Bounds for L^∞ Extremal Polynomials

Maxim Zinchenko

Department of Mathematics and Statistics
University of New Mexico

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Chebyshev Polynomials

Let $\mathsf{K} \subset \mathbb{C}$ be a compact set consisting of infinitely many points and denote the supremum norm on K by $\|f\|_{\mathsf{K}} = \sup_{z\in\mathsf{K}} |f(z)|$.

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If $\mathsf{K} = [-1, 1]$, then $T_n(x) = \frac{1}{2^{n-1}} \cos(n \arccos x)$.

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Fekete [1923], Szegő [1924]

For any compact set K $\subset \mathbb{C}$,

$$\lim_{n\to\infty} \|T_n\|_{\mathsf{K}}^{1/n} = \mathrm{Cap}(\mathsf{K}).$$

Capacity, Equilibrium Measure, Green's Function

The Robin constant of a compact set $\mathsf{K} \subset \mathbb{C}$ is defined by

$$\mathcal{R}(\mathsf{K}) = \inf_{\mathrm{supp}(\mu) \subset \mathsf{K},\, \mu(\mathsf{K})=1} \iint \log \frac{1}{|x-y|}\, d\mu(x) d\mu(y)$$

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If K is nonpolar, the outer domain $\Omega =$ unbounded component of $\overline{\mathbb{C}} \setminus \mathsf{K}$ supports the Green's function - a unique positive harmonic function with a logarithmic pole at infinity and zero boundary values q.e. on $\partial\Omega$.

$$G_{\mathsf{K}}(z) = \log \frac{|z|}{\operatorname{Cap}(\mathsf{K})} \big(1 + o(1)\big) \quad \text{as} \quad z \to \infty.$$

The set K is called regular if $G_{\mathsf{K}}(z) = 0$ everywhere on $\partial\Omega$.

Lower Bounds

To simplify notation, we introduce Widom factors: $W_n(\mathsf{K}) = \dfrac{\|T_n\|_{\mathsf{K}}}{\mathrm{Cap}(\mathsf{K})^n}$.

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Szegő's root asymptotics $W_n(\mathsf{K})^{1/n} \to 1$ allows for sub-exponential growth/decay of $W_n(\mathsf{K})$. But it turns out that:

Szegő [1924]

For any nonpolar compact set $\mathsf{K} \subset \mathbb{C}$,

$$\inf_n W_n(\mathsf{K}) \geq 1.$$

There are sets for which the Szegő inequality is optimal (e.g., smooth closed Jordan curves), but for $\mathsf{K} \subset \mathbb{R}$ the sharp lower bound is larger:

Schiefermayr [2008]

For any nonpolar compact set $\mathsf{K} \subset \mathbb{R}$,

$$\inf_n W_n(\mathsf{K}) \geq 2.$$

Upper Bounds

However, Widom factors can grow sub-exponentially:

Goncharov–Hatinoglu [2015]

For any sequence $D_n \geq 1$ of sub-exponential growth (i.e. $\frac{1}{n} \log D_n \to 0$) there exists a compact set $\mathsf{K} \subset \mathbb{R}$ (a zero measure Cantor-type set) s.t.

$$W_n(\mathsf{K}) \geq D_n, \quad n \geq 1.$$

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Nevertheless, for finitely connected sets we have:

Widom [1969], Totik–Varga [2015], Andrievskii [2016]

If $\mathsf{K} \subset \mathbb{C}$ is a finite disjoint union of quasiconformal arcs/curves, then

$$\sup_n W_n(\mathsf{K}) < \infty.$$

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Open Problem: Is $W_n(\mathsf{K})$ bounded for any connected compact set K?

Parreau–Widom and Homogeneous Sets

A compact set $\mathsf{K} \subset \mathbb{C}$ is called Parreau–Widom if

$$PW(\mathsf{K}) = \sum_{\{c_j \,:\, \nabla G_{\mathsf{K}}(c_j)=0\}} G_{\mathsf{K}}(c_j) < \infty.$$

Parreau–Widom sets $\mathsf{K} \subset \mathbb{R}$ are necessarily of positive Lebesgue measure.

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A compact set $\mathsf{K} \subset \mathbb{R}$ is called homogeneous if there is a uniform lower bound on its Lebesgue density:

$$\exists \delta > 0 \;\text{ s.t. }\; |\mathsf{K} \cap (x-\varepsilon, x+\varepsilon)| > \delta\varepsilon \;\; \forall x \in \mathsf{K},\; 0 < \varepsilon < 1.$$

Every homogeneous set is Parreau–Widom and regular.

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A canonical example of a homogeneous set is a positive measure middle $\{\varepsilon_n\}_{n=1}^{\infty}$ Cantor set, that is, $[0,1]$ with the middle ε_n-th portion removed at step n for a summable sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,1)$.

Upper Bound for Parreau–Widom Sets

Christiansen–Simon–Z [2017]

If $\mathsf{K} \subset \mathbb{R}$ is regular and Parreau–Widom, then

$$\sup_n W_n(\mathsf{K}) \le 2 \exp[PW(\mathsf{K})]$$

and the upper bound is (asymptotically) attained for a generic set.

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A compact set $\mathsf{K} \subset \mathbb{C}$ is said to have rationally independent harmonic measures if for every decomposition $\mathsf{K} = \mathsf{K}_0 \cup \cdots \cup \mathsf{K}_\ell$ into closed disjoint sets, harmonic measures $\rho_{\mathsf{K}}(\mathsf{K}_1), \ldots, \rho_{\mathsf{K}}(\mathsf{K}_\ell)$ are rationally independent.

Christiansen–Simon–Yuditskii–Z [2019]

If $W_n(\mathsf{K})$ is bounded for a compact, regular set $\mathsf{K} \subset \mathbb{C}$ with rationally independent harmonic measures, then K is Parreau–Widom.

Open Problem: Is there a non-PW set K with bounded $W_n(\mathsf{K})$?

Weighted Chebyshev Polynomials

Let $w : \mathsf{K} \to [0, \infty)$ be a bounded weight function which is positive at infinitely many points of K. Then the weighted Chebyshev polynomials $T_{n,w}$ are the unique monic polynomials that minimizes $\|wT_{n,w}\|_{\mathsf{K}}$.

There is a root asymptotics, $\|wT_{n,w}\|_{\mathsf{K}}^{1/n} \to \mathrm{Cap}(\mathsf{K})$ if $w > 0$ ρ_{K}-a.e.

To simplify notation, we introduce the weighted Widom factors and the (exponential of) Szegő/entropy integral:

$$W_n(\mathsf{K}, w) = \frac{\|wT_{n,w}\|_{\mathsf{K}}}{\mathrm{Cap}(\mathsf{K})^n}, \quad S(\mathsf{K}, w) = \exp\left[\int \log w(z)\, d\rho_{\mathsf{K}}(z)\right].$$

Weighted Lower Bounds

Novello–Schiefermayr–Z [2021]

For any nonpolar compact set $\mathsf{K} \subset \mathbb{C}$,

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and the inequality is sharp even for real sets K.

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Alpan–Z [2024x], Christiansen–Simon–Z [2025x]

Let $\mathsf{K} \subset \mathbb{R}$ be a nonpolar compact set. Then

- If $w(x) = \left|\prod_j \frac{x-\alpha_j}{x-\beta_j}\right|$ with $\alpha_j \in \mathsf{K}$, $\beta_j \notin \mathsf{K}$, $\sum_j G_{\mathsf{K}}(\beta_j) < \infty$, then

$$\inf_n W_n(\mathsf{K}, w) \geq 2S(\mathsf{K}, w)$$

- If w is continuous with at most finitely many algebraic zeros, then

$$\liminf_{n\to\infty} W_n(\mathsf{K}, w) \geq 2S(\mathsf{K}, w)$$

Weighted Upper Bounds

Widom [1969], Christiansen–Simon–Z [2025x]

If $\mathsf{K} \subset \mathbb{C}$ is a finite disjoint union of C^{2+} closed curves and w is USC, then

$$\limsup_{n\to\infty} W_n(\mathsf{K}, w) \leq S(\mathsf{K}, w) \exp[PW(\mathsf{K})]$$

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If $\mathsf{K} \subset \mathbb{C}$ is a finite disjoint union of C^{2+} arcs or $\mathsf{K} \subset \mathbb{R}$ is regular and Parreau–Widom and w is USC, then

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Corollary: A Szegő-type Theorem in L^∞

If $\mathsf{K} \subset \mathbb{R}$ is regular and Parreau–Widom or $\mathsf{K} \subset \mathbb{C}$ is a finite disjoint union of C^{2+} arcs/curves and w is USC, then

$$\inf_n W_n(\mathsf{K}, w) > 0 \quad \iff \quad \int \log w(z) d\rho_{\mathsf{K}}(z) > -\infty$$

and if either one holds then also $\sup_n W_n(\mathsf{K}, w) < \infty$

Some Asymptotics

Bucheker-Eichinger-Z [2025x]

If K is a C^{1+} Jordan region and w is USC on K, then

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Lubinsky-Saff [1987], Alpan-Z [2024x]

If K is an interval and either w is USC with $1/w \in L^p$ for all $p < \infty$ or w is continuous with at most finitely many zeros, then

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Thiran–Detaille [1991], Christiansen-Eichinger-Rubin-Z

If K is an arc on the unit circle of angular opening $\alpha \in (0, 2\pi)$ and w is continuous with at most finitely many algebraic zeros, then

$$\lim_{n\to\infty} W_n(\mathsf{K}, w) = \big[1 + \cos(\alpha/4)\big] S(\mathsf{K}, w)$$

Finally. . .

Thank you for your attention!