CORRIGENDUM TO "ON SMOOTH EXTENSIONS OF VECTOR-VALUED FUNCTIONS DEFINED ON CLOSED SUBSETS OF BANACH SPACES" [MATH. ANN. (2013) 355: 1201–1219]

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Abstract. The present note is a corrigendum to the paper "On Smooth extensions of vector-valued functions defined on closed subsets of Banach spaces" [Math. Ann. (2013) 355: 1201–1219].

L. Zajicek kindly pointed us a flaw in the proof of Lema 3.8 in [1]. The present note is devoted to explain how to overcome the flaw in [1] with the additional assumption that (X, Z) has property (E) (we follow the notation of [1]). We will give in this note a list of changes to be made in the proof of Lemma 3.8 in [1] in order to obtain a correct version of this lemma and thus correct versions of Theorem 3.1 and Theorem 3.2 of [1]. In a personal communication L. Zajicek informed us that together with J. Johanis they have obtained a different proof of Theorem 3.1 and Theorem 3.2 in [2] with the additional assumption that the pair of Banach spaces (X, Z) has property (E) as well.
Let us see how to modify Lemma 3.8. In the first place, as it is already mentioned, we additionally should assume in Lemma 3.8 that the pair of Banach spaces (X, Z) has property (E) for a constant $K > 0$. Let us fix a constant $\tau > 2$. Following the notation of the proof of Lemma 3.8, we may assume, when selecting the open covering $\{B_\gamma := B(y_\gamma, r_\gamma)\}_{\gamma\in\Gamma}$ satisfying all conditions specified in the proof of Lemma 3.8 that additionally $\{B(y_\gamma, \frac{r_\gamma}{\tau})\}_{\gamma\in\Gamma}$ is an open cover of A as well. Notice that from [1], $\|D(y)\| \le M$ for all $y \in A$, $\|D(y) - h'(y)\| \le \varepsilon'$ for all $y \in A$ and $\|h'(y) - h'(y_\gamma)\| \le \frac{\varepsilon'}{8C_0}$ for all $y \in B_\gamma$, so we get

$$\|h'(y)\| \le \|h'(y) - h'(y_\gamma)\| + \|h'(y_\gamma) - D(y_\gamma)\| + \|D(y_\gamma)\| \le \frac{\varepsilon'}{8C_0} + \varepsilon' + M, \qquad \text{for all } y \in B_\gamma. \quad (1)$$

Also from [1], $Lip(f - h|_A) < \min\{\frac{\varepsilon'}{C_0(1+2C_0)}, (\frac{\varepsilon'}{2C_0(R+2\varepsilon')})^2\}$ whenever $R < \infty$ and $Lip(f - h|_A) < \frac{\varepsilon'}{C_0}$ whenever $R = \infty$ (property (iii) in the proof of Lemma 3.8), so we get

$$Lip(h|_A) \le \min\{\frac{\varepsilon'}{C_0(1+2C_0)}, (\frac{\varepsilon'}{2C_0(R+2\varepsilon')})^2\} + Lip(f) := L_1 \qquad \text{whenever } R < \infty$$

and $Lip(h|_A) \le \frac{\varepsilon'}{C_0} + Lip(f) := L_2$ whenever $R = \infty$. Since $L_1 \le L_2$, it will be enough for us to consider L_2 for both cases. So $Lip(h|_A) \le L_2$ for any $R \in (0, \infty]$.

Let us define the open balls $V_\gamma := B(y_\gamma, \frac{r_\gamma}{\tau})$ for every $\gamma \in \Gamma$. We consider the open subset $U := \cup_{\gamma\in\Gamma} V_\gamma \subset X$ and the restriction mapping $h|_U$. Let us check that $h|_U$ is Lipschitz: If $x, y \in U$ there are indexes α and γ such that $x \in V_\alpha$ and $y \in V_\gamma$. Then by inequality (1), we have

$$\begin{aligned}\|h(x) - h(y)\| \le& \|h(x) - h(y_\alpha)\| + \|h(y_\alpha) - h(y_\gamma)\| + \|h(y_\gamma) - h(y)\| \le \\ \le& (\frac{\varepsilon'}{8C_0} + \varepsilon' + M)\|x - y_\alpha\| + L_2\|y_\alpha - y_\gamma\| + (\frac{\varepsilon'}{8C_0} + \varepsilon' + M)\|y - y_\gamma\| \le \\ \le& (\frac{\varepsilon'}{8C_0} + \varepsilon' + M)(\|x - y_\alpha\| + \|y - y_\gamma\|) + L_2(\|y_\alpha - x\| + \|x - y\| + \|y - y_\gamma\|).\end{aligned}$$

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Now, define $\delta = \frac{2}{\tau-2} > 0$. If $\|x-y_\alpha\| \le \delta\|x-y\|$ and $\|y-y_\gamma\| \le \delta\|x-y\|$, then

$$\|h(x)-h(y)\| \le \Big[2\delta(\frac{\varepsilon'}{8C_0}+\varepsilon'+M)+L_2(2\delta+1)\Big]\|x-y\|,$$

If $\|x-y_\alpha\| > \delta\|x-y\|$ or $\|y-y_\gamma\| > \delta\|x-y\|$, let us assume for example the first inequality, then $\|x-y\| < \frac{1}{\delta}\|x-y_\alpha\| < \frac{r_\alpha}{\delta\tau}$ and thus $\|y-y_\alpha\| \le \|y-x\|+\|x-y_\alpha\| < \frac{r_\alpha}{\delta\tau}+\frac{r_\alpha}{\tau} = (1+\frac{1}{\delta})\frac{1}{\tau}r_\alpha = \frac{r_\alpha}{2}$ and $y \in B_\alpha$. So $\|h(x)-h(y)\| \le (\frac{\varepsilon'}{8C_0}+\varepsilon'+M)\|x-y\|$. A similar argument works for the second inequality. Therefore,

$$\|h(x)-h(y)\| \le L_3\|x-y\|, \text{ for all } x,y \in U,$$

where

$$L_3 := \max\{2\delta(\frac{\varepsilon'}{8C_0}+\varepsilon'+M)+L_2(2\delta+1),\ \frac{\varepsilon'}{8C_0}+\varepsilon'+M\}.$$

The next change to be done is to define the new open cover of X formed by $V_0 = X \setminus A$ and $\mathcal{C} = \{V_\beta : \beta \in \Sigma = \Gamma \cup \{0\}\}$ and consider an open refinement $\{W_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ of $\mathcal{C} := \{V_\beta : \beta \in \Sigma\}$ and a C^1 smooth and Lipschitz partition of unity $\{\psi_{n,\beta} : n \in \mathbb{N}, \beta \in \Sigma\}$ subordinated to $\mathcal{C}$ satisfying properties (P1)-(P3) specified in the proof of Lemma 3.7 in [1], where among other properties, $\operatorname{supp}\psi_{n,\beta} \subset W_{n,\beta} \subset V_\beta$, for all $n \in \mathbb{N}$ and $\beta \in \Sigma$.
The definitions of the constants $L_{n,\beta} := \max\{Lip(\psi_{n,\beta}),1\}$ for $n \in \mathbb{N}$ and $\beta \in \Sigma$ and the functions $\delta_{n,\gamma}$ obtained by applying property (*) to $T_\gamma - h$ on B_γ for every $n \in \mathbb{N}$, $\gamma \in \Gamma$ remain the same as in [1].
The next modification is to apply property (E) to $h|_U : U \to B_Z(0, R+\varepsilon'+\frac{\varepsilon'}{8C_0})$. In the case, $R=+\infty$ we get a KL_3-Lipschitz function $H : X \to Z$ such that $H|_U = h|_U$. We apply property (*) to H in order to obtain C^1 smooth mappings $F_0^n : X \to Z$ such that $\|H(x)-F_0^n(x)\| \le \frac{\varepsilon'}{2^{n+2}L_{n,0}}$ for all $x \in X$ and $Lip(F_0^n) \le C_0KL_3$ for all $n \in \mathbb{N}$. In the case $R < +\infty$ we consider the auxiliary function $G : U \cup W \to B_Z(0, R+\varepsilon'+\frac{\varepsilon'}{8C_0})$, where $W = \{x \in X : \operatorname{dist}(x,U) \ge (R+\varepsilon'+\frac{\varepsilon'}{8C_0})L_3^{-1}\}$ defined by $G|_U = h|_U$ and $G|_W = 0$. It can be checked that G is L_3-Lispchitz. By property (E) there is a KL_3-Lispchitz extension $L : X \to Z$ of G and thus, in particular, it can be checked that $L : X \to B_Z(0, (K+1)(R+\varepsilon'+\frac{\varepsilon'}{8C_0}))$. Now, let us apply property (*) to L to get C^1 smooth mappings $F_0^n : X \to B_Z(0, (K+1)(R+\varepsilon'+\frac{\varepsilon'}{8C_0})+\frac{\varepsilon'}{2^{n+2}L_{n,0}})$ such that $\|L(z)-F_0^n(z)\| < \frac{\varepsilon'}{2^{n+2}L_{n,0}}$ for all $z \in X$ and $Lip(F_0^n) \le C_0KL_3$.
The definitions of the functions $\{\Delta_\beta^n\}_{n\in\mathbb{N},\beta\in\Sigma}$ in terms of the functions $\{F_0^n\}_{n\in\mathbb{N}}$, $\{T_\gamma - \delta_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$ and the definition of g in terms of $\{\psi_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ is the same as in Lemma 3.8 in [1]. The new upper bound for g is

$$\|g(x)\| \le \sum_{(n,\beta)\in\mathbb{N}\times\Sigma}\psi_{n,\beta}(x)\|\Delta_\beta^n(x)\| \le (K+1)(R+\varepsilon'+\frac{\varepsilon'}{8C_0})+\frac{\varepsilon'}{8} := Q_1, \quad \text{for } x \in X. \tag{C.1}$$

It can be checked that the new upper bound for $\|(\Delta_\beta^n)'(x)\|$ for $x \in X$ is

$$\|(\Delta_\beta^n)'(x)\| \le \max\{Lip(F_0^n),\ M+9\frac{\varepsilon'}{8}\} \le \max\{C_0KL_3,\ M+9\frac{\varepsilon'}{8}\} = C_0KL_3,$$

where $n \in \mathbb{N}$ and $\beta \in \Sigma$. Let us check that g is Lispchitz. Let us denote $S := H$ if $R=+\infty$ and $S := L$ if $R < +\infty$. Since $\operatorname{supp}\psi_{n,\beta} \subset V_\beta$ for all $n \in \mathbb{N}$, $\beta \in \Gamma$ and $S|_U = h$, following the notation of the proof of Lemma 3.8 in [1] we have for $x \in X$,

$$\begin{aligned}||g'(x)|| &\le \sum_{(n,\beta)\in F_x}\|\psi'_{n,\beta}(x)\|\|S(x)-\Delta_\beta^n(x)\| + \sum_{(n,\beta)\in F_x}\psi_{n,\beta}(x)\|(\Delta_\beta^n)'(x)\| \le \\ &\le \sum_{\{n:(n,\beta(n))\in F_x\}} L_{n,\beta(n)}\frac{\varepsilon'}{2^{n+2}L_{n,\beta(n)}} + \sum_{\{n:(n,\beta(n))\in F_x\}}\psi_{n,\beta}(x)C_0KL_3 \le \frac{1}{4}\varepsilon'+C_0KL_3 := L_4,\end{aligned} \tag{C.2}$$

In addition, properties (i), (ii) and (iii) in the statement of Lemma 3.8 in [1] follow in the same way.

The new upper bound in property (iv) in the statement of Lemma 3.8 in [1] is Q_1 given in (C.1). The new upper bound in property (v) in the statement of Lemma 3.8 in [1] is L_4 given in (C.2).
Due to the preceding arguments, we must assume the additional condition that the pair of Banach spaces (X, Z) has property (E) in the statements of Theorem 3.1 and Theorem 3.2 of [1]. In order to apply Lemma 3.8 with the new upper bounds Q_1 and L_4, let us consider simpler upper bounds in terms of M, K, C_0, ε. Recall that, from the assumptions in Lemma 3.8 in [1], $3\varepsilon' < \varepsilon$, and thus after some straightforward calculations it can be checked that

$$\begin{aligned} Q_1 &< (K+1)R + A_1\varepsilon, \\ L_4 &< A_2\varepsilon + C_0K(4\delta+1)\max\{Lip(f), M\}, \end{aligned}$$

where $A_1 := \frac{1}{3}[(K+1)(1+\frac{1}{8C_0}) + \frac{1}{8}],\ A_2 := \frac{1}{3}[\frac{1}{4} + C_0K(1+\frac{2}{C_0}+\frac{1}{8C_0})]$ and we have assumed without loss of generality that $2\delta \leq 1$.

Now, we can reproduce the proofs of Theorem 3.1 and 3.2 given in [1] with any sequence $\{\varepsilon_n\}_n \subset (0,1)$ such that $\sum_n \varepsilon_n < \infty$. For the sake of completeness let us briefly sketch it. Following the notation of the proofs of Theorem 3.1 and 3.2 in [1], in the first step we apply Lemma 3.7 to get a C^1 smooth mapping $G_1 : X \to Z$ such that if $g_1 := G_1|_A$ then (i) $||f(y) - g_1(y)|| < \varepsilon_1$ for all $y \in A$; (ii) $\|D(y) - G_1'(y)\| < \varepsilon_1$ for all $y \in A$ and (iii) $Lip(f - g_1) < \varepsilon_1$. In the second step, the funtion $f - g_1 : A \to B_Z(0, \varepsilon_1)$ satisfies the mean value condition for the function $D - G_1' : A \to \mathcal{L}(X, Z)$ with $\|D(y) - G_1'(y)\| \leq \varepsilon_1$ for all $y \in A$. So Lemma 3.8 applies to get a C^1 smooth function $G_2 : X \to Z$ such that if $g_2 := G_2|_A$, then (i) $\|f(y) - g_1(y) - g_2(y)\| < \varepsilon_2$ for all $y \in A$, (ii) $\|D(y) - G_1'(y) - G_2'(y)\| < \varepsilon_2$ for all $y \in A$, (iii) $Lip(f - g_1 - g_2) < \varepsilon_2$, (iv) $\|G_2(y)\| < (K+1)\varepsilon_1 + A_1\varepsilon_2$ for all $y \in X$ and (iv) $Lip(G_2) < A_2\varepsilon_2 + C_0K(4\delta+1)\varepsilon_1$. In general, in the n-th step ($n \geq 2$) we apply Lemma 3.8 to the Lipschitz function $f - (g_1 + \cdots + g_{n-1}) : A \to B_Z(0, \varepsilon_{n-1})$, which satisfies the mean value condition for the function $D - (G_1' + \cdots + G_{n-1}') : A \to \mathcal{L}(X, Z)$ with $\|D(y) - (G_1' + \cdots + G_{n-1}')(y)\| \leq \varepsilon_{n-1}$ for $y \in A$ and $Lip(f - (g_1 + \cdots + g_{n-1})) < \varepsilon_{n-1}$ to get a C^1 smooth function $G_n : X \to Z$ such that if $g_n := G_n|_A$, then (i) $\|f(y) - (g_1 + \cdots + g_n)(y)\| < \varepsilon_n$ for all $y \in A$, (ii) $\|D(y) - (G_1' + \cdots + G_n')(y)\| < \varepsilon_n$ for all $y \in A$, (iii) $Lip(f - (g_1 + \cdots + g_n)) < \varepsilon_n$, (iv) $\|G_n(y)\| < (K+1)\varepsilon_{n-1} + A_1\varepsilon_n$ for all $y \in X$, and (v) $Lip(G_n) < A_2\varepsilon_n + C_0K(1+4\delta)\varepsilon_{n-1}$.

Then, it can be checked that the sum $G := \sum_n G_n$ is a C^1 smooth extension of f. If the initial function f is Lipschitz and satisfies the mean value condition for a function D with $\|D(y)\| \leq M$ for all $y \in A$, then applying Lemma 3.8 instead of Lemma 3.7 in the first step, we additionally obtain $Lip(G_1) < A_2\varepsilon_1 + C_0K(1+4\delta)\max\{M, Lip(f)\}$. So

$$Lip(G) < C_0K(1+4\delta)\max\{M, Lip(f)\} + \widetilde{\varepsilon},$$

where $\widetilde{\varepsilon} = A_2\varepsilon_1 + \sum_{n\geq 2}(A_2\varepsilon_n + C_0K(1+4\delta)\varepsilon_{n-1})$. Since we can choose from the beginning $\widetilde{\varepsilon} > 0$ and $\delta > 0$ to be as small as we want, we can get G in such a way that $Lip(G)$ is as close to $C_0K\max\{M, Lip(f)\}$ as we need.
So in the statement of Theorem 3.2 in [1] the upper bound for $Lip(G)$ should be replaced by any constant greater than $C_0K\max\{M, Lip(f)\}$.

In addition, the examples given in Corollary 3.4 in [1] are valid since they have property (E) and (*) (or equivalently property (E) and (A)). However, the upper bound for $Lip(G)$ should be replaced by the new one obtained above.

Finally, let us metion that A. Sofi kindly pointed us that Corollary 4.11 does not hold. Although (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$ X^{**} is a $\mathcal{P}_\lambda$-space, the implication (iv)$\Rightarrow$(i) does not hold.

Conflict of interest. The author states that there is no conflict of interest.
Data availability. This note has no additional data.

References

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