# Approximate Morse-Sard type results for non-separable Banach spaces. Smooth functions with no critical points

Mar Jiménez Sevilla
Joint work with D. Azagra and M. García-Bravo

Department of Mathematical Analysis and Applied Mathematics
Faculty of Mathematics
Universidad Complutense de Madrid

Madrid, June 17, 2025
47th SUMMER SYMPOSIUM IN REAL ANALYSIS

# Bump functions and Rolle's theorem

- For a Banach space $X$ a bump $f : X \to \mathbb{R}$ is a non zero continuous function with bounded support.

**Theorem (Asplund, Ekeland, Kurzweil, Lebourg, Leach and Whitfield, Namioka, Phelps, Preiss, Stegall, etc...)**

*In a separable Banach space $X$, T.F.A.E:*

1. *$X$ has a $C^1$ smooth norm,*
2. *$X$ has a $C^1$ smooth bump,*
3. *$X$ is an Asplund space (every continuous convex function on $X$ is Fréchet differentiable on a dense $G_\delta$ subset of $X$),*
4. *$X^*$ has the Radon-Nikodym property (every bounded subset $B$ of $X^*$ is dentable: for all $\varepsilon > 0$, there are $F \in X^{**}$ and $\delta \in \mathbb{R}$ such that the "slice" $\{g \in B : F(g) > \delta\}$ is non-empty with diameter smaller than $\varepsilon$).*
5. *$X^*$ is separable.*

- In a general Banach space $X$, (1)⇒(2)⇒(3)⇔(4)⇔(6): Every separable subspace of $X$ has separable dual.

# Bump functions and Rolle's theorem

**Theorem (James 1972, Enflo 1972, Fabian, Whitfield, Zizler 1983)**

- *A Banach space $X$ is superreflexive $\iff$ $X$ has a bump with uniformly continuous derivative.*
- *A Banach space $X$ is superreflexive $\iff$ $X$ has a bump with locally uniformly continuous derivative and $X$ does not contain $c_0$.*

**Question**

Does Rolle's theorem hold in infinite dimension? If $X$ is an infinite dimensional Banach space, $f : X \to \mathbb{R}$ is a $C^1$ smooth bump, does there exist a point $x_0 \in U = \{x \in X : f(x) \neq 0\} = \operatorname{supp}_0 f$, such that $f'(x_0) = 0$, that is a critical point $x_0 \in U$?

- A body (closed, bounded with non empty interior) subset $A \subset X$ is a starlike body provided there exists a point $x_0 \in \mathring{A}$ (we will assume $x_0 = 0$ by translations) such that each ray emanating from 0 intersect $\partial A$ exactly once.

- A starlike body $A$ is $C^p$ smooth whenever its Minkowski functional $\mu_A$ is $C^p$ smooth on $X \setminus \{0\}$.

**Theorem (Shkarin 1992; Azagra, Dobrowolski 1997; Azagra, J.S., 2001)**

*Let $X$ be a Banach space, $\dim X = \infty$. T.F.A.E:*

1. *$X$ has a $C^p$ smooth bump ($p \in \mathbb{N} \cup \{\infty\}$).*
2. *There is no a "Rolle's theorem" in $X$: There is a $C^p$ smooth bump $f : X \to \mathbb{R}$ such that $\operatorname{supp}_0 f = \{x \in X : f(x) \neq 0\} := U$ is contractible and yet $f$ does not have critical points in $U$.*
3. *For every $C^p$ smooth bounded starlike body $A$, there exists a $C^p$ smooth bump $f$ on $X$ with $\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}} = A$ and yet $f$ does not have critical points in $\mathring{A}$.*
4. *There exists a non-empty (contractible) closed subset $D$ of the unit ball $B_X$ of $X$ and a $C^p$ diffeomorphism $H : X \to X \setminus D$ so that $H|_{X \setminus B_X} = Id_X|_{X \setminus B_X}$. This kind of diffeomorphisms are called "deleting" or "extracting".*

- If, in addition, the bump in (1) is Lipschitz, then the bump in (2) can be constructed to be Lipschitz as well.
- Moreover, the bump in (2) can be constructed to satisfy $f'(U) \cap W = \emptyset$ for any pre-fixed finite-dimensional vector subspace $W \subset X^*$.

## Some consequences (Azagra, J.S. 2001)

The support of bumps not satisfying Rolle's theorem:

3. If $X$ is **separable**, for every bounded and open subset $U \subset E$, there is a Fréchet differentiable bump $f : E \to \mathbb{R}$ with $\operatorname{supp} f = \overline{U}$, $f$ is $C^1$ **smooth on** $U$ and yet $f$ has no critical points in $U$.
4. If $X$ is **separable**, has an **inconditional Schauder basis** and a $\boldsymbol{C^p}$ **smooth Lipschitz bump** ($p > 1$ or $p = \infty$) for every bounded and open set $U \subset E$, there is a **Fréchet differentiable bump** $f : E \to \mathbb{R}$ with $\operatorname{supp} f = \overline{U}$, $f$ is $\boldsymbol{C^p}$ **smooth on** $U$ and yet $f$ has no critical points in $U$.

- (West, 1969) For $E$ separable Banach space with a $C^p$ smooth bump function with bounded $p$ derivatives ($p \in \mathbb{N}$), and for every bounded and open subset $U \subset E$ there is a $C^p$ smooth bump on $E$ such that $\operatorname{supp} f = \overline{U}$.

Let us recall the following equivalences:

**Theorem (Bonic and Frampton, Kurzweil, Torunzcyk, Godefroy, Troyanski, Withfield, Zizler, etc...)**

*Let $X$ be an infinite dimensional Banach space WCG (Weakly Compactly Generated; for example, separable or reflexive spaces) and $k \in \mathbb{N} \cup \{\infty\}$. T.F.A.E.:*

1. *$X$ has a $C^k$ smooth bump.*
2. *$X$ has "uniform approximations by $C^k$ smooth functions": For any pair of continuous functions $f : X \to F$ ($F$ any Banach space) and $\varepsilon : X \to (0, \infty)$ there is a $C^k$ smooth function $g : X \to F$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in X$.*
3. *There is a homeomorphic embedding $H : X \to c_0(\Gamma)$ for some set of indices $\Gamma$ such that each "coordinate function" $H_\gamma : X \to \mathbb{R}$ is $C^k$ smooth, where $\gamma \in \Gamma$.*
4. *$X$ has $C^k$-partitions of unity.*

- In a general Banach space (1)⇐(2)⇔(3)⇔(4).
- Not every Banach space with condition (2) is WCG.

# Differentiable functions with no critical points

**Question**

- Let us consider $k \in \mathbb{N} \cup \{\infty\}$ and a Banach space $X$ admitting "uniform approximations by $C^k$ smooth functions".

Can we uniformly approximate every continuous function $f : X \to F$ (being $F$ any non zero quotient of $E$) by $C^k$ smooth functions with no critical points?

- We say that $x \in X$ is a critical point of a differentiable function $g : X \to F$ if the bounded operator $g'(x) : X \to F$ is NOT surjective.

A positive answer to this problem provides "approximate versions of the Morse-Sard theorem" for certain infinite dimensional Banach spaces.

**Theorem (Morse-Sard Theorem, 1942)**

*If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a $C^r$ smooth function, with $r > \max\{n-m, 0\}$, and $\mathcal{C}_f$ is the set of critical points of $f$, then the set of critical values $f(\mathcal{C}_f)$ has Lebesgue measure zero in $\mathbb{R}^m$.*

- An infinite-dimensional version of the Morse-Sard's theorem:

**Theorem (Smale's Theorem, 1965)**

*If $X, Y$ are Banach spaces and $f : X \to Y$ a $C^r$ smooth Fredholm mapping (that is, $\dim(\ker f'(x)) < \infty$, $f'(x)(X)$ is closed and $\operatorname{codim}(f'(x)(X)) < \infty$ for every $x \in X$).*

*Then, $f(\mathcal{C}_f)$ is of first Baire category and, in particular $f(\mathcal{C}_f)$ has no interior points, provided that $r > \max\{\operatorname{index}(f'(x)), 0\}$ for all $x \in X$, where $\operatorname{index}(f'(x)) := \dim(\ker f'(x)) - \operatorname{codim}(f'(x)(X))$.*

- If $\dim X = \infty$ the above assumptions imply that $\dim Y = \infty$. Thus, Smale's theorem cannot be applied to functions $f : X \to \mathbb{R}$.

**Theorem (Kupka's Theorem, 1965)**

*(A counterexample to the Morse-Sard theorem on infinite dimensional spaces). There are $C^\infty$ smooth functions $f : \ell_2 \to \mathbb{R}$ such that their set of critical values $f(\mathcal{C}_f)$ contain intervals (and in particular, they have positive measure).*

**Example (Bates, Moreira 2001)**

There are polynomials (of degree three) $p : \ell_2 \to \mathbb{R}$ such that their set of critical values $p(\mathcal{C}_p) = [0, 1]$.

**Theorem (Eells, McAlpin, 1968)**

*For every pair of continuous functions $f : \ell_2 \to \mathbb{R}$ and $\varepsilon : \ell_2 \to (0, \infty)$, there is a $C^1$ smooth function $g : \ell_2 \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon(x), \text{ for all } x \in \ell_2,$$

*and the set of critical values $g(\mathcal{C}_g)$ is of (Lebesgue) measure zero.*

# Approximate Morse-Sard results in separable Banach spaces

**Theorem (Azagra, Cepedello, 2004)**

*For every pair of continuous functions $f : \ell_2 \to \mathbb{R}^n (n \in \mathbb{N})$ and $\varepsilon : \ell_2 \to (0, \infty)$ there exists a $C^\infty$ smooth function $g : X \to \mathbb{R}^n$ such that*

$$|f(x) - g(x)| < \varepsilon(x) \text{ for all } x \text{ and } g \text{ has no critical points.}$$

**Theorem (Hajek, 1998)**

*Any (Fréchet) differentiable function $f : c_0 \to \mathbb{R}$ with locally uniformly continuous derivative has locally compact derivative (thus $f'(X)$ is contained in a $K_\sigma$ subset).*

**Theorem (Hajek, Johanis, 2003)**

*Let $X$ be a separable Banach space with a $C^p$ smooth bump ($p \in \mathbb{N} \cup \{\infty\}$) and containing $c_0$.*

*Then, for every pair of continuous functions $f : X \to \mathbb{R}$, $\varepsilon : X \to (0, \infty)$ and any pre-fixed countable set $N \subset X^*$, there exists a $C^p$ smooth function $g : X \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon(x) \text{ for all } x \in X,$$

$$g'(X) \text{ is of first Baire category and } g'(X) \cap N = \emptyset.$$

**Definition**

A norm $\| \cdot \|$ in a Banach space $X$ is **LUR** (locally uniformly rotund) if $\|x_n - x\| \xrightarrow{n} 0$ whenever $\|\frac{x_n + x}{2}\| \xrightarrow{n} 1$ and $\|x_n\| = \|x\| = 1$ for all $n \in \mathbb{N}$.

- Recall that LUR $\Rightarrow$ **Rotund (or strictly convex)** (the unit sphere of the norm does not have segments).

## Theorem (2007, Azagra, J.S.)

*Let $X$ be a separable Banach space, $\dim X = \infty$, with a LUR and $C^p$ smooth norm $||\cdot||$ ($p \in \mathbb{N} \cup \{\infty\}$).*

*For every pair of continuous functions $f : X \to \mathbb{R}$ and $\varepsilon : X \to (0, \infty)$, there exists $g : X \to \mathbb{R}$ $C^p$ smooth such that*

$$|f(x) - g(x)| < \varepsilon(x) \text{ and } g \text{ has no critical points.}$$

- [If $p > 1$ the above conditions imply superreflexivity of $X$].

## Examples

1. $X$ Banach space with separable dual (i.e. $X$ separable Asplund space), $\dim X = \infty$ and $p = 1$.
2. $X = \ell_r, L_r[0,1]$ $(1 < r < \infty)$, where $p = \infty$ if $r$ even, $p = r - 1$ if $r$ is odd and $p = [r]$ is not an integer.
3. (Azagra, J.S. 2007; Moulis 1971) $X$ separable Banach space, $\dim X = \infty$, $X$ with a unconditional basis and a $C^p$ smooth and Lipschitz bump with $p > 1$.

## Some consequences/motivations

For a Banach space $X$ satisfying "the approximation property by $C^p$ smooth functions $g : X \to \mathbb{R}$ with no critical points" ($p \in \mathbb{N} \cup \{\infty\}$), we have:

1. A "non-linear $C^p$ smooth Hahn-Banach separation result": For any pair of disjoint closed subset $C_1$, $C_2 \subset X$ there is a $C^p$ smooth function $g : E \to \mathbb{R}$ with no critical points "separating $C_1$ and $C_2$": that is, $g^{-1}(0)$ is a 1-codimensional $C^p$ smooth submanifold in $E$ separating $C_1$ from $C_2$ (that is $C_1 \subset g^{-1}((0, \infty))$ and $C_2 \subset g^{-1}((-\infty, 0))$.

2. "$C^p$ smooth approximation of closed sets": For any closed subset $C \subset E$ and any open subset $W \subset E$ with $C \subset W$, there is an open subset $U$ such that $C \subset U \subset W$ and $U$ is $C^p$ smooth (that is, $\partial U$ is a 1-codimensional $C^p$ smooth submanifold of $E$).

# Approximate M-S results in separable spaces: vector-valued

**Theorem (Azagra, Dobrowolski, García-Bravo, 2019)**

*Let $X = c_0, \ell_r, L_r[0,1]$ $(1 < r < \infty)$, $p = \infty$ if $X = c_0$ or $r$ is even, $p = r - 1$ if $r$ is odd and $p = [r]$ if $r$ is not an integer and $F$ any (non zero) quotient of $X$. Then, for every pair of continuous functions $f : X \to F$ and $\varepsilon : X \to (0, \infty)$ there is a $C^p$ smooth function $g : X \to F$ such that*

$$\|f(x) - g(x)\| < \varepsilon(x) \text{ for all } x \in X \text{ and } g \text{ has no critical points.}$$

**Theorem (Azagra, Dobrowolski, García-Bravo, 2019)**

*Let $X$ be a separable reflexive such that $X \simeq X \oplus X$ and $F$ is any (non zero) quotient of $X$. Then, for every pair of continuous functions $f : X \to F$ and $\varepsilon : X \to (0, \infty)$ there is a $C^1$ smooth function $g : X \to F$ such that*

$$\|f(x) - g(x)\| < \varepsilon(x) \text{ for all } x \in X \text{ and } g \text{ has no critical points.}$$

- The condition $X \simeq X \oplus X$ is only required when $\dim F = \infty$.

## Theorem ( Azagra, Dobrowolski, García-Bravo, 2019.)

*Let $X$ be a separable space with*

1. *a $C^1$ smooth and LUR norm $\|\cdot\|$,*
2. *a 1-unconditional Schauder basis $\{e_j\}_{j\in\mathbb{N}}$ (in the sense that $\|\sum_{j\in H}\lambda_j e_j\| \leq \|\sum_{j\in H'}\lambda_j e_j\|$ whenever $H \subset H' \subset \mathbb{N}$ are finite subsets and any set $\{\lambda_j\}_{j\in H'} \subset \mathbb{R}$),*
3. *$F$ is any (non zero) quotient of $X$.*
4. *If $F$ is infinite-dimensional, there exists a subset $\mathbb{E}$ of $\mathbb{N}$ with $|\mathbb{E}| = |\mathbb{N}\setminus\mathbb{E}|$ and, for every infinite subset $J \subset \mathbb{E}$, $F$ is a quotient of $\overline{\text{span}}\{e_j : j \in J\}$.*

*Then, for every pair of continuous functions $f : X \to F$ and $\varepsilon : X \to (0,\infty)$ there is a $C^1$ smooth function $g : X \to F$ such that*

$$\|f(x) - g(x)\| < \varepsilon(x) \text{ for all } x \in X \text{ and } g \text{ has no critical points.}$$

### Examples

More technical results are obtained by Azagra, Dobrowolski and García-Bravo (2019) in separable Banach spaces. As a consequence, uniform approximation by $C^p$ smooth functions with no critical points is obtained for

1. $X$ separable Banach space containing $c_0$ with a shrinking Schauder basis, $p = 1$ and $F$ any quotient of $X$.
2. $X = J, J^*$, where $J$ is the James space, $p = 1$, and $F$ any quotient of $X$.
3. $X$ is a finite direct sum of the classical Banach spaces $c_0$, $\ell_r$ or $L_r[0,1]$, $(1 < r < \infty)$, $p$ depending on the minimum $r$ in the decomposition, $F$ is a quotient of $X$.
4. $X = C(K)$ separable with $K$ countable compact, $p = 1$ and $F$ any quotient of $X$.

# Approximate/extension results in separable spaces

**Definition**

- An infinite-dimensional Banach space is said to be polyhedral if the unit ball of any of its finite-dimensional subspaces is a polyhedron.

- **(Fabian, Fonf, Whitfield, Zizler)** Polyhedral spaces are know to be $c_0$-saturated and Asplund spaces.
- **(Hajek)** Every separable polyhedral space $X$ has an equivalent $C^\infty$ smooth and LFC-norm $\|\cdot\|$, that is, the norm locally depends on a finite number of functionals (except at 0), i.e. the norm is locally of the form

$$\|y\| = \varphi(f_1(y), \ldots, f_n(y))$$

for certain $C^\infty$ smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and functionals $\{f_1, \ldots, f_n\} \subset X^*$. More precisely, for every $x \in X \setminus \{0\}$ there is a neighborhood $V_x$ of $x$ such that $\|y\| = \varphi(f_1(y), \ldots, f_n(y))$ for all $y \in V_x$ for suitable function $\varphi$ and functionals $\{f_1, \ldots, f_n\}$ (determined locally).

# Approximate/extension results in separable spaces

**Theorem (García-Bravo, 2020)**

*Let $X$ be a separable isomorphically polyhedral Banach space, $\dim X = \infty$, with unconditional basis.*

*Then, for every $C^1$ smooth function $f : X \to \mathbb{R}^n$ and every continuous function $\varepsilon : X \to (0, \infty)$, every open set $U \subset X$ such that $\mathcal{C}_f \subset U$, where $\mathcal{C}_f$ is the set of critical points of $f$, there exists a $C^1$ smooth function $g : X \to \mathbb{R}^n$ such that*

- $\|f(x) - g(x)\| \leq \varepsilon(x)$ *for all* $x \in X$;
- $f(x) = g(x)$ *for all* $x \in X \setminus U$,
- $\|f'(x) - g'(x)\| \leq \varepsilon(x)$ *for all* $x \in X$, *and*
- $g$ *has no critical points.*

## Theorem (García-Bravo, 2020)

*Let $X$ be a separable Banach space with a $C^1$ smooth and strictly convex equivalent norm $\|\cdot\|$ and 1-unconditional basis $\{e_j\}_{j\in\mathbb{N}}$ ($\|\sum_{j\in H}\lambda_j e_j\| \le \|\sum_{j\in H'}\lambda_j e_j\|$ whenever $H \subset H' \subset \mathbb{N}$ are finite subsets and any set $\{\lambda_j\}_{j\in H'} \subset \mathbb{R}$).*

*Then, for every $C^1$ smooth function $f : X \to \mathbb{R}^n$ ($n \in \mathbb{N}$), every continuous function $\varepsilon : X \to (0,\infty)$, every open set $U \subset X$ such that $\mathcal{C}_f \subset U$, where $\mathcal{C}_f$ is the set of critical points of $f$, there exists a $C^1$ smooth function $g : X \to \mathbb{R}^n$ such that*

- $\|f(x) - g(x)\| \le \varepsilon(x)$ *for all* $x \in X$;
- $f(x) = g(x)$ *for all* $x \in X \setminus U$, *and*
- $g$ *has no critical points.*

## Examples

- $c_0$, $d_*(\omega, 1)$ (preduals of Lorentz spaces).
- $\ell_p$ $(1 < p < \infty)$.

# Approximate Morse-Sard results for non-separable Banach (general target space)

**Theorem (Azagra, García-Bravo, J.S., 2024)**

*Consider $E = c_0(\Gamma), \ell_p(\Gamma)$ $(1 < p < \infty)$, where $\Gamma$ is an infinite set and $F$ a (non zero) quotient of $E$.*

*Then, for every pair of continuous functions $f : E \to F$ and $\varepsilon : E \to (0, \infty)$ there is a $C^k$ smooth $g : E \to F$ such that*

$$\|f(x) - g(x)\| \leq \varepsilon(x) \text{ and } g \text{ has no critical points}$$

*where $k = \infty$ if $E = c_0(\Gamma)$ or $p$ is even, $k = p - 1$ if $p$ is odd and $k = [p]$ if $p$ is not integer.*

**Definition**

A Banach space $Y$ has a decomposition of the form $Y = \bigoplus_{n\in\mathbb{N}} Y_n$ if

1. $Y_n$ is a closed subspace of $Y$, for every $n$,
2. $Y_n \cap Y_m = \{0\}$ for $n \neq m$,
3. every $y \in Y$ can be written in a unique way $y = \sum_{n=1}^{\infty} y_n$ with $y_n \in Y_n$ for all $n$,
4. the canonical projections $P_n : Y \to Y_n$, $P_n(y) = y_n$ are continuous for all $n$. Thus, $Y_n$ is complemented in $Y$ for all $n$.

**Definition**

A Banach space $X$ has $C^k$-partitions of unity ($k \in \mathbb{N} \cup \{\infty\}$) if for every open cover $\{U_\alpha\}_{\beta\in\Omega}$ of $X$, there is a family of $C^k$ smooth functions $\{\psi_i\}_{i\in\Delta}$, $\psi_i : X \to [0,\infty)$ for all $i$, such that

- $\{\psi_i\}_{i\in\Delta}$ is locally finite: for every $x \in X$ there is a neighbourhood $V_x$ of $x$ such that $V_x$ intersects $\operatorname{supp}\psi_i$ only for finitely many $i \in \Delta$,
- $\{\psi_i\}_{i\in\Delta}$ is subordinated to $\{U_\alpha\}_{\alpha\in\Omega}$: for every $i$ there is $\alpha$ such that $\operatorname{supp}\psi_i \subset U_\alpha$,
- $\sum_{i\in\Delta}\psi_i(x) = 1$ for all $x \in X$.

- A Banach space $X$ has $C^k$-partitions of unity in a Banach space $X \Leftrightarrow$ for every pair of continuous function $f : X \to Z$ ($Z$ any Banach space) and $\varepsilon : X \to (0,\infty)$ there is a $C^k$ smooth function $g : X \to Z$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in X$.

### Theorem (Azagra, García-Bravo, J.S., 2024)

*Let $X, Y, F$ be Banach spaces, $E = X \oplus Y$, and $k \in \mathbb{N} \cup \{\infty\}$ such that*

1. *$X$ has $C^k$-partitions of unity,*
2. $\dim Y = \infty$ *and has a $C^k$ smooth and LUR norm.*
3. *$Y$ is reflexive, $Y = \oplus_{n\in\mathbb{N}} Y_n$,*
4. *$F$ is a (non zero) quotient of $Y_n$ for every $n$.*
5. *The canonical projection $Q : Y \to \oplus_{i\ odd} Y_i$, given by $Q(y) = \sum_{i\ odd} y_i$ for every $y = \sum_{i\in\mathbb{N}} y_i \in Y$ (with $y_i \in Y_i$ for all $i$) is well defined and continuous.*

*Then, for every pair of continuous functions $f : E \to F$ and $\varepsilon : E \to (0, \infty)$ there is $g : E \to F$ $C^k$ smooth such that*

$$\|f(z) - g(z)\| < \varepsilon(z) \text{ for all } z \in E \text{ and } g \text{ has no critical points.}$$

- [Recall that for $k > 1$, condition (2) $\Rightarrow$ $Y$ is superreflexive].

### Definition

Let $Y$ be a Banach space. A biorthogonal system $\{e_i, e_i^*\}_{i\in\Gamma} \subset Y \times Y^*$ is a M-basis (Markushevich basis) if $\overline{\text{span}}\{e_i : i \in \Gamma\} = Y$. It is called shrinking provided $Y^* = \overline{\text{span}}\{e_i^* :\in \Gamma\}$.

## Theorem (Azagra, García-Bravo, J.S., 2024)

*Let $X, Y, F$ be Banach spaces, $E = X \oplus Y$, such that*

1. $X$ *has* $C^1$*-partitions of unity,*
2. $\dim Y = \infty$ *and* $Y = \oplus_{n\in\mathbb{N}} Y_n$,
3. *each* $Y_n$ *has a shrinking M-basis and the union of all these M-bases is a shrinking M-basis in* $Y$,
4. $F$ *is a (non zero) quotient of* $Y_n$ *for every* $n$.
5. $Y$ *has a* $C^1$ *smooth and LUR norm satisfying that the canonical projections* $Q_m : Y \to (\oplus_{i=1}^m Y_i) \oplus (\oplus_{\substack{i \text{ odd} \\ i>m}} Y_i)$, *given by* $Q(y) = \sum_{i=1}^m y_i + \sum_{\substack{i \text{ odd} \\ i>m}} y_i$ *for every* $y = \sum_i y_i \in Y$ $(y_i \in Y_i$ *for all* $i)$ *are well defined and have norm one for all* $m$.

*Then, for every pair of continuous functions* $f : E \to F$ *and* $\varepsilon : E \to (0,\infty)$ *there is* $g : E \to F$ $C^1$ *smooth such that*

$$\|f(z) - g(z)\| < \varepsilon(z) \text{ for all } z \in E \text{ and } g \text{ has no critical points.}$$

## Example

$E = X \oplus Y$ satisfies the "approximation property by $C^k$ smooth functions with no critical points" ($k \in \mathbb{N} \cup \{\infty\}$) for

1. $E = X \oplus c_0(\Gamma)$ ($\Gamma$ infinite) and $F$ any (non zero) quotient of $c_0(\Gamma)$, where $X$ has $C^k$ smooth partitions of unity.
2. $E = X \oplus \ell_p(\Gamma)$ ($\Gamma$ infinite, $1 < p < \infty$) and $F$ any (non zero) quotient of $\ell_p(\Gamma)$, where $X$ has $C^k$ smooth partitions of unity.
3. $E = X \oplus (\oplus_p Z)$ ($1 < p < \infty$) and $F$ any (non zero) quotient of $Z$, where $Y := \oplus_p Z$ is a reflexive space with $C^k$ smooth LUR (equivalent) norm with the required continuity on the mentioned canonical projection and $X$ has $C^k$ smooth partitions of unity.

In (2) and (3) $k \leq p - 1$ if $p$ is odd and $k \leq [p]$ if $p$ is not an integer.

## Example

$E = X \oplus Y$ satisfies the "approximation property by $C^1$ smooth functions with no critical points" for

4. $E = X \oplus (\oplus_p Z)$ $(1 < p < \infty)$, where $Y := \oplus_p Z$ and $F$ any (non zero) quotient of $Z$, the Banach space $Z$ has a shrinking M-basis (equivalently, $Z$ is a WCG Asplund Banach space) with a $C^1$ smooth and LUR norm with the required conditions on the norm one projections, and $X$ has $C^1$ smooth partitions of unity.

# Approximate Morse-Sard results for non-separable Banach (finite dimensional target space)

**Theorem (Azagra, García-Bravo, J.S.)**

*Let $X, Y$ be Banach spaces, $E = X \oplus Y$ and $k \in \mathbb{N} \cup \{\infty\}$, such that*

1. *$X$ has $C^k$ smooth partitions of unity,*
2. *$\dim Y = \infty$, $Y$ separable, and $Y$ has a $C^k$ smooth LUR norm,*

*Then, for every pair of continuous functions $f : E \to \mathbb{R}^n$ and $\varepsilon : E \to (0, \infty)$ there is $C^k$ smooth $g : E \to \mathbb{R}^n$ such that*

$$\|f(z) - g(z)\| < \varepsilon(z) \text{ for all } z \in E \text{ and } g \text{ has no critical points.}$$

- [Recall that for $k > 1$, condition (2) $\Rightarrow$ $Y$ is superreflexive].

## Example

$E$ satisfies the "approximation property by $C^1$ smooth functions with no critical points" (for a finite dimensional target space) if:

- $E = X \oplus Y$, where
  - $X$ has $C^1$ smooth partitions of unity,
  - $Y$ Banach space with separable dual and $\dim Y = \infty$.

- $E = \widetilde{X} \oplus \widetilde{Y}$, where
  - $\widetilde{X}$ has $C^1$ smooth partitions of unity,
  - $\widetilde{Y}$ Asplund and WCG Banach space and $\dim \widetilde{Y} = \infty$. So, in particular, for $\widetilde{Y}$ reflexive space, $\dim \widetilde{Y} = \infty$.

# Tools for the case of $c_0(\Gamma)$

The proof for functions $f : E := c_0(\Gamma) \to F$, being $F$ a quotient of $c_0(\Gamma)$ relays on: **(Toruńczyk)** the existence of $C^\infty$ **partitions of unity** $\{\psi_i\}_{i\in\Delta}$ in $c_0(\Gamma)$, where each mapping $\psi_i$ is **locally** of the form

$$\psi_i(y) = \varphi(e^*_{i_1}(y), \dots, e^*_{i_n}(y))$$

$(y \in E)$ for a suitable $C^\infty$ smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and a finite number of functionals $\{e^*_{i_1}, \dots, e^*_{i_n}\}$ (locally determined), where $\{e^*_i\}_{i\in\Gamma}$ are the functionals associated with the canonical basis of $c_0(\Gamma)$. The approximating functions with no critical points are of the form

$$g(x) = \sum_{i\in\Delta}(f(x_i) + T(x - x_i))\psi_i(x), \text{ where}$$

- $x_i \in \operatorname{supp}\psi_i$, $T = \sum_{n\in\mathbb{N}} T_n \circ P_n$ bounded operator,
- $T_n : c_0(\Gamma_n) \to F$ surjective bounded operator,
- $P_n : c_0(\Gamma) \to c_0(\Gamma_n)$ are the canonical projections,
- $\Gamma = \cup_{n\in\mathbb{N}}\Gamma_n$ (disjoint union) with $|\Gamma| = |\Gamma_n|$ for all $n \in \mathbb{N}$.

# Tools for the case of $\ell_p(\Gamma)$

The proof for functions $f : E = l_p(\Gamma) \to F$ $\mathbf{(1 < p < \infty)}$, being $F$ a quotient of $E$ relays on: **(Toruńczyk)** the existence of $\boldsymbol{C^k}$**-partitions of unity** $\boldsymbol{\{\psi_i\}_{i\in\Delta}}$ in $\ell_p(\Gamma)$, where each function $\psi_i$ is **locally** of the form

$$\boldsymbol{\psi_i(y) = \varphi(\|y\|_p^p, e_{i_1}^*(y), \dots, e_{i_n}^*(y))}$$

$(y \in E)$ for a suitable $C^k$ smooth function $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ and a finite number of functionals $\{e_{i_1}^*, \dots, e_{i_n}^*\}$ (locally determined), being $\{e_i^*\}_{i\in\Gamma}$ the functionals associated with the canonical basis of $\ell_p(\Gamma)$. The approximating functions with no critical points are of the form

$$\boldsymbol{g = h \circ d}, \text{ where}$$

- $h : E \to F$, $h(x) = \sum_{i\in\Delta}(f(x_i) + T(x - x_i))\psi_i(x)$,
- $x_i \in \operatorname{supp}\psi_i$ , $T = \sum_{m \text{ odd}} T_m \circ P_m$ is a bounded operator,
- $T_m : \ell_p(\Gamma_m) \to F$ surjective bounded operator,
- $P_m : \ell_p(\Gamma) \to \ell_p(\Gamma_m)$ are the canonical projections,
- $\Gamma = \cup_{n\in\mathbb{N}}\Gamma_n$ (disjoint union) with $|\Gamma| = |\Gamma_n|$,

# Tools for the case of $\ell_p(\Gamma)$

- **(Azagra, Dobrowolski, García-Bravo)** $d : E \to E \setminus \mathcal{C}_h$ is a $\boldsymbol{C^k}$ **"deleting (or extracting) diffeomorphism"** (close to the identity), being $\mathcal{C}_h$ the closed subset of critical points of $h$.
- So, by the chain rule $g'(x) = h'(d(x)) \circ d'(x)$ is a surjective operator for all $x \in E$.
- It is crucial that $\mathcal{C}_h$ is locally contained in $\ell_p((\cup_{i=1}^m \Gamma_i) \cup (\cup_{i \text{ odd}} \Gamma_i))$ for a suitable $m$ (locally determined), that is, $\boldsymbol{\mathcal{C}_h}$ **is locally contained in a infinite codimensional and complemented closed subspace**.
- Recall that $k = \infty$ if $p$ is even, $k = p - 1$ if $p$ is odd and $k = [p]$ if $p$ is not an integer.

# Tools for the general cases

For continuous functions $f : E = X \oplus Y \to F$, with $X, Y, F$ under any of the assumptions given above, we need:

- **(Toruńczyk) $C^k$-partitions of unity $\{\psi_i\}_{i\in\Delta}$** ($k$ depending on $X$ and $Y$), where each $\psi_i$ is locally of the form,

$$\psi_i(x,y) = \varphi(x, \theta_{k_1}(\|y\|), \ldots, \theta_{k_m}(\|y\|), e^*_{i_1}(y), \ldots, e^*_{i_n}(y))$$

(here $(x,y) \in X \oplus Y$) for a suitable $C^k$ smooth function $\varphi : X \oplus \mathbb{R}^{n+m} \to \mathbb{R}$, a finite number of functionals $\{e^*_{i_1}, \cdots, e^*_{i_n}\}$ and functions $\{\theta_{k_1}, \cdots, \theta_{k_m}\}$, being

★ $\{e^*_i\}_{i\in\Gamma} \subset Y^*$ funcionals associated with a (suitable) shrinking M-basis in $Y$,

★ $\{\theta_j\}_{j\in\mathbb{N}}$ $C^\infty$ smooth non-decreasing functions $\theta_j : \mathbb{R} \to [0,\infty)$ with $\theta_j(t) = t$ if $t > \frac{1}{j}$, $\theta_j(t) = 0$ if $t < \frac{1}{2j}$ and $|\theta'_j(t)| \leq 3$ for all $t \in \mathbb{R}$ and all $j \in \mathbb{N}$.

# Tools for the general cases

The approximating functions with no critical points are of the form

$$\boldsymbol{g} = \boldsymbol{h} \circ \boldsymbol{d}, \text{ where}$$

- $h : E = X \oplus Y \to F$,

$$h(x,y) = \sum_{i \in \Delta} (f(x_i, y_i) + T(y - y_i))\psi_i(x,y),$$

- $(x_i, y_i) \in \operatorname{supp} \psi_i$,
- $T : Y \to F$ is a suitable surjective bounded operator (several technical conditions are required, depending on $F$, finite or infinite dimensional and the decomposition of $Y = \bigoplus_{n \in \mathbb{N}} Y_n$),
- **(Azagra, Dobrowolski, García-Bravo) $C^k$ "Deleting diffeomorphism"** $d : E \to E \setminus \mathcal{C}_h$, where $\mathcal{C}_h$ is the closed subset of critical points of $h$. Here, it is crucial that **$\mathcal{C}_h$ is locally contained in the graph of a continuous function** $c : M \to N$, being $M, N$ closed subspaces and $E = M \oplus N$, $M$ with $C^k$-partitions of unity, $N$ with a $C^k$ smooth norm and $\dim N = \infty$ (locally determined).

# Tools for the general cases (finite dimensional target space)

- For $f : E = X \oplus Y \to \mathbb{R}^n$, $Y$ non-reflexive we need a renorming result:

**Proposition (Azagra, García-Bravo, J.S.)**

*Let $(Y, \|\cdot\|)$ be a Banach space and let $W$ be a $K_\sigma$ subset in the unit sphere of $Y^*$. Then, the set of (equivalent) norms $|||\cdot|||$ on $Y$ such that their dual norms $|||\cdot|||^*$ are Fréchet differentiable at the points of $W$ is residual in $(\mathcal{N}_Y, \rho)$, the metric space of all equivalent norms on $Y$ with the usual metric*

$$\rho(p, q) = \sup\{|p(x) - q(x)| : \ ||x|| = 1\}, \ p, q \in \mathcal{N}_Y.$$

*In particular, for any of these norms $|||\cdot|||$, every functional $f \in W$ attains its $|||\cdot|||^*$-norm.*

**Corollary (Fabian, Zajicek, Zizler 1982; Azagra, García-Bravo, J.S.)**

*Let $Y$ be a Banach space with a LUR norm $\|\cdot\|$ whose dual norm $\|\cdot\|^*$ is LUR and let $W$ be a $K_\sigma$ subset in the unit sphere of $Y^*$.*

*Then, the set of (equivalent) norms $|||\cdot|||$ on $Y$ such that $|||\cdot|||$ and $|||\cdot|||^*$ are LUR and $|||\cdot|||^*$ is Fréchet differentiable at the points of $W$ is residual in $(\mathcal{N}_Y, \rho)$. In particular, for any of these norms $|||\cdot|||$, every functional $f \in W$ attains its $|||\cdot|||^*$-norm.*

# Some references


- D. Azagra, T. Dobrowolski and M. García-Bravo, *Smooth approximations without critical points of continuous mappings between Banach spaces and diffeomorphis stractions of sets*, Adv. Math. 354 (2019).
- D. Azagra, M. García-Bravo, M. Jiménez-Sevilla, *Approximate Morse-Sard type results for non-separable Banach spaces*, J. Funct. Anal. (2024).
- D. Azagra, M. Jiménez-Sevilla, *Approximation by smooth functions with no critical points on separable infinite-dimensional Banach spaces*, J. Funct. Anal. (2007).
- M. García-Bravo, *Extraction of critical points of smooth functions on Banach spaces*, J. Math. Anal. Appl. 482, (2020).
- P. Hájek and M. Johanis, *Smooth approximations without critical points*, Cent. Eur. J. Math. 1 (2003), no. 3.
- H. Toruńczyk, *Smooth partitions of unity on some non-separable Banach spaces*, Studia Math. 46 (1973).

# Thank you