A Mazurkiewicz Set as the union of two Sierpiński-Zygmund functions

47th Summer Symposium in Real Analysis

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Outlines

- Cantor sets and their algebraic differences... Continued

1. Introduction
   - Sierpiński-Zygmund Function
   - Mazurkiewicz Set
2. A Mazurkiewicz set containing
   - No Sierpiński-Zygmund function
   - One Sierpiński-Zygmund function
   - Two Sierpiński-Zygmund functions

[Kha24a] A. Kharazishvili, *A Mazurkiewicz set containing the graph of a Sierpiński-Zygmund function*, Georgian Math. J., posted on 2024, DOI 10.1515/gmj-2024-2023.

[Pana] C.-H. Pan, *Mazurkiewicz Sets and Containment of Sierpiński-Zygmund Functions under Rotations*. arXiv:2504.12603.

Cantor sets and their algebraic differences... Continued

Nowakowski Theorem(s)

- Under some condition, $\mathcal{C}(\mathbf{a}) - \mathcal{C}(\mathbf{a})$ is a Cantorval.
- Under some other condition, $\mathcal{C}(\mathbf{a}) - \mathcal{C}(\mathbf{a})$ is a Cantorval.
- Classification of alg difference of special affine Cantor sets.

Cantor sets and their algebraic differences... Continued

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Nowakowski-Pan Theorem (Submitted)

If $\mathcal{C} \subseteq [0,1]$ is a Cantor set that contains 0 and 1, then

$$\sup m(\mathcal{C}^c - \mathcal{C}) = 2, \qquad \inf m(\mathcal{C}^c - \mathcal{C}) = \frac{3}{2}.$$

[NPb] P. Nowakowski and C.-H. Pan, *The algebraic difference of a Cantor set and its complement*. arXiv.2505.03170.

Sierpiński-Zygmund Function

Definition

$f : \mathbb{R} \to \mathbb{R}$ is a **Sierpiński-Zygmund** function provided

- $f \restriction_S$ is **NOT continuous** whenever $|S| = \mathfrak{c}$.

f intersects every continuous partial function in $< \mathfrak{c}$-many points.

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Let $\mathcal{G} := \{g \in \mathbb{R}^G : G \subseteq \mathbb{R}$ is G_δ and g is continiuous$\}$.

[f is Sierpiński-Zygmund] iff [$|f \cap g| < \mathfrak{c}$ for every $g \in \mathcal{G}$]

Theorem (Kuratowski)

If $g : S \to \mathbb{R}$ is **continuous** *for some $S \subseteq \mathbb{R}$, then g has a* **continuous extension** *$\bar{g} : G \to \mathbb{R}$ for some G_δ set $S \subseteq G \subseteq \bar{S}$.*

Sierpiński-Zygmund Function

Question (Darji)

Can a Sierpiński-Zygmund function have **Darboux property**?

[Dar93] U. B. Darji, *A Sierpiński-Zygmund function which has a perfect road at each point*, Colloq. Math. **64** (1993), no. 2, 159–162, DOI 10.4064/cm-64-2-159-162. MR1218479

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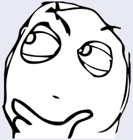
Answer (Balcerzak, Ciesielski, Natkaniec)

The answer $\mathrm{SZ} \cap \mathrm{D} \neq \emptyset$ or $\mathrm{SZ} \cap \mathrm{D} = \emptyset$ is **independent** of ZFC.

[BCN97] M. Balcerzak, K. Ciesielski, and T. Natkaniec, *Sierpiński-Zygmund functions that are Darboux, almost continuous, or have a perfect road*, Arch. Math. Logic **37** (1997), no. 1, 29–35, DOI 10.1007/s001530050080. MR1485861

Mazurkiewicz Set

Stefan Mazurkiewicz — Student of Sierpiński



There is a **plane subset** that...

intersects **every straight line** in **exactly two points**.

Mazurkiewicz Set

Definition

A set $\mathfrak{M} \subseteq \mathbb{R}^2$ is called a **Mazurkiewicz set**, or a **two-point set** provided that it intersects *every straight line* of $\mathbb{R}^2$ in *two points*.

Notation

- $\mathbb{L}$ denotes the collection of all straight lines of $\mathbb{R}^2$.
- $\mathcal{L}(A) \coloneqq \{\ell \in \mathbb{L} \colon |\ell \cap A| = 2\}$. $\quad$ ($|A| < \mathfrak{c}$ implies $|\mathcal{L}(A)| < \mathfrak{c}$)

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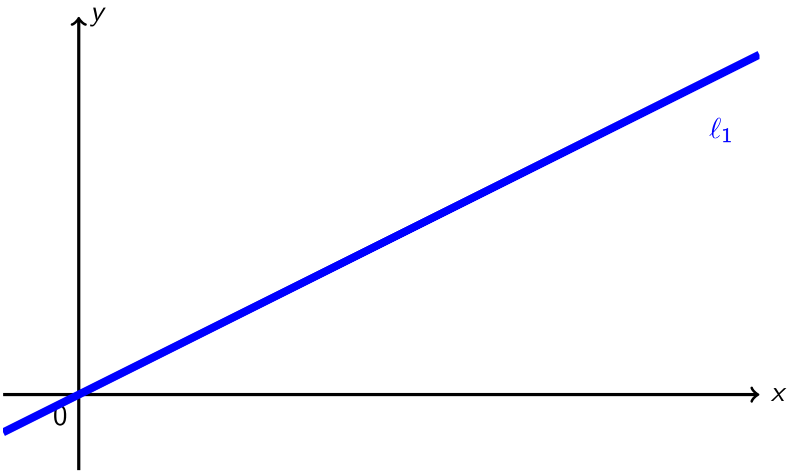
Construction

Let $\{\ell_\xi\}_{\xi<\mathfrak{c}}$ be an enumeration of $\mathbb{L}$. For every $\xi < \mathfrak{c}$, choose

$$E_\xi \subseteq \ell_\xi \setminus \bigcup \mathcal{L}(\bigcup_{\zeta<\xi} E_\zeta) \text{ such that } |\ell_\xi \cap \bigcup_{\zeta\leq\xi} E_\xi| = 2.$$

$\mathfrak{M} := \bigcup_{\xi<\mathfrak{c}} E_\xi$ is as needed.

Mazurkiewicz Set



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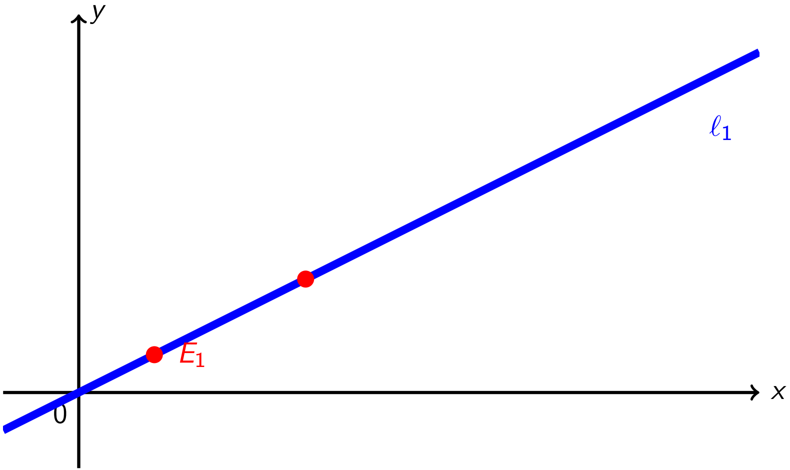

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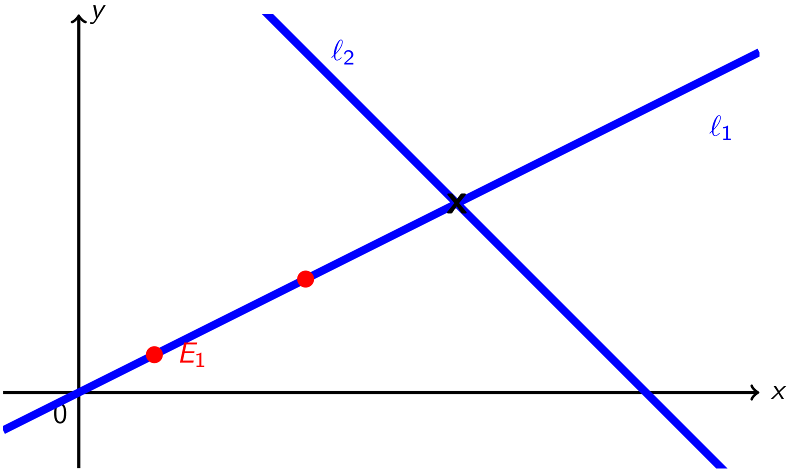

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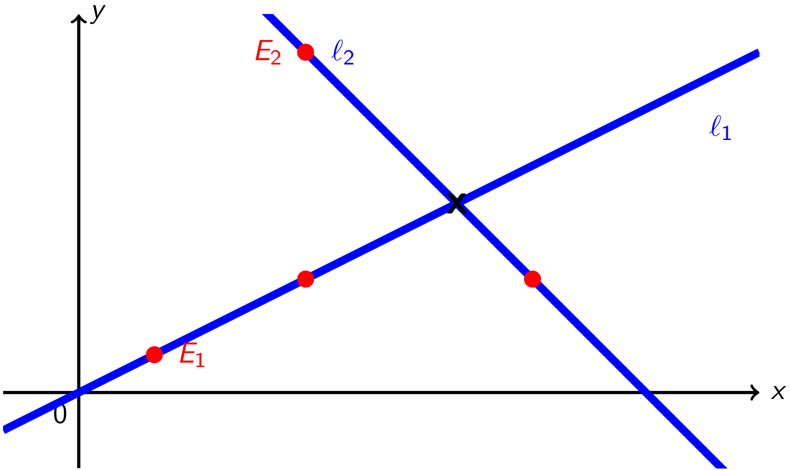

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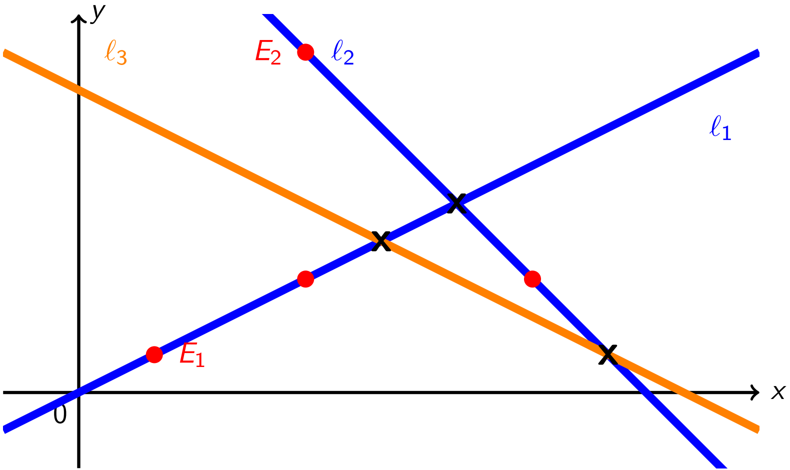

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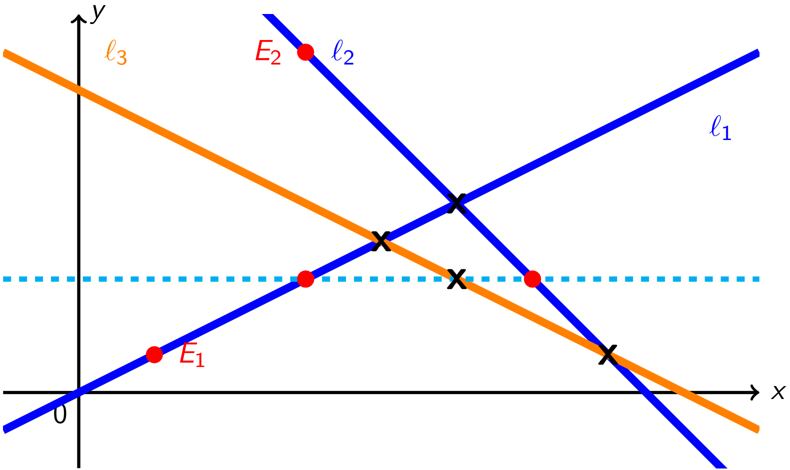

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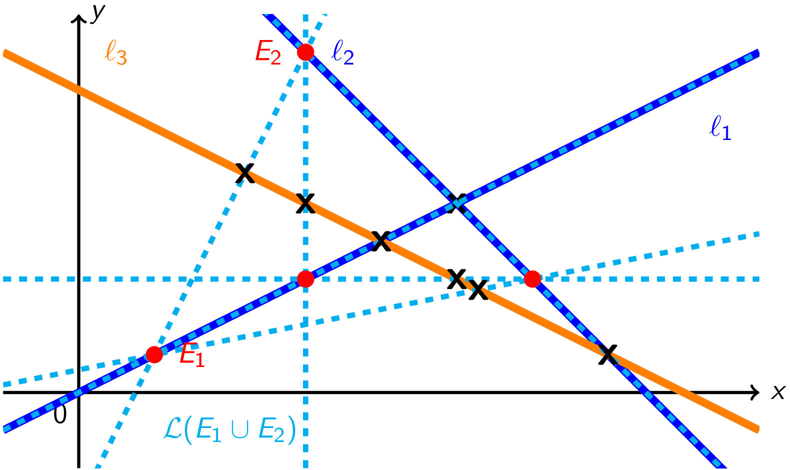

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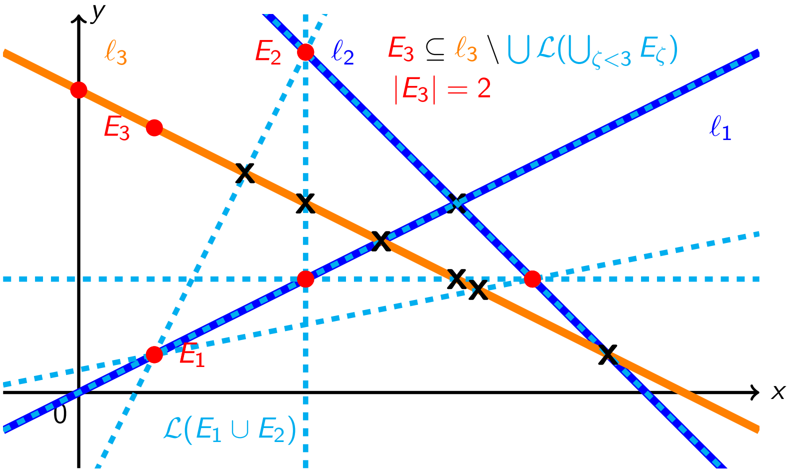

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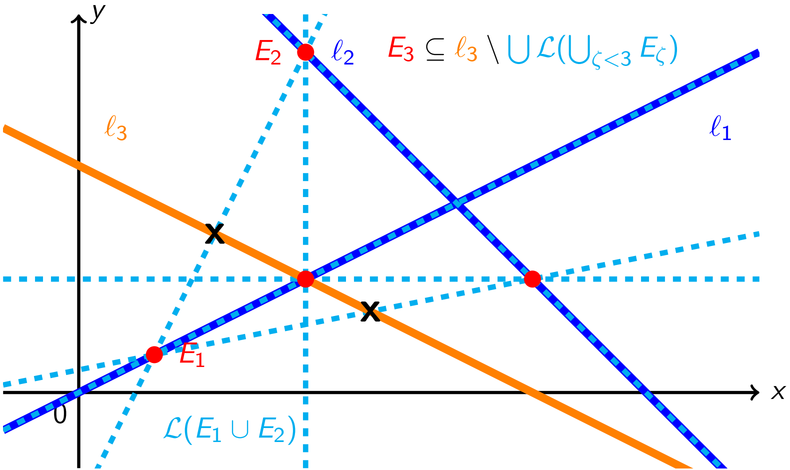

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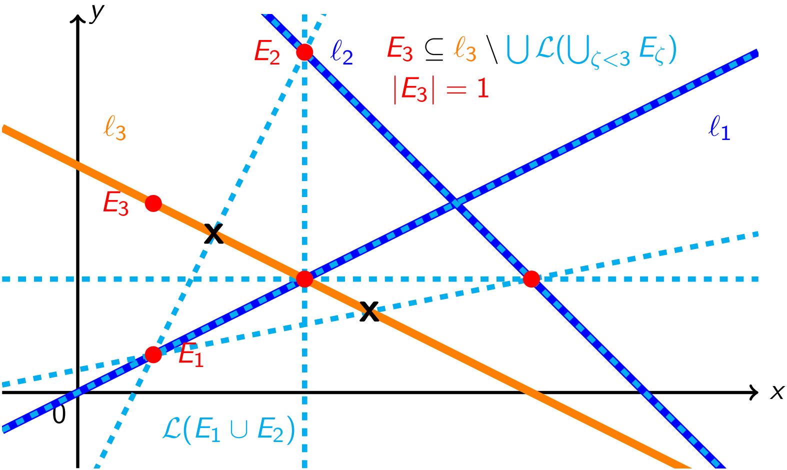

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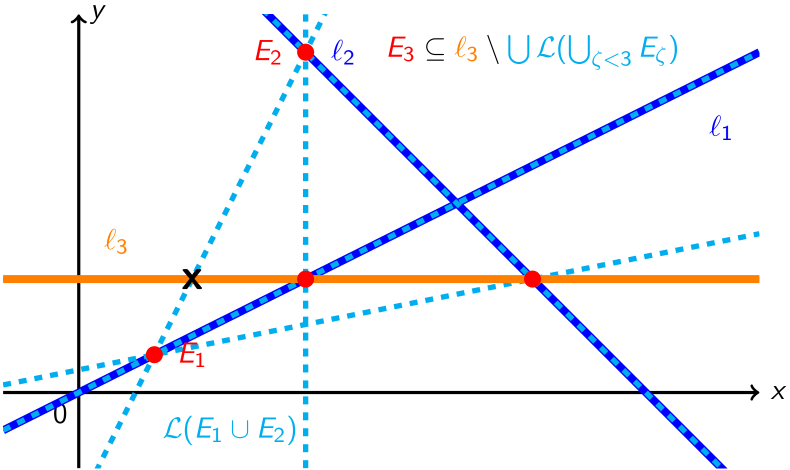

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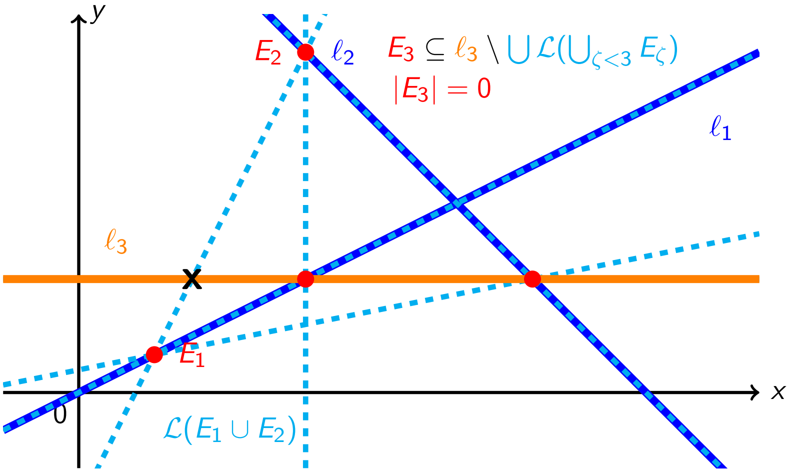


Mazurkiewicz Set

Lebesgue measurable?

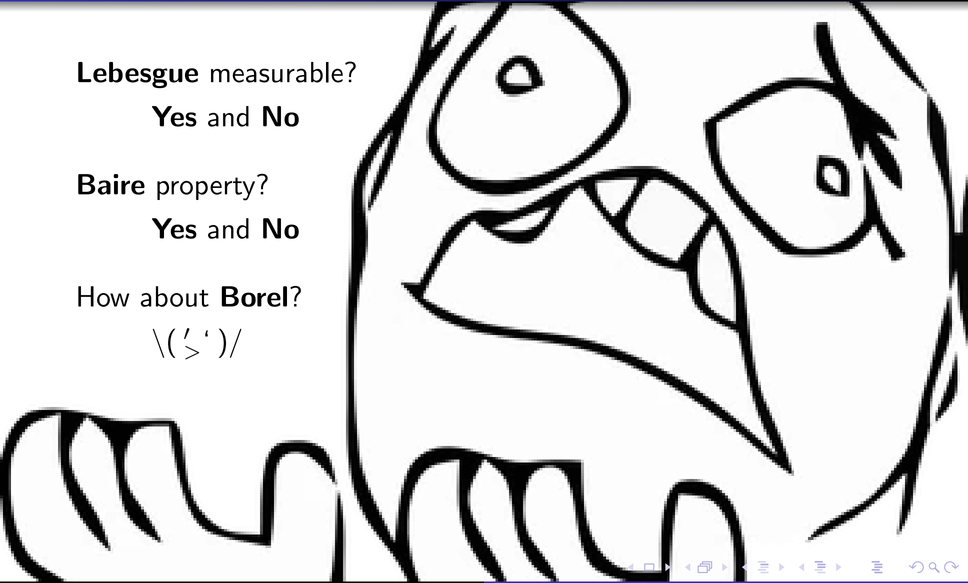
Yes and **No**

Baire property?

Yes and **No**

How about **Borel**?

\(′>‘)/



Mazurkiewicz Set

Union of

Two Functions!

Closed under

Rotations

No Sierpiński-Zygmund function

Example

There is an $\mathfrak{M}$ set containing **no** $\mathcal{SZ}$ function.

Notation

- $\mathbb{L}$ denotes the collection of all straight lines of $\mathbb{R}^2$.
- $\mathbb{L}_\circ := \{\ell^* \in \mathbb{L} \colon \ell^*$ is vertical and $|\ell^* \cap \bigcirc| = 2\}$
- $\mathcal{L}(A) := \{\ell \in \mathbb{L} \colon |\ell \cap A| = 2\}$. $\quad$ ($|A| < \mathfrak{c}$ implies $|\mathcal{L}(A)| < \mathfrak{c}$)

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Construction

Let $\{\ell_\xi\}_{\xi<\mathfrak{c}}$ be an enumeration of $\mathbb{L}$. For every $\xi < \mathfrak{c}$, choose

- $E'_\xi \subseteq \ell_\xi \setminus \bigcup \mathcal{L}(\bigcup_{\zeta<\xi} E_\zeta)$ such that $|\ell_\xi \cap \bigcup_{\zeta\leq\xi} E_\zeta| = 2$.

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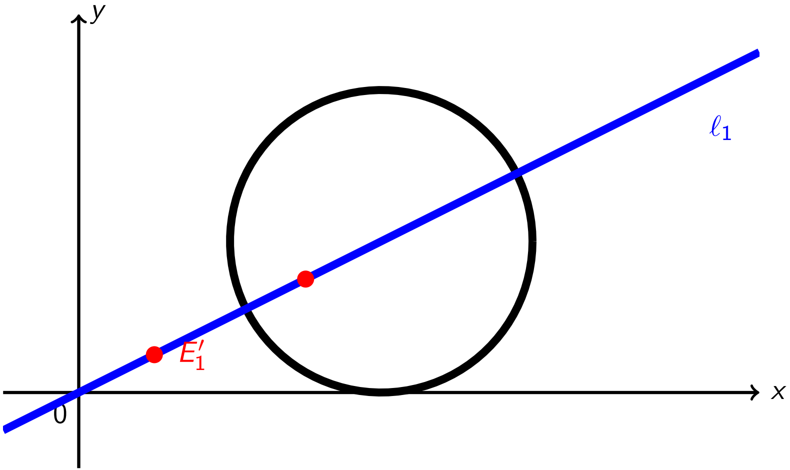
Construction

Let $\{\ell_\xi\}_{\xi<\mathfrak{c}}$ be an enumeration of $\mathbb{L}$. For every $\xi < \mathfrak{c}$, choose

- $E'_\xi \subseteq \ell_\xi \setminus \bigcup \mathcal{L}(\bigcup_{\zeta<\xi} E_\zeta)$ such that $|\ell_\xi \cap \bigcup_{\zeta\leq\xi} E_\zeta| = 2$.
- $\ell^* \in \mathbb{L}_\circ$ such that $|\ell^* \cap \bigcirc \setminus \mathcal{L}(E'_\xi \cup \bigcup_{\zeta<\xi} E_\zeta)| = 2$.

Let $E_\xi := E'_\xi \cup (\ell^* \cap \bigcirc)$ and $\mathfrak{M} := \bigcup_{\xi<\mathfrak{c}} E_\xi$ is as needed.

No Sierpiński-Zygmund function



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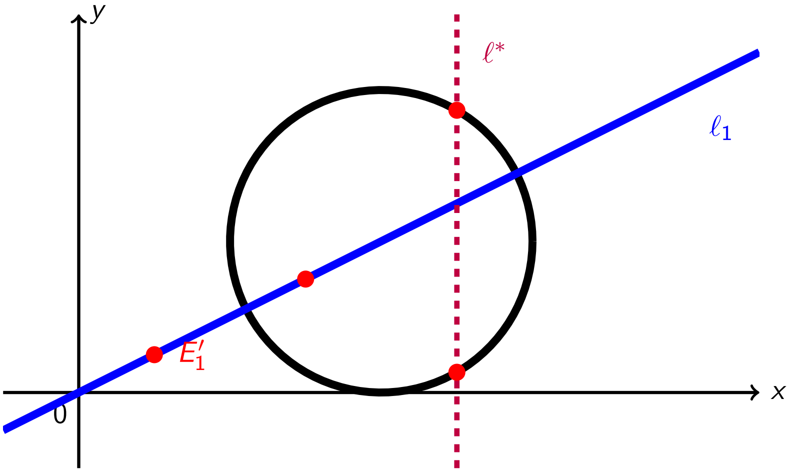

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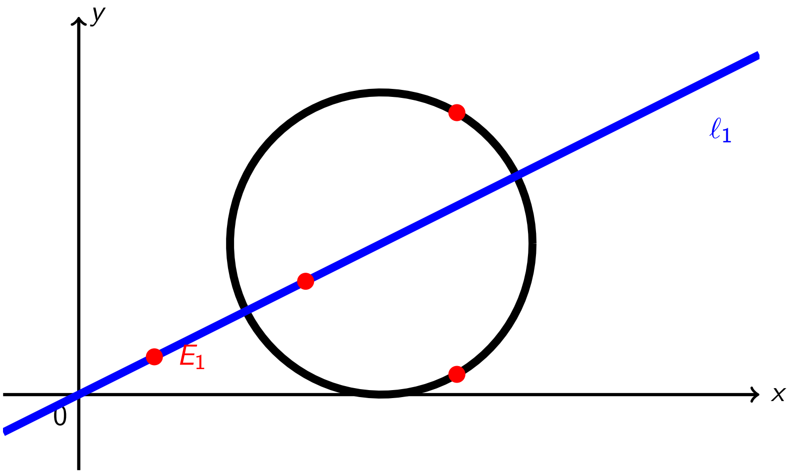

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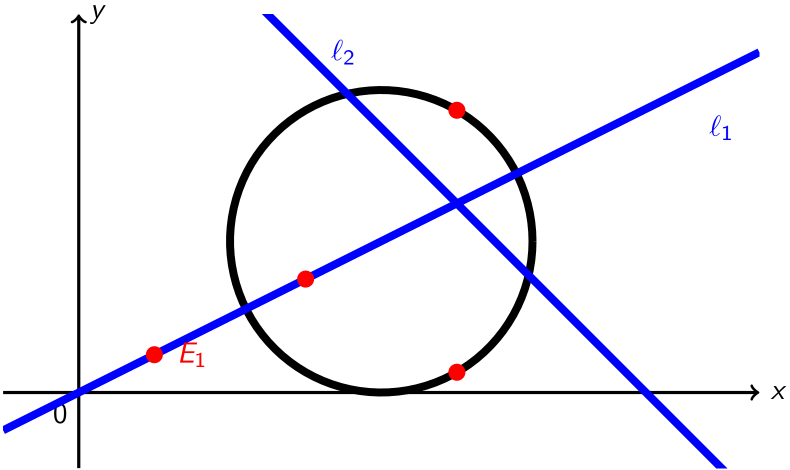

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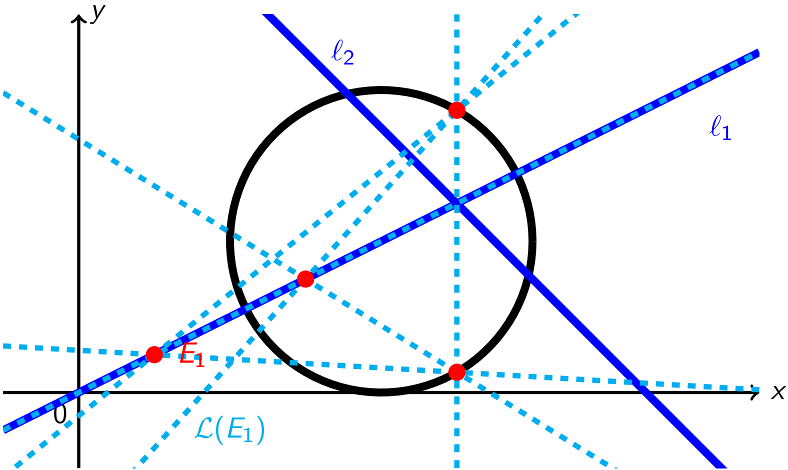

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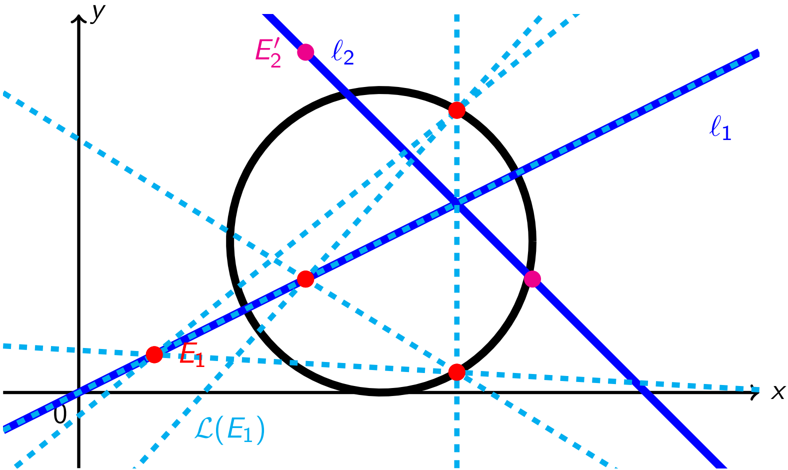

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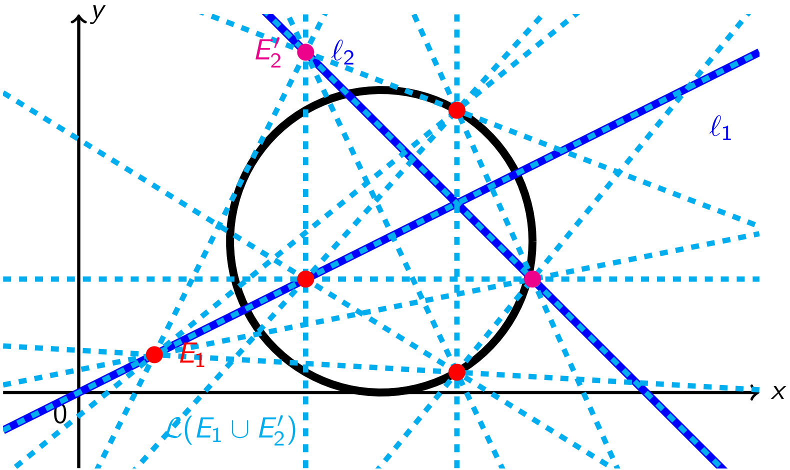

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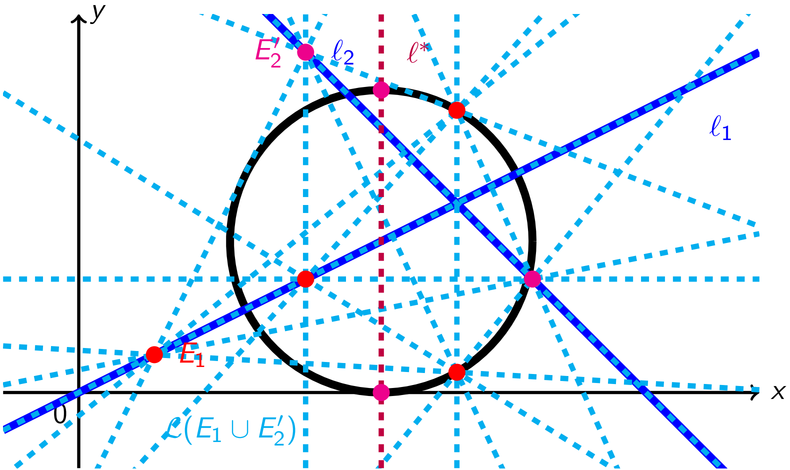

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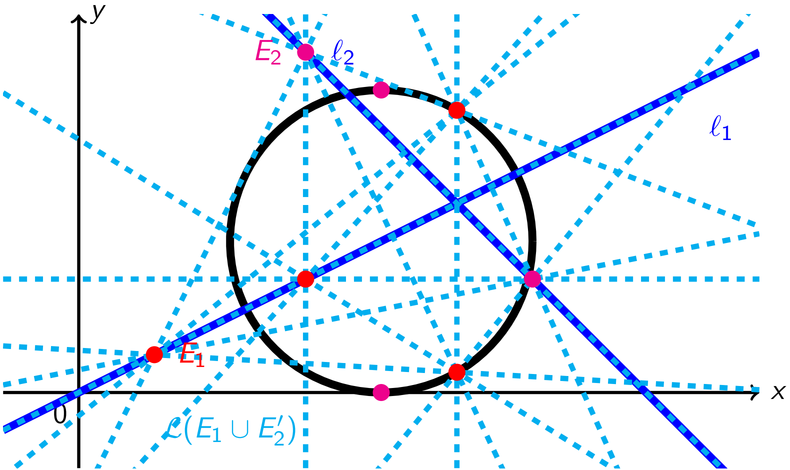

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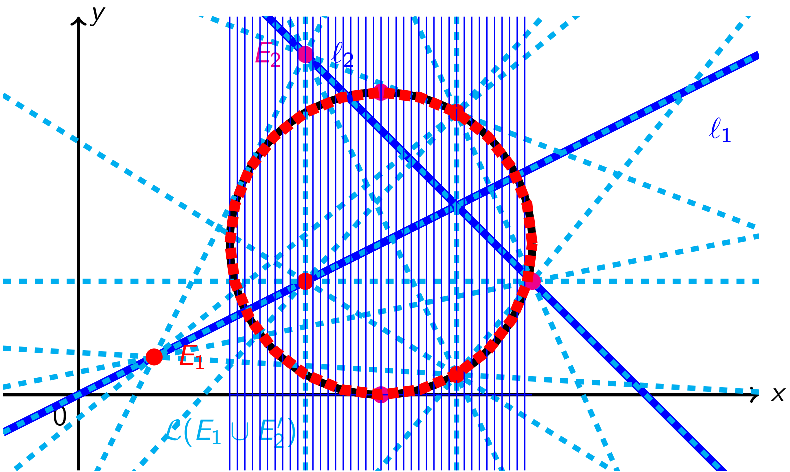

One Sierpiński-Zygmund function

Recall the notation

- $\mathbb{L}$ denotes the collection of all straight lines of $\mathbb{R}^2$.
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Recall the construction of an $\mathfrak{M}$ set (**No control**)

Let $\{\ell_\xi\}_{\xi<\mathfrak{c}}$ be an enumeration of $\mathbb{L}$. For every $\xi < \mathfrak{c}$, choose

$$E_\xi \subseteq \ell_\xi \setminus \bigcup \mathcal{L}(\bigcup_{\zeta<\xi} E_\zeta) \text{ such that } |\ell_\xi \cap \bigcup_{\zeta\leq\xi} E_\xi| = 2.$$

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If $B \subseteq \mathbb{R}^2$ and $|B \cap \ell| = \mathfrak{c}$ for all $\ell \in \mathbb{L}$, there exists an $\mathfrak{M}$ set in B

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Theorem (Kharazishvili)

There is an $\mathfrak{M}$ set containing **one** $\mathcal{SZ}$ function.

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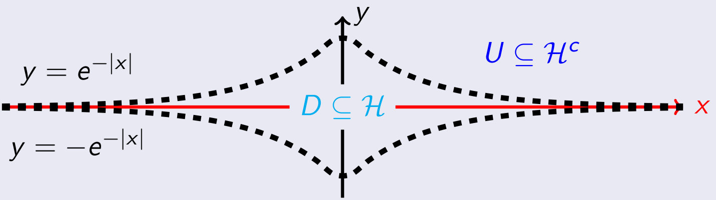
One Sierpiński-Zygmund function

Kharazishvili's argument

Let $\mathcal{H} := \{(x,y) \in \mathbb{R}^2 \colon e^{-|x|} \leq y \leq e^{|x|}]\}$. There is a $D \subseteq \mathcal{H}$ and a $U \subseteq \mathcal{H}^c$ such that

- ℓ is vertical implies $|D \cap \ell| = \mathfrak{c}$ and $|U \cap \ell| = 1$.
- ℓ is nonvertical implies $|U \cap \ell| = \mathfrak{c}$ except x-axis.
- Any $f \subseteq D$ is $\mathcal{SZ}$.



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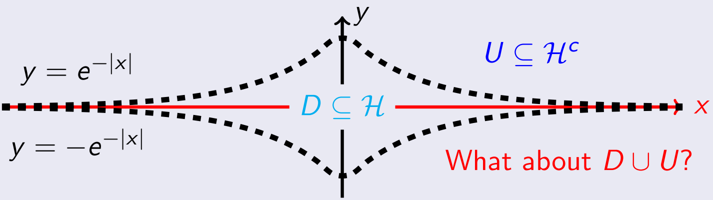

One Sierpiński-Zygmund function

Lemma (Kharazishvili)

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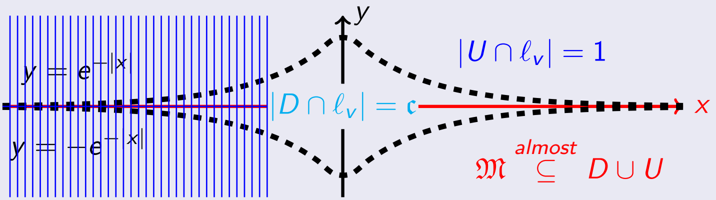

Two Sierpiński-Zygmund functions

Example

Assume CH. There is an $\mathfrak{M}$ set containing **two** $\mathcal{SZ}$ functions.

Recall

Let $\mathcal{G} := \{g \in \mathbb{R}^G \colon G \subseteq \mathbb{R}$ is G_δ, $|G| = \mathfrak{c}$, g is continiuous$\}$.

[f is an $\mathcal{SZ}$ function] iff [$|f \cap g| < \mathfrak{c}$ for every $g \in \mathcal{G}$]

Two Sierpiński-Zygmund functions

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$$[f \text{ is an } \mathcal{SZ} \text{ function}] \quad \text{iff} \quad [|f \cap g| < \mathfrak{c} \text{ for every } g \in \mathcal{G}]$$

Definition (Płotka)

$$[M \subseteq \mathbb{R}^2 \text{ is a } \mathcal{SZ} \text{ set}] \quad \text{iff} \quad [|M \cap g| < \mathfrak{c} \text{ for every } g \in \mathcal{G}]$$

[Pło02b] K. Płotka, *Sum of Sierpiński–Zygmund and Darboux like functions*, Topology Appl. **122** (2002), no. 3, 547–564, DOI 10.1016/S0166-8641(01)00184-5. MR1911699

Two Sierpiński-Zygmund functions

Example

Assume CH. There is an $\mathcal{SZ}$ $\mathfrak{M}$ set.

Notation

- $\{g_\xi\}_{\xi<\mathfrak{c}}$ enumerates $\mathcal{G}$, and $\{\ell_\xi\}_{\xi<\mathfrak{c}}$ enumerates $\mathbb{L}$.

Two Sierpiński-Zygmund functions

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- $g^*_\xi := g_\xi \setminus \bigcup \mathbb{L}_\xi$,

Two Sierpiński-Zygmund functions

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Construction

[CH] $\Rightarrow$ [$\bigcup_{\zeta<\xi} g_\zeta^*$ is meager in ℓ_ξ] $\Rightarrow$ [ℓ_ξ^* is comeager] $\Rightarrow$ [$|\ell_\xi^*| = \mathfrak{c}$]

Two Sierpiński-Zygmund functions

Example

Assume CH. There is an $\mathcal{SZ}$ $\mathfrak{M}$ set.

Notation

- $\{g_\xi\}_{\xi<\mathfrak{c}}$ enumerates $\mathcal{G}$, and $\{\ell_\xi\}_{\xi<\mathfrak{c}}$ enumerates $\mathbb{L}$.
- $\mathbb{L}_\xi := \{\ell \in \mathbb{L} \colon g_\xi$ is somewhere dense in $\ell\}$.
- $g_\xi^* := g_\xi \setminus \bigcup \mathbb{L}_\xi$, and $\ell_\xi^* := \ell_\xi \setminus \bigcup_{\zeta<\xi} g_\zeta^*$.

Construction

[CH] $\Rightarrow$ [$\bigcup_{\zeta<\xi} g_\zeta^*$ is meager in ℓ_ξ] $\Rightarrow$ [ℓ_ξ^* is comeager] $\Rightarrow$ [$|\ell_\xi^*| = \mathfrak{c}$] $\Rightarrow$ [$\bigcup_{\xi<\mathfrak{c}} \ell_\xi^*$ intersects every straight line in a set of cardinality $\mathfrak{c}$]

Two Sierpiński-Zygmund functions

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Construction

[CH] $\Rightarrow$ [$\bigcup_{\zeta<\xi} g^*_\zeta$ is meager in ℓ_ξ] $\Rightarrow$ [ℓ^*_ξ is comeager] $\Rightarrow$ [$|\ell^*_\xi| = \mathfrak{c}$]
$\Rightarrow$ [$\bigcup_{\xi<\mathfrak{c}} \ell^*_\xi$ intersects every straight line in a set of cardinality $\mathfrak{c}$]
$\Rightarrow$ [$\bigcup_{\xi<\mathfrak{c}} \ell^*_\xi$ contains some $\mathfrak{M}$ set] $\overset{\textit{Need to show}}{\Rightarrow}$ [$\mathfrak{M}$ is also $\mathcal{SZ}$]

Two Sierpiński-Zygmund functions

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Claim: $\mathfrak{M} \subseteq \bigcup_{\xi<\mathfrak{c}} \ell^*_\xi$ is also $\mathcal{SZ}$

- Fix any $g_\xi \in \mathcal{G}$. (Goal: $|\mathfrak{M} \cap g_\xi| < \mathfrak{c}$)

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Claim: $\mathfrak{M} \subseteq \bigcup_{\xi<\mathfrak{c}} \ell^*_\xi$ is also $\mathcal{SZ}$

- Fix any $g_\xi \in \mathcal{G}$. (Goal: $|\mathfrak{M} \cap g_\xi| < \mathfrak{c}$) (Note: $g_\xi \subseteq g^*_\xi \cup \bigcup \mathbb{L}_\xi$)
- $|\mathfrak{M} \cap g^*_\xi|$
- $|\mathfrak{M} \cap \bigcup \mathbb{L}_\xi|$

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Claim: $\mathfrak{M} \subseteq \bigcup_{\xi<\mathfrak{c}} \ell_\xi^*$ is also $\mathcal{SZ}$

- Fix any $g_\xi \in \mathcal{G}$. (Goal: $|\mathfrak{M} \cap g_\xi| < \mathfrak{c}$) (Note: $g_\xi \subseteq g_\xi^* \cup \bigcup \mathbb{L}_\xi$)
- $|\mathfrak{M} \cap g_\xi^*| = |\mathfrak{M} \cap \bigcup_{\zeta<\mathfrak{c}} \ell_\zeta^* \cap g_\xi^*|$
- $|\mathfrak{M} \cap \bigcup \mathbb{L}_\xi|$

Two Sierpiński-Zygmund functions

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Two Sierpiński-Zygmund functions

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Claim: $\mathfrak{M} \subseteq \bigcup_{\xi<\mathfrak{c}} \ell^*_\xi$ is also $\mathcal{SZ}$

- Fix any $g_\xi \in \mathcal{G}$. (Goal: $|\mathfrak{M} \cap g_\xi| < \mathfrak{c}$) (Note: $g_\xi \subseteq g^*_\xi \cup \bigcup \mathbb{L}_\xi$)
- $|\mathfrak{M} \cap g^*_\xi| = |\mathfrak{M} \cap \bigcup_{\zeta\leq\xi} \ell^*_\zeta \cap g^*_\xi| \leq 2 \otimes \xi = \xi < \mathfrak{c}$
- $|\mathfrak{M} \cap \bigcup \mathbb{L}_\xi|$

Two Sierpiński-Zygmund functions

Example

Assume CH. There is an $\mathcal{SZ}$ $\mathfrak{M}$ set.

Notation

- $\{g_\xi\}_{\xi<\mathfrak{c}}$ enumerates $\mathcal{G}$, and $\{\ell_\xi\}_{\xi<\mathfrak{c}}$ enumerates $\mathbb{L}$.
- $\mathbb{L}_\xi := \{\ell \in \mathbb{L} : g_\xi$ is somewhere dense in $\ell\}$.
- $g_\xi^* := g_\xi \setminus \bigcup \mathbb{L}_\xi$, and $\ell_\xi^* := \ell_\xi \setminus \bigcup_{\zeta<\xi} g_\zeta^*$.

Claim: $\mathfrak{M} \subseteq \bigcup_{\xi<\mathfrak{c}} \ell_\xi^*$ is also $\mathcal{SZ}$

- Fix any $g_\xi \in \mathcal{G}$. (Goal: $|\mathfrak{M} \cap g_\xi| < \mathfrak{c}$) (Note: $g_\xi \subseteq g_\xi^* \cup \bigcup \mathbb{L}_\xi$)
- $|\mathfrak{M} \cap g_\xi^*| = |\mathfrak{M} \cap \bigcup_{\zeta\leq\xi} \ell_\zeta^* \cap g_\xi^*| \leq 2 \otimes \xi = \xi < \mathfrak{c}$
- $|\mathfrak{M} \cap \bigcup \mathbb{L}_\xi| \leq 2 \otimes |\mathbb{L}_\xi|$

Two Sierpiński-Zygmund functions

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Claim: $\mathfrak{M} \subseteq \bigcup_{\xi<\mathfrak{c}} \ell^*_\xi$ is also $\mathcal{SZ}$

- Fix any $g_\xi \in \mathcal{G}$. (Goal: $|\mathfrak{M} \cap g_\xi| < \mathfrak{c}$) (Note: $g_\xi \subseteq g^*_\xi \cup \bigcup \mathbb{L}_\xi$)
- $|\mathfrak{M} \cap g^*_\xi| = |\mathfrak{M} \cap \bigcup_{\zeta\leq\xi} \ell^*_\zeta \cap g^*_\xi| \leq 2 \otimes \xi = \xi < \mathfrak{c}$
- $|\mathfrak{M} \cap \bigcup \mathbb{L}_\xi| \leq 2 \otimes |\mathbb{L}_\xi|$ (Final Claim: $|\mathbb{L}_\xi| \leq \omega$)

Two Sierpiński-Zygmund functions

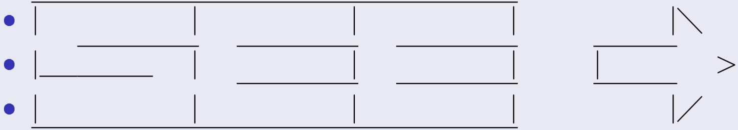
Example

Assume CH. There is an $\mathcal{S}\mathcal{Z}$ $\mathfrak{M}$ set.

Notation

- $\{g_\xi\}_{\xi<\mathfrak{c}}$ enumerates $\mathcal{G}$, and $\{\ell_\xi\}_{\xi<\mathfrak{c}}$ enumerates $\mathbb{L}$.
- $\mathbb{L}_\xi := \{\ell \in \mathbb{L} \colon g_\xi$ is somewhere dense in $\ell\}$.
- $g^*_\xi := g_\xi \setminus \bigcup \mathbb{L}_\xi$, and $\ell^*_\xi := \ell_\xi \setminus \bigcup_{\zeta<\xi} g^*_\zeta$.

Final Claim: $\mathbb{L}_\xi$ is at most countable



Two Sierpiński-Zygmund functions

Example

There is an $\mathfrak{M}$ set containing **no** $\mathcal{SZ}$ function.

Theorem (Kharazishvili)

There is an $\mathfrak{M}$ set containing **one** $\mathcal{SZ}$ function.

Example

Assume CH. There is an $\mathfrak{M}$ set containing **two** $\mathcal{SZ}$ functions.

Two Sierpiński-Zygmund functions

Theorem (Pan, submitted)

There is an $\mathfrak{M}$ set containing **no** $\mathcal{SZ}$ function **in every direction**!

Theorem (Pan, submitted)

There is an $\mathfrak{M}$ set containing **one** $\mathcal{SZ}$ function **in every direction**!

Theorem (Pan, submitted)

Assume $\text{cov}(\mathcal{M}) = \mathfrak{c}$. There is an $\mathfrak{M}$ set containing **two** $\mathcal{SZ}$ functions **in every direction**!

Two Sierpiński-Zygmund functions

Theorem (Pan, submitted)

There is an $\mathfrak{M}$ set containing **no** $\mathcal{SZ}$ function **in every direction**!

Theorem (Pan, submitted)

There is an $\mathfrak{M}$ set containing **one** $\mathcal{SZ}$ function **in every direction**!

Theorem (Pan, submitted)

Assume $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. There is an $\mathfrak{M}$ set containing **two** $\mathcal{SZ}$ functions **in every direction**!

Theorem (Pan, submitted)

Assume $\mathrm{SZ} \cap \mathcal{D} = \emptyset$. No $\mathfrak{M}$ set contains **two** $\mathcal{SZ}$ functions.

References

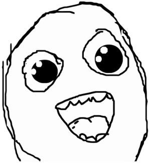
[1] U. B. Darji, *A Sierpiński-Zygmund function which has a perfect road at each point*, Colloq. Math. **64** (1993), no. 2, 159–162, DOI 10.4064/cm-64-2-159-162. MR1218479

[2] M. Balcerzak, K. Ciesielski, and T. Natkaniec, *Sierpiński-Zygmund functions that are Darboux, almost continuous, or have a perfect road*, Arch. Math. Logic **37** (1997), no. 1, 29–35, DOI 10.1007/s001530050080. MR1485861

[3] A. Kharazishvili, *A Mazurkiewicz set containing the graph of a Sierpiński-Zygmund function*, Georgian Math. J., posted on 2024, DOI 10.1515/gmj-2024-2023.

[4] C.-H. Pan, *Mazurkiewicz Sets and Containment of Sierpiński-Zygmund Functions under Rotations*. arXiv:2504.12603.

[5] K. Płotka, *Sum of Sierpiński–Zygmund and Darboux like functions*, Topology Appl. **122** (2002), no. 3, 547–564, DOI 10.1016/S0166-8641(01)00184-5. MR1911699

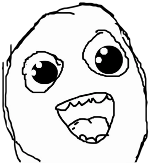


HELLO!

Say **hello** from me

to conference participants

I just started my travels and will be busy for the next 3 weeks



Thank you

for your attention!

Special Last Name Wanted

- Ciesielski-Pan Theorem
- Gurung-Pan Theorem
- Nowakowski-Pan Theorem
- Albkwre-Pan Theorem

- ____?____-Pan Theorem