Modes of convergence of sequences of real or complex functions: a linear point of view

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AIM OF THIS TALK: To compare, under a linear perspective to be specified later:

(a) 3 important modes of convergence of sequences of holomorphic functions, and

(b) 6 important modes of convergence of sequences of measurable functions.

OUR SETTINGS

(a) Given a domain $\Omega \subset \mathbb{C}$, the sequences (f_n) will be members of $H(\Omega)^{\mathbb{N}}$, where $H(\Omega)$ is the vector space of all holomorphic functions $\Omega \longrightarrow \mathbb{C}$.

(b) Given a positive measure space $(\Omega, \mathcal{A}, \mu)$, the sequences (f_n) will be members of $L_0^{\mathbb{N}}$, where L_0 is the vector space of all [μ-classes of] measurable functions $\Omega \longrightarrow \mathbb{R}$.

In the case $H(\Omega)^{\mathbb{N}}$ the concepts of convergence we will deal make sense in more general environments:

Let X be a nonempty set, and $f_n, f : X \to \mathbb{R}$ or $\mathbb{C}$ $(n \geq 1)$. Then we say that:

- $f_n \longrightarrow f$ pointwisely on X provided that
  $f_n(x) \longrightarrow f(x) \quad \forall x \in X.$
- $f_n \longrightarrow f$ uniformly on X provided that
  $\text{lím}_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0.$
- [assuming, additionally, that X is a TS]
  $f_n \longrightarrow f$ compactly on X if $f_n \to f$ uniformly
  on each compact $K \subset X$.
  [equivalent to local uniform convergence if X is T_2-loc. comp].

$f_n \to f$ uniformly $\Longrightarrow$ $f_n \to f$ compactly $\Longrightarrow$ $f_n \to f$ pointwisely.

In the case $X \subset \mathbb{R}$, it is not hard to provide examples showing that both reverse implications are FALSE, even with (f_n) consisting entirely of real analytic functions:

- $\frac{x}{n} \longrightarrow 0$ compactly on $\mathbb{R}$, but not uniformly.
- $nx^2 e^{-nx^2} \longrightarrow 0$ pointwisely on $\mathbb{R}$, but not compactly.

In the case $X = \Omega \subset \mathbb{C}$, it is also easy to find counterexamples [with the f_n's holomorphic] to the reverse of the1st implication:

- $\frac{z}{n} \longrightarrow 0$ compactly on $\mathbb{C}$, but not uniformly.
- If $\Omega \neq \mathbb{C}$ and $a \in \partial\Omega$, then $\frac{1}{n(z-a)} \longrightarrow 0$ compactly on Ω, but not uniformly.

- However, finding counterexamples to the converse of the 2nd implication is not so easy.
- The reason is, maybe, that in the holomorphic setting both types of convergence are not too far from each other, as the two following theorems show. That's why the construction of pointwise convergent seqs of hol fs not converging compactly requires, in general, the use of approximation theorems.

Vitali–Porter's theorem

Let $\Omega \subset \mathbb{C}$ be a domain and $(f_n) \subset H(\Omega)$. Assume that (f_n) is uniformly bounded on compacta and that $\exists S \subset \Omega$ with $S' \cap \Omega \neq \varnothing$ such that $(f_n(z))$ converges $\forall z \in S$ $\Longrightarrow \exists f \in H(\Omega)$ such that $f_n \longrightarrow f$ compactly on Ω.

Osgood's theorem

Let $\Omega \subset \mathbb{C}$ be a domain and $(f_n) \subset H(\Omega)$ be a sequence converging pointwisely in $\Omega \Longrightarrow \exists$ dense open subset $G \subset \Omega$ and $f \in H(G)$ such that $f_n \longrightarrow f$ compactly on G.

Construction of a sequence $f_n \longrightarrow 0$ point. but not comp.:
Assume w.l.o.g. that $0 \in \Omega$. Let
$R := \sup\{x \in \mathbb{R} : \ x \geq 0 \text{ and } [0,x] \subset \Omega\} \in (0,+\infty]$ and
$G := \Omega \setminus [0,R)$, that is open in $\mathbb{C} \Longrightarrow \exists$ seq $\{K_n : \ n \in \mathbb{N}\}$ of compact subsets of G satisfying:

- $G = \bigcup_{n\in\mathbb{N}} K_n$.
- Each K_n is contained in K_{n+1}°.
- For each $n \in \mathbb{N}$, every connected component of $\mathbb{C}_\infty \setminus K_n$ contains a connected component of $\mathbb{C}_\infty \setminus G$.

Choose $(s_n), (t_n) \subset (0,+\infty)$ such that

- $0 < \cdots < s_3 < s_2 < s_1 < t_1 < t_2 < t_3 < \cdots < R$, and
- $s_n \to 0$, $t_n \to R$,

and define the compacta $L_n := K_n \cup \{0\} \cup \{s_{n+1}\} \cup [s_n, t_n] \subset \Omega$.

Select $r_n > 0$ such that:

- $G_n := L_n + D(0, r_n) \subset \Omega$
- $\big(K_n + D(0, r_n)\big) \cap \big([0, t_n] + D(0, r_n)\big) = \varnothing$,
- $D(0, r_n) \cap D(s_{n+1}, r_n) = \varnothing$, and $D(s_{n+1}, r_n) \cap D(s_n, r_n) = \varnothing$.

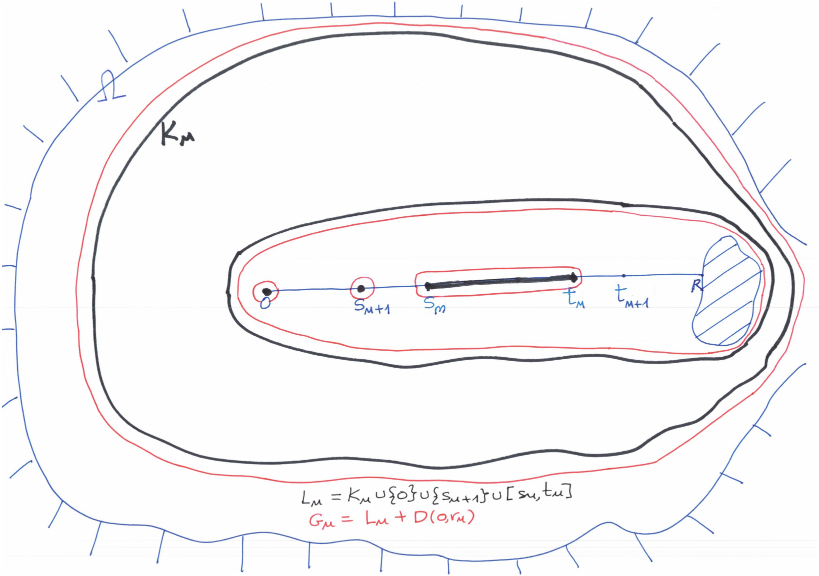
Define $g_n \in H(G_n)$ as

$$g_n(z) = \begin{cases} 0 & \text{if } z \in \big(K_n + D(0, r_n)\big) \cup D(0, r_n) \cup \big([s_n, t_n] + D(0, r_n)\big) \\ n & \text{if } z \in D(s_{n+1}, r_n). \end{cases}$$

Each component of $\mathbb{C}_\infty \setminus L_n$ contains a component of $\mathbb{C}_\infty \setminus \Omega$ $\Longrightarrow$ [**Runge's Approximation Theorem**] $\exists\, f_n \in H(\Omega)$ such that

$$|f_n(z) - g_n(z)| < \frac{1}{n} \quad \forall z \in L_n.$$

Then: $f_n \longrightarrow 0$ point. on Ω but $f_n \not\longrightarrow 0$ unif. on U $[\forall U \in \mathcal{N}(0)]$.


Ω
K_M
0
s_{M+1}
s_M
t_M
t_{M+1}
R
$L_M = K_M \cup \{0\} \cup \{s_{M+1}\} \cup [s_M, t_M]$
$G_M = L_M + D(0, r_M)$

Definitions for (b)

In the case $L_0^{\mathbb{N}}$, we consider the ff. modes of convergence:

- $f_n \longrightarrow f$ pointwisely a.e. provided that $\exists\, Z \in \mathcal{A}$ with $\mu(Z) = 0$ s.t. $f_n(x) \to f(x)\ \forall x \in \Omega \setminus Z$.
- $f_n \longrightarrow f$ uniformly a.e. provided that $\exists\, Z \in \mathcal{A}$ with $\mu(Z) = 0$ s.t. $f_n \to f$ uniformly on $\Omega \setminus Z$.
- $f_n \longrightarrow f$ in measure whenever $\lim_{n\to\infty} \mu(\{x \in \Omega : |f_n(x) - f(x)| > \varepsilon\}) = 0\ \forall \varepsilon > 0$.
- $f_n \longrightarrow f$ almost uniformly if $\forall \varepsilon > 0\ \exists\, Z_\varepsilon \in \mathcal{A}$ with $\mu(Z_\varepsilon) < \varepsilon$ s.t. $f_n \to f$ uniformly on $\Omega \setminus Z_\varepsilon$.
- $f_n \longrightarrow f$ in q-norm (or in q-mean), where $q \in (0, +\infty)$, if $\lim_{n\to\infty} \int_\Omega |f_n - f|^q \, d\mu = 0$.
- $f_n \longrightarrow f$ completely if $\sum_{n=1}^{\infty} \mu(\{x \in \Omega : |f_n(x) - f(x)| > \varepsilon\}) < \infty\ \forall \varepsilon > 0$.

Implications in setting (b)

- $f_n \to f$ uniformly a.e. $\Longrightarrow$ $f_n \to f$ almost uniformly.
- $f_n \to f$ almost uniformly $\Longrightarrow$ $f_n \to f$ pointwisely a.e.
- $f_n \to f$ almost uniformly $\Longrightarrow$ $f_n \to f$ in measure.
- For $q \in (0, +\infty)$, $f_n \to f$ in q-mean $\Longrightarrow$ $f_n \to f$ in measure.
- $f_n \to f$ completely $\Longrightarrow$ $f_n \to f$ in measure.
- $f_n \to f$ completely $\Longrightarrow$ $f_n \to f$ pointwisely a.e.
- If μ is finite: $f_n \to f$ uniformly a.e. $\Longrightarrow$ $f_n \to f$ completely.
- If μ is finite: $f_n \to f$ uniformly a.e. $\Longrightarrow$ $f_n \to f$ in q-mean.
- **[Egoroff's theorem]** If μ is finite: $f_n \to f$ pointwisely a.e. $\Longleftrightarrow$ $f_n \to f$ almost uniformly.

Concrete examples of the **failure** of each of the **opposite implications** have been furnished, mostly by choosing the **Lebesgue measure** on some interval of $\mathbb{R}$ or the **counting measure** on $\mathbb{N}$. For instance:

- In $(\Omega, \mathcal{A}, \mu) = ([0,1], \mathcal{L}, \lambda)$, we have $\chi_{[0,1/n]} \longrightarrow 0$ almost unif. and in q-mean ($q > 0$) but not unif. a.e., and $n \cdot \chi_{[0,1/n]} \longrightarrow 0$ almost unif. but not in q-mean ($q \geq 1$).
- In $(\Omega, \mathcal{A}, \mu) = ([0,1], \mathcal{L}, \lambda)$, the "typewriter sequence" (f_n) defined by $f_{2^k+h} = \chi_{[h/2^k,(h+1)/2^k]}$ $(k = 0, 1, 2, \ldots;\ h = 0, 1, \ldots, 2^k - 1)$ satisfies $f_n \longrightarrow 0$ in measure and even in q-mean ($q > 0$) but not point. a.e.
- In $(\Omega, \mathcal{A}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathrm{card})$, we have $(f_n) := \chi_{\{n,n+1,n+2,\ldots\}} \longrightarrow 0$ point. a.e. but not in measure.
- In $(\Omega, \mathcal{A}, \mu) = ([0,1], \mathcal{L}, \lambda)$, we have $\chi_{[0,2^{-n}]} \longrightarrow 0$ almost unif., point. a.e, in measure, in q-mean ($q > 0$), and completely but not uniformly a.e.

- As said before, in specific measure spaces (planar domains) it is relatively easy to construct sequences of measurable (holomorphic, resp.) fs converging in one mode but not in another mode, **BUT** ...

... WE WANT TO GO A STEP FURTHER:

Could we find large algebraic/algebraic-topological structures inside the families of seqs of measurable (hol., resp.) fs converging in a given sense but not in another sense?

Lineability: definitions, I

- It is convenient to introduce a number of concepts coming from the modern theory of **Lineability**.
- This is justified by the fact that, in the current millenium, there has been a rapid development of results in which many families of mathematical entities have been found to be large (or very small) from an algebraic point of view, regardless their topological size. The notions have been coined starting from **V. Gurariy**:

Vladimir Gurariy (1935-2005)

Aron, Bayart, Gurariy, PérezGª, Quarta, Seoane, Bartoszewicz, Glab, LBG 2004–13

Assume that X is a TVS and that α is a cardinal number. A subset $A \subset X$ is called:

- α-lineable if $A \cup \{0\}$ contains a vector space M with $\dim(M) = \alpha$,
- α-dense-lineable if $A \cup \{0\}$ contains a dense vector subspace M of X with $\dim(M) = \alpha$,
- spaceable whenever $A \cup \{0\}$ contains a closed infinite dimensional vector subspace of X,
- algebrable if X is contained in some linear algebra and $A \cup \{0\}$ contains some infinitely generated algebra, and
- strongly α-algebrable if X is contained in some commutative linear algebra and $A \cup \{0\}$ contains some α-generated free algebra
  [$\Longleftrightarrow$ $\exists B \subset X$ with $\mathrm{card}(B) = \alpha$ s.t. $\forall$ polynomial $P \neq 0$ in N variables with $P(0) = 0$ and any different $b_1, \dots, b_N \in B$, we have $P(b_1, \dots, b_N) \in A \setminus \{0\}$].

- A reference for background on Lineability:

R. Aron, D. Pellegrino, J.B. Seoane and LBG, Lineability: The Search for Linearity in Mathematics, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2016.

- Before going on, it is worth remarking that a number of important positive as well as negative results are known, as for instance the following ones (some of them, rather old), that we write in the language of lineability:

Examples

(a) Levine and Milman (1940):
$CBV[0,1]$ is not spaceable in $C[0,1]$.

(b) Gurariy (1966):
$D[0,1] := \{\text{derivable fs } [0,1] \to \mathbb{R}\}$ is not spaceable in $C[0,1]$.

(c) Herrero, Bourdon, Bès, Wengenroth (1991-2003):
If T is a hypercyclic operator on a TVS X, then

$$HC(T) := \{\text{dense orbit vectors}\}$$

is dense-lineable in X.

Examples

(d) Aron, García and Maestre (2001):
If $\Omega \subset \mathbb{C}$ is a domain, then

$$\{f \in H(\Omega) : f \text{ is not extendable beyond } \partial G\}$$

is dense-lineable and algebrable (with a closed subalgebra, hence it is spaceable in $H(\Omega)$).

(e) Gurariy and Quarta (2004):

$$\widehat{C}[0,1] := \{f \in C[0,1] : f \text{ attains its maximum at exactly one point}\}$$

does not contain a 2-dim vector space.

Examples

(f) Aron, Conejero, Peris, Seoane (2007):
$HC(\tau_a)$ $[\tau_a(f) := f(\cdot + a)]$ is not algebrable in $H(\mathbb{C})$.

(g) Bartoszewicz, Bienias, Filipczak, Glab (2014):
{nowhere monotone differentiable fs $\mathbb{R} \longrightarrow \mathbb{R}$}
is strongly $\mathfrak{c}$-algebrable.

Warning!: One might think that topological largeness $\Longrightarrow$ algebraic largeness.

This is far from being true!

For instance, $\widehat{C}[0,1]$ is residual in $C[0,1]$ but it is highly non-lineable.

As far as I know, until now only **2** papers deal with the **linear comparison** between the diverse modes of convergence of sequences of holomorphic functions:

- M.C. Calderón-Moreno, J. López-Salazar, J.A. Prado-Bassas and LBG, Modes of convergence of sequences of holomorphic functions: a linear point of view, Mediterr. J. Math. 22:55 (2025), 20 pp.
- M.C. Calderón-Moreno, J. López-Salazar, J.A. Prado-Bassas and LBG, Spaceability of special families of null sequences of holomorphic functions, Preprint (2025).

However, it is fair to say that a lot of recent works due to several authors **[Araújo, Bartoszewicz, Calderón, Conejero, Fenoy, Fdez-Sánchez, Filipczak, Glab, Gerlach, López-Salazar, Muñoz-Fdez, Murillo, Ordóñez, Prado, Seoane, Trutschnig, Vecina, LBG, among others]** have been devoted to study the **linear comparison** between the diverse kinds of convergence of sequences of measurable real functions defined on measure [mainly, probability] spaces:

- G. Araújo, G.A. Muñoz, J.A. Prado, J.B. Seoane and LBG, Lineability in sequence and function spaces, Stud. Math. 237 (2017), 119–136.
- G. Araújo, M. Fenoy, J. Fernández-S, J. López-S, J.B. Seoane and J.M. Vecina, Modes of convergence of random variables and algebraic genericity, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 118:63 (2024), 24 pp.
- A. Bartoszewicz, M. Bienias and S. Glab, Lineability within Peano curves, martingales, and integral theory, J. Funct. Spaces 2018, Art. ID 9762491, 8 pp.
- A. Bartoszewicz, M. Filipczak and S. Glab, Algebraic structures in the set of sequences of independent random variables, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM 117:45, 16 pp.
- M.C. Calderón-Moreno and LBG, Anti-Fubini and pseudo-Fubini functions, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 115:127 (2021), 16 pp.
- M.C. Calderón-Moreno, M. Murillo-Arcila, J.A. Prado and LBG, Undominated sequences of integrable functions, Mediterr. J. Math. 17:179 (2020), 17 pp.
- M.C. Calderón-Moreno, P. Gerlach and J.A. Prado, Lineability and modes of convergence, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 114:18 (2020), 12 pp.
- J.A. Conejero, M. Fenoy, M. Murillo-Arcila and J.B. Seoane, Lineability within probability theory settings, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 111 (2017), 673–684.
- J. Fernández-S, J.B. Seoane and W. Trutschnig, Lineability, algebrability, and sequences of random variables, Math. Nachr. 295 (2022), 861–875.
- M. Ordóñez and LBG, Lineability criteria, with applications, J. Funct. Anal. 266 (2014), 3997–4025.

Hands to work

- Under the terminology of lineability, we want to know **to what extent** the family of seqs of (measurable/hol.) fs converging in a given mode but not in another one **is algebraically large.**
- Since $f_n \to f \iff f_n - f \to 0$, we can reduce the question to convergence to **0**.

Notation for (a)

- $\mathcal{S}_u := \{(f_n) \in H(\Omega)^{\mathbb{N}} : f_n \to 0 \text{ uniformly on } \Omega\}$.
- $\mathcal{S}_{uc} := \{(f_n) \in H(\Omega)^{\mathbb{N}} : f_n \to 0 \text{ compactly on } \Omega\}$.
- $\mathcal{S}_p := \{(f_n) \in H(\Omega)^{\mathbb{N}} : f_n \to 0 \text{ pointwisely on } \Omega\}$.

$$\mathcal{S}_u \subset \mathcal{S}_{uc} \subset \mathcal{S}_p.$$

Notation

Notation for (b)

- $\mathcal{S}_p := \{\mathbf{f} = (f_n) \in L_0^{\mathbb{N}} : \ f_n \to 0 \text{ pointwisely a.e.}\}$
- $\mathcal{S}_u := \{\mathbf{f} = (f_n) \in L_0^{\mathbb{N}} : \ f_n \to 0 \ \text{ uniformly a.e.}\}$
- $\mathcal{S}_{au} := \{\mathbf{f} = (f_n) \in L_0^{\mathbb{N}} : \ f_n \to 0 \ \text{ almost uniformly}\}$
- $\mathcal{S}_m := \{\mathbf{f} = (f_n) \in L_0^{\mathbb{N}} : \ f_n \to 0 \ \text{ in measure}\}$
- $\mathcal{S}_{L_q} := \{\mathbf{f} = (f_n) \in L_0^{\mathbb{N}} : \ f_n \to 0 \ \text{ in } q\text{-mean}\}$
- $\mathcal{S}_c := \{\mathbf{f} = (f_n) \in L_0^{\mathbb{N}} : \ f_n \to 0 \ \text{ completely}\}$.

- $\mathcal{S}_u \subset \mathcal{S}_{au} \subset \mathcal{S}_p$, $\mathcal{S}_{L_q} \cup \mathcal{S}_{au} \subset \mathcal{S}_m$ and $\mathcal{S}_c \subset \mathcal{S}_p$.
- If μ is finite, then $\mathcal{S}_p = \mathcal{S}_{au}$ and $\mathcal{S}_u \subset \mathcal{S}_c \cap \mathcal{S}_{L_q}$.

Our specific goal is to study the **algebraic-topological size** of the difference sets $\mathcal{C} \setminus \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ run over all pairs of previous families [cases (a) and (b)] satisfying $\mathcal{C} \not\subset \mathcal{D}$.

In view of this, we need to endow $L_0^{\mathbb{N}}$ and $H(\Omega)^{\mathbb{N}}$ with respective **natural topologies**:

- In $H(\Omega)$ we consider the compact-open topology, and in $H(\Omega)^{\mathbb{N}}$, the corresponding **product topology**. Then $H(\Omega)^{\mathbb{N}}$ becomes a separable metrizable TVS.
- In L_0 we consider the topology τ of local convergence in measure, and in $L_0^{\mathbb{N}}$, the corresponding **product topology**.

The topologies we will use, II

- (L^0, τ) is metrizable $\iff$ μ is σ-finite.
- **(S)** := $\exists\, \mathcal{A}_0$ countable $\subset \mathcal{A}$ satisfying: $\forall M \in \mathcal{A}$ with $\mu(M) < \infty$ and $\forall \varepsilon > 0$, $\exists S = S_{M,\varepsilon} \in \mathcal{A}_0$ s.t. $\mu(M \Delta S) < \varepsilon$.

  We have: **(S)** $\Longrightarrow$ L_0 is separable.
- **(S)** + μ σ**-finite** $\Longrightarrow$ $L_0^{\mathbb{N}}$ is a separable metrizable TVS.

Araújo-Fenoy-FdezSánchez-LópezS-Seoane-Vecina 2024

Let V be a vector space and $\mathcal{C}, \mathcal{D} \subset V^{\mathbb{N}}$ satisfying the following properties, where $\mathcal{E}$ is $\mathcal{C}$ or $\mathcal{D}$:

(a) $\lambda\mathcal{E} \subset \mathcal{E}\ \forall\lambda \in \mathbb{R}$.

(b) $\mathbf{0} := (0, 0, 0, \dots) \in \mathcal{E}$.

(c) If a seq $\mathbf{f} \in V^{\mathbb{N}}$ can be decomposed into finitely many subseqs $\in \mathcal{E}$, then $\mathbf{f} \in \mathcal{E}$.

(d) If $\mathbf{f} \in \mathcal{E}$, then all its subseqs $\in \mathcal{E}$ too.

(e) If $\mathbf{f} \in \mathcal{E}$ and finitely many terms are deleted from (or added to) $\mathbf{f}$, then the new sequence $\in \mathcal{E}$ too.

(f) $\mathcal{C} \not\subset \mathcal{D}$.

Then $\mathcal{C} \setminus \mathcal{D}$ is **$\mathfrak{c}$-lineable**.

- Every family $\mathcal{E} \in \{\mathcal{S}_{uc}, \mathcal{S}_p, \mathcal{S}_m, \mathcal{S}_u, \mathcal{S}_c, \mathcal{S}_{au}, \mathcal{S}_{L_q}\}$ in $H(\Omega)^{\mathbb{N}}$ or $L_0^{\mathbb{N}}$ satisfies (a) to (e) of the previous theorem AFFLSV.

Aron-GªPacheco-Ordóñez-PérezGª-Seoane-LBG 2008–2014

Let α be an infinite cardinal number. Assume that X is a metrizable separable TVS and that $A, B \subset X$ fulfill the following:

(i) A is stronger than B: $A + B \subset A$.

(ii) $A \cap B = \varnothing$.

(iii) A is α-lineable.

(iv) B is dense-lineable.

Then A is **α-dense-lineable**.

- For $X = H(\Omega)^{\mathbb{N}}$ or $L_0^{\mathbb{N}}$ [if separable], the set $B = c_{00}(X) := \{\mathbf{f} = (f_n) \in X : \exists\, k = k(\mathbf{f}) \in \mathbb{N} \text{ s.t. } f_n = 0\ \forall n > k\}$ is **dense-lineable**, and satisfies $\mathcal{E} \cap B = \varnothing$ and $\mathcal{E} + B \subset \mathcal{E}\ \forall \mathcal{E}$ as above.

Calderón-LópezSalazar-Prado-LBG 2025

- $\mathcal{S}_{uc} \setminus \mathcal{S}_u$ is $\mathfrak{c}$-dense-lineable in $H(\Omega)^{\mathbb{N}}$.
- $\mathcal{S}_p \setminus \mathcal{S}_{uc}$ is $\mathfrak{c}$-dense-lineable in $H(\Omega)^{\mathbb{N}}$.
- $\mathcal{S}_{uc} \setminus \mathcal{S}_u$ is strongly $\mathfrak{c}$-algebrable.
- $\mathcal{S}_p \setminus \mathcal{S}_{uc}$ is strongly $\mathfrak{c}$-algebrable.
- $\mathcal{S}_{uc} \setminus \mathcal{S}_u$ is spaceable in $H(\Omega)^{\mathbb{N}}$.
- $\mathcal{S}_p \setminus \mathcal{S}_{uc}$ is spaceable in $H(\Omega)^{\mathbb{N}}$.

Idea of the proof of some of these results:

• $\mathcal{S}_p \setminus \mathcal{S}_{uc}$ is $\mathfrak{c}$-dense-lineable in $H(\Omega)^{\mathbb{N}}$:

Apply the machine theorems AFFLSV and AGOPSB with

$V = H(\Omega)$, $\mathcal{C} = \mathcal{S}_p$, $\mathcal{D} = \mathcal{S}_{uc}$, $X = H(\Omega)^{\mathbb{N}}$, $A = \mathcal{S}_p \setminus \mathcal{S}_{uc}$, $B = c_{00}(H(\Omega))$.

• $\mathcal{S}_p \setminus \mathcal{S}_{uc}$ is strongly $\mathfrak{c}$-algebrable:

As in p. 6, define $g_{c,n} \in H(G_n)$ [$c \in H :=$ a maximal $\mathbb{Q}$-l.indep. subset of $(0, +\infty)$] as

$$g_{c,n}(z) = \begin{cases} 0 & \text{if } z \in \big(K_n + D(0,r_n)\big) \cup D(0,r_n) \cup \big([s_n,t_n] + D(0,r_n)\big) \\ e^{cn} & \text{if } z \in D(s_{n+1}, r_n). \end{cases}$$

and use Runge's Th. to obtain $f_{c,n} \in H(\Omega)$ s.t. $|f_{c,n}(z) - g_{c,n}(z)| < e^{-n^2} \quad \forall z \in L_n$.

Then $\{(f_{c,n}) : c \in H\}$ has card = $\mathfrak{c}$ and generates a free algebra $\subset \{0\} \cup (\mathcal{S}_p \setminus \mathcal{S}_{uc})$.

- $\mathcal{S}_{uc} \setminus \mathcal{S}_u$ is spaceable in $H(\Omega)^{\mathbb{N}}$:

Take $\mathbf{f} = (f_n) \in \mathcal{S}_{uc} \setminus \mathcal{S}_u$. WLOG we can assume that $\overline{\mathbb{D}} \subset \Omega$.

Then **Arakelian approximation theorem** + **basis perturbation theorem** [as applied on $L^2(\mathbb{T})$] provide a sequence $(\varphi_n) \subset H(\Omega)$ each of whose members is "large" in a point z_n (with $z_n \to \partial_\infty \Omega$). Then

$$\mathbf{M} := \left\{ (f_n \cdot \Phi) : \Phi \in \overline{\text{span}}\{\varphi_k : k \in \mathbb{N}\} \right\}.$$

is a closed inf-dim subspace $\subset (\mathcal{S}_{uc} \setminus \mathcal{S}_u) \cup \{0\}$.

- μ is said to be nonatomic if it lacks atoms. An atom is a set $A \in \mathcal{A}$ with $\mu(A) > 0$ s.t. there do not $\exists B, C \in \mathcal{A}$ with
$$\mu(B) > 0 < \mu(C),\ B \cap C = \varnothing \text{ and } B \cup C = A.$$
- μ is said to be semifinite if for each $A \in \mathcal{A}$ we have $\mu(A) = \mathbf{sup}\{\mu(B) : B \in \mathcal{A},\ B \subset A, \text{ and } \mu(B) < \infty\}$.
- μ finite $\Longrightarrow$ μ σ-finite $\Longrightarrow$ μ semifinite.
- No converse is true. Example: Ω uncountable $\Longrightarrow$ counting measure is semifinite but not σ-finite.

- μ is said to satisfy (P) if $\exists$ a family $\{A_n : n \in \mathbb{N}\} \subset \mathcal{A}$ s.t. $A_n \cap A_k = \varnothing$ $(n \neq k)$ and $\mathbf{inf}\{\mu(A_n) : n \in \mathbb{N}\} > 0$.
- μ is said to satf. (Q) if $\mathbf{sup}\{\mu(S) : S \in \mathcal{A}, \mu(S) < \infty\} = \infty$.
- μ is said to satf. (R) if $\mathbf{inf}\{\mu(S) : S \in \mathcal{A}, \mu(S) > 0\} = 0$.
- (Q) $\Longrightarrow$ (P). The converse is false: $\mu(A) := \infty$ if $\varnothing \neq A \subset \mathbb{N}$.
- μ nonatomic + semifinite + $[\mu(\Omega) = \infty]$ $\Longrightarrow$ (Q).
- μ nonatomic + semifinite $\Longrightarrow$ (R).
- The counting measure on $\mathbb{N}$ satisfies (Q) but not (R), and is (purely) atomic.
- $\mu(A) := \sum_{n \in A} 2^{-n}$ $(A \subset \mathbb{N})$: finite, (purely) atomic, (R), non-(P).

- M.C. Calderón-Moreno, P. Gerlach-Mena, J.A. Prado-Bassas and LBG, **Almost uniform convergence vs. pointwise convergence from a linear point of view**, Preprint (2025).

(a) $[\mu\ \sigma$-finite + nonatomic + (S)$] \Longrightarrow \left(\bigcap_{q>0} \mathcal{S}_{L_q}\right) \setminus \mathcal{S}_p$ (hence $\mathcal{S}_m \setminus \mathcal{S}_p$) is $\mathfrak{c}$-dense-lineable in $L_0^{\mathbb{N}}$.

(b) $[\mu$ semifinite + nonatomic$] \Longrightarrow \left(\bigcap_{q>0} \mathcal{S}_{L_q}\right) \setminus \mathcal{S}_p$ (hence $\mathcal{S}_m \setminus \mathcal{S}_p$) is strongly $\mathfrak{c}$-algebrable.

(c) $[\mu$ semifinite + nonatomic$] \Longrightarrow \mathcal{S}_m \setminus \mathcal{S}_p$ is spaceable.

(d) $[\mu\ \sigma$-finite + (S) + (P)$] \Longrightarrow \mathcal{S}_p \setminus \mathcal{S}_m$ is $\mathfrak{c}$-dense-lineable.

(e) (P) $\Longrightarrow$ $\mathcal{S}_p \setminus \mathcal{S}_m$ is strongly $\mathfrak{c}$-algebrable.

(f) (Q) $\Longrightarrow$ $\mathcal{S}_p \setminus \mathcal{S}_m$ is spaceable.

(g) [μ σ-finite + (S) + (R)] $\Longrightarrow$ $(\mathcal{S}_c \cap \mathcal{S}_{au}) \setminus \mathcal{S}_u$ is $\mathfrak{c}$-dense-lineable.

(h) (R) $\Longrightarrow$ $(\mathcal{S}_c \cap \mathcal{S}_{au}) \setminus \mathcal{S}_u$ is strongly $\mathfrak{c}$-algebrable.

(i) (R) $\Longrightarrow$ $(\mathcal{S}_c \cap \mathcal{S}_{au}) \setminus \mathcal{S}_u$ is spaceable.

(j) [μ σ-finite + (S) + $\mu(\Omega) = \infty$] $\Longrightarrow$ $\mathcal{S}_u \setminus \bigcup_{q>0} \mathcal{S}_{L_q}$ is $\mathfrak{c}$-dense-lineable.

(k) $\mu(\Omega) = \infty$ $\Longrightarrow$ $\mathcal{S}_u \setminus \bigcup_{q>0} \mathcal{S}_{L_q}$ is strongly $\mathfrak{c}$-algebrable.

Idea of the proof of some of these results :

• (d) [μ σ-finite + (S) + (P)] $\Longrightarrow$ $\mathcal{S}_p \setminus \mathcal{S}_m$ is $\mathfrak{c}$-dense-lineable:

(P) $\Longrightarrow \exists\, A_n$'s pairwise disjoint s.t. $\mu(A_n) \geq \alpha > 0$ $\forall n \in \mathbb{N}$. Define $\mathbf{f} = (\chi_{A_n})$.
Then $\mathbf{f} \in \mathcal{S}_p \setminus \mathcal{S}_m$. Apply the machine theorems AFFLSV and AGOPSB with
$V = L_0$, $X = L_0^{\mathbb{N}}$, $\mathcal{C} = \mathcal{S}_p$, $\mathcal{D} = \mathcal{S}_m$, $A = \mathcal{S}_p \setminus \mathcal{S}_m$ and $B = c_{00}(L_0)$.

• (e) (P) $\Longrightarrow$ $\mathcal{S}_p \setminus \mathcal{S}_m$ is strongly $\mathfrak{c}$-algebrable:

Let $H \subset (0, +\infty)$ be $\mathbb{Q}$-linearly independent with card$(H) = \mathfrak{c}$.
Then $\{(e^{cn} \cdot \chi_{A_n})_{n\geq 1} : c \in H\}$ is algebraically free and generates an algebra $\subset \{0\} \cup (\mathcal{S}_p \setminus \mathcal{S}_m)$.

• (f) (Q) $\Longrightarrow$ $\mathcal{S}_p \setminus \mathcal{S}_m$ is spaceable:

(Q) $\Longrightarrow \exists\, (A_{k,n})_{k,n}$ mutually disjoint s.t. $1 \leq \mu(A_{k,n}) < \infty$ $\forall k, n$.
Then $\overline{\text{span}}\{(\chi_{A_{k,n}})_n : k \in \mathbb{N}\}$ is a closed inf-dim subspace $\subset \{0\} \cup (\mathcal{S}_p \setminus \mathcal{S}_m)$.

Questions: **1.** When the L_q's appear in the stage, the situation happens to be refractory to spaceability:

- Is $\bigcap_{q>0} \mathcal{S}_{L_q} \setminus \mathcal{S}_p$ spaceable in $L_0^{\mathbb{N}}$?
- Is $\mathcal{S}_u \setminus \bigcup_{q>0} \mathcal{S}_{L_q}$ spaceable in $L_0^{\mathbb{N}}$?

2. In the **holomorphic setting**: Comparative study of $\mathcal{S}_{distrib}$.

I ENCOURAGE THE INTERESTED PEOPLE (IF ANY!) TO INVESTIGATE FURTHER IN THESE LINES.

THAT'S ALL FOR NOW. THANK YOU !