_0^* is weak*-separable (as E^* is), according to [17, Theorem 2.1] there exists a closed subspace $Z \subset E$ such that $R \cap Z = \{0\}$ and Z is a proper quasicomplement of Y_0. Set $V = R + Z$. Then $X \subset V$ and $V \in \mathcal{R}_d(E)$. Therefore, by Lemma 3.4, E/X has a separable quotient, and Theorem 4.6 applies.

Now, suppose that R is finite-codimensional in E. Then R is a closed subspace of E, and $R = X \oplus Y_0$. Let F be a finite-dimensional subspace of E such that $R \cap F = \{0\}$ and $R + F = E$. Define $Y_1 = Y_0 + F$. Since Y_0 has a separable quotient, so does Y_1. Taking into account that E/X is isomorphic to Y_1 it follows that E/X has a separable quotient as well, and a new appeal to Theorem 4.6 yields the desired conclusion. □

5. Operator ranges in weakly Lindelöf determined Banach spaces

In this section, we establish the impossibility of extending some of the previous results to a wide class of Banach spaces with no weak*-separable dual. In particular, we show that some weak forms of the theorems of Shevchik and Cross-Shevchik do not hold in any nonseparable weakly Lindelöf determined space. Recall that a Banach space E is said to be **weakly Lindelöf determined** (**WLD**, for short) if there exist a set Γ and a one-to-one (weak*, τ_p)-continuous operator $T : E^* \to \ell_\infty^c(\Gamma)$, where $\ell_\infty^c(\Gamma)$ is the space of all $x \in \ell_\infty(\Gamma)$ with countable support (that is, card $(\{\gamma \in \Gamma : x(\gamma) \neq 0\}) \leq \aleph_0$), and τ_p denotes the topology of pointwise convergence on that space. WLD spaces constitute a large class that contains the one of WCG (weakly compactly generated) spaces. It is well-known that if E is a WLD space, then E is **DENS**, that is, the weak*-density character of E^* agrees with the density character of E (cf. [15, Proposition 5.40]), and that if X is a closed subspace of E, then X is WLD (cf. [15, Corollary 5.43]) and has a quasicomplementary subspace (cf. [15, Corollary 5.74]). From the latter and the results in [24] it follows that if E is a WLD space, then E has a separable quotient, and $\mathcal{R}_d(E) \neq \emptyset$. We also notice that, by the very definition, the class of WLD spaces is stable for quotients. For more information on this class we refer to the paper [3] and the monograph [15].

The results of the two previous sections yield the following characterization of separable spaces among WLD spaces in terms of essential disjointness properties of operator ranges.

Fact 5.1. *If E is a WLD space, then the following assertions are equivalent:*

(1) E is separable.
(2) For each $R \in \mathcal{R}_d(E)$ there exists a one-to-one endomorphism $T : E \to E$ such that $T(E) \cap R = \{0\}$.
(3) For each $R \in \mathcal{R}_d(E)$ and each $\varepsilon > 0$ there exists an isomorphism $\varphi : E \to E$ such that $\|\varphi - I_E\| < \varepsilon$ and $\varphi(R) \cap R = \{0\}$.
(4) For each $R \in \mathcal{R}_d(E)$ there exists a closed subspace $Y \subset E$ such that E/Y is separable and $R \cap Y = \{0\}$.
(5) For each $R \in \mathcal{R}_d(E)$ there exist two closed isomorphic mutually nuclearly adjacent quasicomplementary subspaces $Y, Z \subset E$ such that $R \cap (X + Y) = \{0\}$.
(6) For each closed subspace $X \subset E$ of infinite codimension there exist two closed isomorphic mutually nuclearly adjacent quasicomplemented subspaces $Y, Z \subset E$ such that $X \cap (Y + Z) = \{0\}$.

Proof. Since WLD spaces are DENS and have a separable quotient, the equivalence between properties (1), (3) and (4) is a consequence of Theorem 3.2, (1) $\Rightarrow$ (2) is a particular case of Shevchik's theorem, and (2) $\Rightarrow$ (1) follows again from Theorem 3.2. The implication (1) $\Rightarrow$ (5) is Theorem 4.5, and (1) $\Rightarrow$ (6) follows from Theorem 4.6. On the other hand, if E contains a couple of mutually nuclearly adjacent quasicomplementary subspaces, then E^* is weak*-separable (cf. [17, Corollary 2.14]), thus assertions (5) and (6) both imply (1). □

Next, we provide some refinements of Fact 5.1. In particular, we will show that a WLD space E is separable if, and only if, for every $R \in \mathcal{R}_d(E)$ there exists a dense-range operator $T : E \to E$ such that $T(E) \cap R = \{0\}$. We start with an example of this situation in the Hilbert space setting.

Example 5.2. Let Γ be any uncountable set, consider the Hilbert space $H = \ell_2(\Gamma) \oplus \ell_2(\mathbb{N})$, let R be a proper dense operator range in $\ell_2(\mathbb{N})$, and let Q be the operator range in H defined as $Q = \ell_2(\Gamma) \oplus R$. Then, there is no continuous linear operator $T : H \to H$ with dense range such that $T(H) \cap Q = \{0\}$.

Indeed, suppose that such an operator T exists. Set $W = H/\ker T$ and let $\widehat{T} : W \to H$ be the operator defined for each $\hat{x} \in W$ as $\widehat{T}(\hat{x}) = T(x)$. Then, $\widehat{T}$ is injective and has the same dense range as T. Since $\widehat{T}$ is dense in H it follows that W is a nonseparable Hilbert space. Now, consider the composition $\pi \circ \widehat{T} : W \to \ell_2(\mathbb{N})$, where π is the orthogonal projection of H onto $\ell_2(\mathbb{N})$. Then, $\pi \circ \widehat{T} : W \to \ell_2(\mathbb{N})$ is one-to-one. Indeed, if $\pi \circ \widehat{T}(x) = 0$ for some $x \in W$, then $\widehat{T}(x) \in \ell_2(\Gamma)$. Since $\ell_2(\Gamma) \subset Q$ and $\widehat{T}(W) \cap Q = \{0\}$, we have that $\widehat{T}(x) = 0$ and thus $x = 0$ (because of the injectivity of $\widehat{T}$). Thus, if $\{e_n^*\}_n$ is the sequence of biothogonal functionals associated with the canonical basis of $\ell_2(\mathbb{N})$, then the sequence $\{(\pi \circ \widehat{T})^*(e_n^*)\}_n$ is total over W, therefore W is separable, a contradiction. □

Now, we state the promised result. As regards assertions $(2)-(5)$ in the next theorem we point out that, in general, the class of operator ranges in a Banach space is strictly larger than the one of endomorphism ranges (cf. [7, 13]).

Theorem 5.3. *If E is a WLD space, then the following assertions are equivalent:*

(1) E is separable.
(2) For each proper dense endomorphism range R in E there exists a one-to-one endomorphism $T : E \to E$ such that $T(R) \cap R = \{0\}$.
(3) For each proper dense endomorphism range R in E there exists an endomorphism $T : E \to E$ such that $\overline{T(E)} = E$ and $T(R) \cap R = \{0\}$.
(4) For each proper dense endomorphism range R in E there exists a closed subspace $Y \subset E$ such that E/Y is separable and $R \cap Y = \{0\}$.
(5) For each proper dense endomorphism range R in E there exist two closed subspaces $Y, Z \subset E$ such that $Y + Z$ is dense in E and $R \cap Y = R \cap Z = \{0\}$.
(6) For each closed subspace $X \subset E$ with $\operatorname{codim}_E X = \infty$ there exist two closed subspaces $Y, Z \subset E$ such that $Y + Z$ is dense in E and $X \cap Y = X \cap Z = \{0\}$.

In the proof of this theorem we shall use the following easy fact.

Lemma 5.4. *If E is a WLD space, then E contains a proper dense endomorphism range R and a closed subspace X such that E/X is separable and $X \subset R$. In particular, if Z is any closed subspace of E such that $R \cap Z = \{0\}$, then Z is separable.*

Proof. Being WLD, the space E contains a separable complemented subspace, say Y (cf. [15, p. 105]). Consider a proper dense endomorphism range R_0 in Y, let $\pi : E \to Y$ be a projection onto Y and set $X = \ker \pi$. Then, the sum $R = R_0 + X$ is a proper dense linear subspace of E. Moreover, because of the separability of Y we have that E/X is separable. On the other hand, if $U_0 : Y \to Y$ is an endomorphism with $U_0(Y) = R_0$ and $U : E \to E$ is the operator defined as $U = U_0 \circ \pi + (I_E - \pi)$, then $R = U(E)$, thus R is an endomorphism range in E.

Now, consider a closed subspace $Z \subset E$ such that $R \cap Z = \{0\}$. Then $X \cap Z = \{0\}$, therefore the restriction to Z of the quotient map $Q : E \to E/X$ is one-to-one, and taking into account that E/X is separable it follows that Z^* is weak*-separable. Since Z is a WLD space, it is DENS, consequently Z is separable. □

Proof of Theorem 5.3. The fact that (1) implies the rest of assertions is already known. Let us prove the reverse implications. Along the proof, the symbols R and X denote a proper dense endomorphism range in E and a closed subspace $X \subset R$ satisfying the properties specified by the previous lemma.

Suppose that assertion (2) holds. Then there exists a one-to-one operator $T : E \to E$ such that $T(R) \cap R = \{0\}$, in particular $T(X) \cap X = \{0\}$. Since E/X is separable, thanks to Theorem 3.2 we deduce that E^* is weak*-separable, and taking into account that every WLD space is DENS, it follows that E is separable, thus (2) ⇒ (1).

Next, assume that assertion (3) is satisfied. Let $T : E \to E$ be a dense-range operator such that $T(R) \cap R = \{0\}$. Since E/X is separable, according to [17, Corollary 2.7], we have that X admits a separable quasicomplementary subspace, say Y. Then $T(Y)$ is separable. Thus, bearing in mind that $X + Y$ is dense in E and $\overline{T(E)} = E$, in order to prove that E is separable it is enough to show that $T(X)$ is separable. Let us write $T_1 = T|_X : X \to E$, and let $\widehat{T_1} : X/\ker T_1 \to E$ be the one-to-one operator defined by the formula

$$\widehat{T_1}(\hat{x}) = T_1(x) = T(x), \quad \hat{x} \in X/\ker \widehat{T_1}.$$

Consider the composition $S = Q \circ \widehat{T_1} : X/\ker T_1 \to E/X$, where $Q : E \to E/X$ denotes the quotient map onto E/X. As $\widehat{T_1}$ is one-to-one and $T(X) \cap X = \{0\}$ it follows that S is one-to-one. Therefore, its adjoint $S^* : (E/X)^* \to (X/\ker T_1)^*$ has weak*-dense range, and taking into account that $(E/X)^*$ is weak*-separable we deduce that $(X/\ker T_1)^*$ is weak*-separable as well. On the other hand, since the class of WLD spaces is stable for quotients, we have that $X/\ker T_1$ is WLD, hence $X/\ker T_1$ is separable, and bearing in mind that $\widehat{T_1}\,(X/\ker T_1) = T_1(X) = T(X)$ we deduce that $T(X)$ is separable, consequently (3) ⇒ (1).

Now, suppose that property (4) holds. Then, there exists a closed subspace $Y \subset E$ such that E/Y is separable and $R \cap Y = \{0\}$. By the previous lemma, Y must be separable. Hence E is separable, so (4) ⇒ (1). Now, if (5) is satisfied, then we can find closed subspaces $Y, Z \subset E$ such that the sum $Y + Z$ is dense in E and $R \cap Y = R \cap Z = \{0\}$. Lemma implies that Y and Z are separable, and since $Y + Z$ is dense in E we have that E is separable, too, hence (5) ⇒ (1). Following the same argument we achieve (6) ⇒ (1). □

In [14, Corollary 2.6] (see also [8, Proposition 5.1]), it was shown that if H is a separable Hilbert space, then for every $R \in \mathcal{R}(E)$ there exist two closed subspaces H_0 and H_1 of H such that $H = H_0 \oplus_\perp H_1$ and $H_0 \cap R = H_1 \cap R = \{0\}$. As an immediate consequence of this result and Theorem 5.3 we obtain the following characterization of separable Hilbert spaces.

Corollary 5.5. *If H is a Hilbert space, then the following assertions are equivalent:*

(1) H is separable.
(2) For every $R \in \mathcal{R}(H)$ there exists an orthogonal decomposition $H = H_0 \oplus_\perp H_1$ such that $H_0 \cap R = H_1 \cap R = \{0\}$.
(3) For every proper dense endomorphism range R in H there exists an orthogonal decomposition $H = H_0 \oplus_\perp H_1$ such that $H_0 \cap R = H_1 \cap R = \{0\}$.

As we mentioned in the Introduction, Plichko [22] proved that every Banach space with a fundamental biorthogonal system contains a couple of dense essentially disjoint operator ranges. On the other hand, Theorem 5.3 yields that if E is a nonseparable WLD space, then there exists an operator range $R \in \mathcal{R}_d(E)$ such that $R \cap T(R) \neq \{0\}$ for any one-to-one endomorphism $T : E \to E$. Bearing in mind that every WLD has a fundamental biorthogonal system (cf. [15, Theorem 5.37]), it is natural to ask if, for any WLD space E, there exist a proper dense operator range $V \subset E$ and an isomorphism (or a one-to-one operator) $T : E \to E$ such that $T(V) \cap V = \{0\}$. The next example provides a first result in this direction.

Example 5.6. For every set Γ and every $\varepsilon > 0$ there exist a proper dense endomorphism range W in $c_0(\Gamma)$ and an isomorphism $\Phi : c_0(\Gamma) \to c_0(\Gamma)$ such that $\|\Phi - I_{c_0(\Gamma)}\| < \varepsilon$ and $\Phi(W) \cap W = \{0\}$.

Indeed, consider $c_0(\Gamma)$ partitioned in the following way

$$c_0(\Gamma) = \left\{ (x_\alpha)_{\alpha\in\Gamma} : \, x_\alpha \in c_0 \text{ for all } \alpha \in \Gamma \text{ and } \lim_{\alpha\in\Gamma} \|x_\alpha\|_\infty = 0 \right\},$$

with the norm $\|(x_\alpha)_{\alpha\in\Gamma}\|_\infty = \sup_{\alpha\in\Gamma} \|x_\alpha\|_\infty$. Let V be any proper dense endomorphism range in c_0. According to Theorem 3.1, there exists an isomorphism $\varphi : c_0 \to c_0$ such that $\|\varphi - I_{c_0}\| < \varepsilon$ and $\varphi(V) \cap V = \{0\}$. Now, let $T : c_0 \to c_0$ be an operator such that $T(c_0) = V$. Then, the formula

$$\widehat{T}\left((x_\alpha)_{\alpha\in\Gamma}\right) = \left(T(x_\alpha)\right)_{\alpha\in\Gamma}, \quad (x_\alpha)_{\alpha\in\Gamma} \in c_0(\Gamma)$$

defines an endomorphism $\widehat{T} : c_0(\Gamma) \to c_0(\Gamma)$. Let us write $W = \widehat{T}\left(c_0(\Gamma)\right)$. We claim that W is dense in $c_0(\Gamma)$. Indeed, choose a vector $(x_\alpha)_{\alpha\in\Gamma} \in c_0(\Gamma)$ and $\delta > 0$. Then, there are finitely many indices $\alpha_1, \dots, \alpha_n$ such that $\|x_\alpha\|_\infty < \delta$ for all $\alpha \in \Gamma \setminus \{\alpha_1, \dots, \alpha_n\}$. Since $\overline{T(c_0)} = c_0$ we can find vectors $u_{\alpha_1}, \dots, u_{\alpha_n} \in c_0$ such that $\left\|T(u_{\alpha_j}) - x_{\alpha_j}\right\|_\infty < \delta$ for each $j = 1, \dots, n$. Define $u_\alpha = 0$ if $\alpha \in \Gamma \setminus \{\alpha_1, \dots, \alpha_n\}$. Then, $(u_\alpha)_{\alpha\in\Gamma} \in c_0(\Gamma)$, and $\|\widehat{T}((u_\alpha)_\alpha) - (x_\alpha)_\alpha\|_\infty < \delta$. Therefore, W is a dense endomorphism range in $c_0(\Gamma)$. Now, let $\Phi : c_0(\Gamma) \to c_0(\Gamma)$ be the operator defined by the formula

$$\Phi\left((x_\alpha)_{\alpha\in\Gamma}\right) = \left(\varphi(x_\alpha)\right)_{\alpha\in\Gamma}, \quad (x_\alpha)_{\alpha\in\Gamma} \in c_0(\Gamma).$$

Since $\varphi : c_0 \to c_0$ is an isomorphism we may assume that there is $0 < m \leq 1$ such that

$$m\|x\|_\infty \leq \|\varphi(x)\|_\infty \leq \|x\|_\infty \quad \text{for all} \quad x \in c_0.$$

This yields $m\|(x_\alpha)_\alpha\|_\infty \leq \|\Phi\left((x_\alpha)_\alpha\right)\|_\infty \leq \|(x_\alpha)_\alpha\|_\infty$, for all $(x_\alpha) \in c_0(\Gamma)$. Therefore, Φ is an isomorphic embedding. Now, pick a vector $(y_\alpha)_{\alpha\in\Gamma} \in c_0(\Gamma)$, and set, for each $\alpha \in \Gamma$, $x_\alpha = \varphi^{-1}(y_\alpha)$. Then, $(x_\alpha)_\alpha \in c_0(\Gamma)$ and $\Phi\left((x_\alpha)_\alpha\right) = (y_\alpha)_\alpha$, thus Φ is surjective, and so an isomorphism. Notice also that $\|\Phi - I_{c_0(\Gamma)}\| = \|\varphi - I_{c_0}\| < \varepsilon$. To finish, take $(x_\alpha)_\alpha \in W$ such that $\Phi\left((x_\alpha)_\alpha\right) \in W$. Then $\varphi(x_\alpha) \in V \cap \varphi(V)$ for all $\alpha \in \Gamma$. Since $\varphi(V) \cap V = \{0\}$ we get $\varphi(x_\alpha) = 0$, and thus $x_\alpha = 0$ for all $\alpha \in \Gamma$. Consequently, $\Phi(W) \cap W = \{0\}$, as we wanted. $\Box$

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