DO THE CHANGE OF VARIABLE FORMULA INTEGRALS HAVE EQUAL VALUE?

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QUESTION: ARE THE EQUALITIES

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f\big(\varphi(t)\big)\varphi'(t)dt.$$

and

$$\int_{a}^{b} f(x)dx = \int_{\alpha}^{\beta} f\big(\varphi(t)\big)\varphi'(t)dt$$

TRUE?

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SOME COMMENTS

We show three *Changes of Variable for the Riemann Integral*:

(1) Theorem 1 uses a minimum of *Lebesgue Integration*, but not an actual Lebesgue integral. It covers practical cases.

(2) Theorem 2 has no conditions on φ'. Its proof relies a bit more on *Lebesgue's Integration*. The base function f can be unbounded outside its interval of integration.

(3) Theorem 3 is general, with a trivial statement.

(4) We don't deal with Kestelman (see also Davies) and Preiss and Uher's Changes of Variable (see **[2], [8], [12], and [9]**). But we comment a little on them.

KNOWN VERSIONS OF THE FORMULA

(Apostol, Lang, Spivak) *If f is* **continuous** *on the image set $\varphi\big([\alpha, \beta]\big)$ and φ, φ' are* **continuous**, *then*

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f\big(\varphi(t)\big)\varphi'(t)\,dt\,.$$

(Knapp) *Let f* **be integrable** *on* $[a, b]$*, and* $\varphi : [\alpha, \beta] \to [a, b]$ *be* **strictly increasing continuous and onto***. Suppose that* φ *is differentiable on* (α, β)*, with* φ' **uniformly continuous***.*

(Rudin) *Let f be* **integrable** *on* $[a, b]$*, and* $\varphi : [\alpha, \beta] \to [a, b]$ *be* **strictly increasing, continuous, and onto***. Assume that* φ' *is* **integrable** *on* $[\alpha, \beta]$*.*

In Knapp's and Rudin's versions, the product $(f \circ \varphi)\varphi'$ *is then integrable and*

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\bigl(\varphi(t)\bigr)\varphi'(t)\,dt\,.$$

("General Version") (H. Kestelman [8], 1961; R. O. Davies [2], 1961) *Let $g : [\alpha, \beta] \to \mathbb{R}$ be* **integrable**. *Let us fix $\gamma \in [\alpha, \beta]$. Given $t \in [\alpha, \beta]$, we put*

$$G(t) = \int_{\gamma}^{t} g(\tau)\, d\tau.$$

Let $f : G\big([\alpha, \beta]\big) \to \mathbb{R}$ be **integrable**. *Then, $(f \circ G)g$ is integrable and*

$$\int_{G(\alpha)}^{G(\beta)} f(x)\, dx = \int_{\alpha}^{\beta} f\big(G(t)\big) g(t)\, dt.$$

Remark. This formulation of the Change of the Variable Theorem, for a monotone G, started with Lebesgue in 1909 (see Sarkhel and Výborný **[15]**).

(The reverse of Kestelman's "general version") (Preiss and Uher [12], 1970; Kuleshov's remark [9], 2021) *Consider an* **integrable** $g : [\alpha, \beta] \to \mathbb{R}$. *Let us fix a point* $\gamma \in [\alpha, \beta]$. *Given* $t \in [\alpha, \beta]$, *we put*

$$G(t) = \int_{\gamma}^{t} g(\tau)\, d\tau.$$

Let $(f \circ G)g$ *be* **integrable**, *with* f **bounded** *on* J, *the closed interval with endpoints* $G(\alpha)$ *and* $G(\beta)$. *Then,* f *is integrable on* J *and*

$$\int_{G(\alpha)}^{G(\beta)} f(x)\, dx = \int_{\alpha}^{\beta} f\big(G(t)\big) g(t)\, dt.$$

COMPARING THE SUBSTITUTION MAPS φ AND G

- φ is continuous and differentiable at every point in (α, β).
- G is given by an integral.
  G is a Lipschitz map, differentiable a.e., and G' is integrable.

HYPOTHESIS ABOUT THE DERIVATIVE φ'

- The majority of the change of variable theorems mentioned along this text have the hypothesis that the substitution map (φ or G) is: **"differentiable + something else".**
- Our wish is to get rid of the **"something else".**

TABLE OF USUAL HYPOTHESES

	f	φ & φ' or G & g
— — — — — —	— — — — — —	— — — — — — — — — — — —
Usual	continuous	φ' continuous
Knapp	integrable	φ bicontinuous φ' unif. cont. on the int.
Rudin	integrable	φ bicontinuous φ' **integrable**
Kestelman, Davies	integrable	g **integrable**
Preiss & Uher	bounded, $(f \circ G)g$ *int*.	g **integrable**

.

THIS ARTICLE TABLE OF HYPOTHESES

	$\mathbf{f} : \mathbf{I} \to \mathbb{R}$	$\varphi' : (\alpha, \beta) \to \mathbb{R}$
— — — — —	— — — — — ——	— — — — — — — ——
Theorem 0	integrable **has primitive**	**no conditions**
Theorem 1	integrable	continuous a. e. .
Theorem 2	**bounded on** $[\varphi(\alpha), \varphi(\beta)]$ continuous a.e.	**no conditions**
Theorem 3	integrable	**no conditions**
Improper Integrals	improper integrable	**no conditions**

NOTATION

The letter I indicates an arbitrary interval.
Given a bounded $f : [a, b] \to \mathbb{R}$, its *lower Darboux sum* is

$$s(f, \mathcal{P}) = \sum m_i \Delta x_i,$$

where $\mathcal{P} = \{a = x_0 \leq x_1 \leq \cdots \leq x_n = b\}$ is a *partition* of the interval $[a, b]$, $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$, and $\Delta x_i = x_i - x_{i-1}$, for each $i = 1, \ldots, n$.
The *norm* of $\mathcal{P}$ is $|\mathcal{P}| = \max\{\Delta x_1, \ldots, \Delta x_n\}$.
If f is a Riemann integrable, we have

$$\lim_{|\mathcal{P}| \to 0} s(f, \mathcal{P}) = \int_a^b f(x)\, dx.$$

APPROXIMATING ANY $f : [a, b] \to [0, +\infty)$ FROM BELOW

Lemma

(Approximation Lemma) *There exists a sequence $f_n : [a, b] \to [0, +\infty)$ with the following properties.*

1. *We have $0 \leq f_n \leq f$, for all n.*
2. *Each f_n is piecewise linear continuous.*
3. *If f is continuous at p, then $\lim f_n(p) = f(p)$.*
4. *If f is integrable on the sub-interval $[c, d]$, then*

$$\int_c^d |f_n(x) - f(x)| dx \longrightarrow 0.$$

Proof. From an lower Darboux sum $s(f, \mathcal{P}) = \sum m_i \Delta x_i$, we create a piecewise linear continuous $f_n : [a, b] \to [0, +\infty)$ approximating f from below..

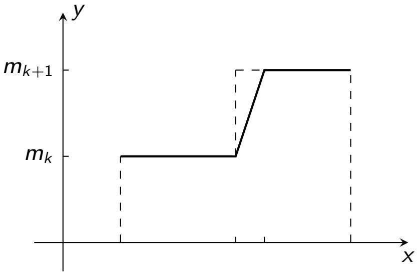

Figure 1: Graph of f_n on "up level sub-intervals".

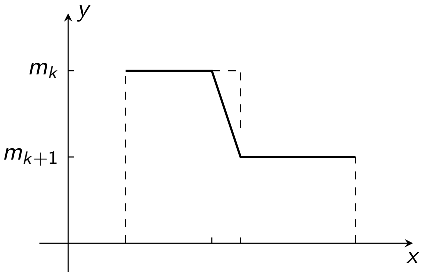

Figure 2: Graph of f_n on “down level sub-intervals”.

THEOREMS USED

Lemma

(Theorem Zero, [3]) *Consider $f : I \longrightarrow \mathbb{R}$ and a continuous $\varphi : [\alpha, \beta] \longrightarrow I$, differentiable on (α, β). Suppose that f has a primitive. The following is true.*

- *We have the identity, provided the finitess of the integrals in it,*

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt.$$

Proof. With F the primitive of f, the proof follows from

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} f(\varphi(t))\varphi'(t)dt = \int_{\alpha+\epsilon}^{\beta-\epsilon} (F \circ \varphi)'(t)dt = F(\varphi(\beta - \epsilon)) - F(\varphi(\alpha + \epsilon)). \quad \square$$

Lemma

(Serrin and Varberg's Theorem on Critical Values, [16]) *Suppose that* $\varphi : \mathbb{R} \to \mathbb{R}$ *has derivative (finite or infinite) on a set* E*, with* $m[\varphi(E)] = 0$*. Then, we have*

$$\varphi' = 0 \text{ almost everywhere on } E.$$

Theorem

(Theorem 1) *Consider an* **integrable** $f : I \to \mathbb{R}$ *and a continuous* $\varphi : [\alpha, \beta] \to I$ *that is differentiable on* (α, β), *with* φ' **continuous. a. e.** *The following is true.*

- *If the product* $(f \circ \varphi)\varphi'$ *is integrable, then we have*

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt.$$

Proof. By steps.

1. **The setup**. We may assume $I = \varphi([\alpha, \beta])$ and $f \geq 0$.
   We may define $\varphi'(\alpha)$ and $\varphi'(\beta)$ at will.
   We assume that the two integrals under scrutiny are finite.

2. **Approximating** f. The **Approx. Lemma** gives $f_n : \varphi([\alpha, \beta]) \to [0, +\infty)$ with f_n continuous, $0 \leq f_n \leq f$, and $f_n(x) \to f(x)$ if f is continuous at x.
3. **Approximating** $(f \circ \varphi)\varphi'$. Let $\mathcal{N}$ be the null set of all points of discontinuity of f. If $\mathcal{M} = \varphi^{-1}(\mathcal{N})$, then $\varphi(\mathcal{M}) \subset \mathcal{N}$. Hence $\varphi(\mathcal{M})$ is a null set and we have (by **Serrin and Varberg**)

$$\begin{cases} \varphi' = 0 \text{ a. e. on } \mathcal{M}, \text{ and} \\ f_n(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t) = 0 \text{ a. e. on } \mathcal{M}. \end{cases}$$

If $t \notin \mathcal{M}$, then f is continuous at $\varphi(t)$ and $f_n(\varphi(t)) \to f(\varphi(t))$. Thus,

$$f_n(\varphi(t))\varphi'(t) \longrightarrow f(\varphi(t))\varphi'(t), \text{ a. e.}.$$

4. **The key integral identity**. Since φ' is continuous a. e, the product $(f_n \circ \varphi)\varphi'$ is continuous a. e. We also have

$$|f_n(\varphi(t))\varphi'(t)| \leq |f(\varphi(t))\varphi'(t)| \text{ on } [\alpha, \beta].$$

Hence, $(f_n \circ \varphi)\varphi'$ is bounded, since $(f \circ \varphi)\varphi'$ is integrable. Thus, $(f_n \circ \varphi)\varphi'$ is integrable. Yet, f_n is continuous, has a primitive, and we may apply **Theorem 0 (an elementary change of variable)**. We get

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f_n(x)dx = \int_{\alpha}^{\beta} f_n(\varphi(t))\varphi'(t)dt.$$

5. **The convergence of the integral of** (f_n). The **first lemma** shows that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f_n(x)dx \longrightarrow \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx.$$

6. **The convergence of the integral of** $\Big((f_n \circ \varphi)\varphi'\Big)$. We saw that $(f_n \circ \varphi)\varphi'$ converges pointwise to $(f \circ \varphi)\varphi'$ a. e. We saw that $|(f_n \circ \varphi)\varphi'|$ is $\mathcal{R}$-integrable and bounded by the $\mathcal{R}$-integrable function $|(f \circ \varphi)\varphi'|$. **Lebesgue's Dominated Convergence Theorem** implies that

$$\int_{\alpha}^{\beta} f_n(\varphi(t))\varphi'(t)dt \longrightarrow \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt.$$

7. **Conclusion**. Just combine steps $4, 5$, and 6. $\square$

Corollary

We may suppose $f = 0$ *outside its interval of integration. Let* f *and* φ *be as in Theorem 1. Suppose that* $m[\varphi^{-1}(\{\varphi(\alpha), \varphi(\beta)\})] = 0$. *Define* $g : I \to \mathbb{R}$ *as*

$$g = \begin{cases} f, & \textit{on the closed interval with endpoints } \varphi(\alpha) \textit{ and } \varphi(\beta), \\ 0, & \textit{elsewhere.} \end{cases}$$

Then, we have

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} g(x)dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t) = \int_{\alpha}^{\beta} g(\varphi(t))\varphi'(t)dt.$$

THEOREM 2

Theorem

Consider a continuous $\varphi : [\alpha, \beta] \to I$ *that is* **differentiable** *on* (α, β). *Let* $f : I \to \mathbb{R}$ *be* **continuous a.e.** *Let* J *be the bounded and closed interval with endpoints* $\varphi(\alpha)$ *and* $\varphi(\beta)$. *The following is true.*

- *If* $(f \circ \varphi)\varphi'$ *is integrable on* $[\alpha, \beta]$, *and* f *is bounded on* J, *then*

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt.$$

THEOREM 3

Theorem

Let us consider a function $f : [a, b] \to \mathbb{R}$ *and a continuous* $\varphi : [\alpha, \beta] \to [a, b]$ *that is* **differentiable** *on* (α, β). *Let us suppose that* $\varphi(\alpha) = a$ *and* $\varphi(\beta) = b$.

- *If* f *is* **integrable** *on* $[a, b]$ *and* $(f \circ \varphi)\varphi'$ *is integrable on* $[\alpha, \beta]$, *then*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

CHANGE OF VARIABLE FOR IMPROPER INTEGRALS

Corollary

Consider any $f : (a, b) \to \mathbb{R}$, with (a, b) any open interval. Suppose that f is **improper integrable or** *$f : [a, b] \to \mathbb{R}$ is* **integrable.** *Let $\varphi : (\alpha, \beta) \to (a, b)$ be* **differentiable**, *with (α, β) any open interval, $\varphi(\alpha+) = a$ and $\varphi(\beta-) = b$.*

- *If $(f \circ \varphi)\varphi'$ is improper integrable on (α, β) or integrable on $[\alpha, \beta]$, then*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

Example 1 Consider

$$f(x) = x^3, \text{ if } x \in \left[0, \frac{2}{\pi}\right], \qquad \text{and} \qquad \varphi(t) = \begin{cases} 0, & \text{if } t = 0, \\ t \sin \frac{1}{t}, & \text{if } t \in \left(0, \frac{2}{\pi}\right]. \end{cases}$$

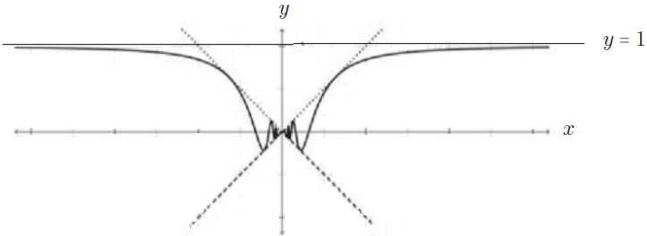

Figure 3: The graph of φ.

Clearly, f is integrable while φ is continuous and oscillates near zero. We have

$$\varphi'(t) = \sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}.$$

- φ' is unbounded and **not integrable** on $[0, 2/\pi]$.
- $(f \circ \varphi)\varphi'$ is **integrable** on $[0, 2/\pi]$, since

$$(f \circ \varphi)(t)\varphi'(t) = t^3 \left(\sin^3 \frac{1}{t}\right) \left(\sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right).$$

By Theorem 3, we have

$$\int_0^{\frac{2}{\pi}} x^3\,dx = \int_0^{\frac{2}{\pi}} [\varphi(t)]^3 \varphi'(t)\,dt = \left.\frac{\varphi^4(t)}{4}\right|_0^{\frac{2}{\pi}} = \frac{4}{\pi^4}. \quad \square$$

Example 2 Consider

$$f(x) = \frac{x}{x^4+1}, \ x \in [0,1], \quad \text{and} \quad \varphi(t) = \sqrt{t}, \ t \in [0,1].$$

Clearly, f is continuous and does integrable.
The map φ is continuous on $[0,1]$ and differentiable on $(0,1]$, with

$$\varphi'(t) = \frac{1}{2\sqrt{t}} \ \textbf{not integrable}.$$

Moreover, $\varphi(0) = 0$ and $\varphi(1) = 1$. We also have

$$(f \circ \varphi)(t)\varphi'(t) = \frac{1/2}{t^2+1}.$$

Thus, $(f \circ \varphi)\varphi'$ is **integrable** on $[0,1]$.

Thus, we may apply Theorem 3 and then write

$$\int_0^1 \frac{x}{x^4+1}\,dx = \int_0^1 f(\varphi(t))\varphi'(t)\,dt.$$

By developing the right-hand side of the equation right above, we find

$$\begin{aligned}\int_0^1 \frac{x}{x^4+1}\,dx &= \frac{1}{2}\int_0^1 \frac{dt}{t^2+1}\\ &= \left.\frac{\arctan t}{2}\right|_{t=0}^{t=1} = \frac{\pi}{8}.\end{aligned}$$

Example 3. Consider

$$f(x) = \frac{1}{x^2+1}, \text{ where } x \in (-\infty, +\infty),$$

and $\varphi(\theta) = \tan(\theta)$, with $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then, f is improper integrable and φ is differentiable, with $\varphi(\theta) \to \pm\infty$ if $\theta \to \pm\frac{\pi}{2}$. We have
$f(\varphi(\theta))\varphi'(\theta) = \frac{\sec^2\theta}{1+\tan^2(\theta)} = 1, \text{ for all } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$
Thus, $(f \circ \varphi)\varphi'$ is integrable on $[-\pi/2, \pi/2]$ and

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, d\theta = \pi.$$

REMARKS ON SUBSTITUTIONS MAPS

(1) **(Volterra's maps, see [18], [5], [6], [21].)** Theorem 1 requires φ' a.e. continuous. In practical problems, it is improbable that one will try a substitution φ with φ' bounded and not integrable (a Volterra map, usually hard to build).

(2) **(Pompei derivatives are not substitution maps, see [13], [10], [7].)** There exists a strictly increasing, bicontinuous, and differentiable map $\varphi : [0,1] \to [0,1]$, with $\varphi' = 0$ at a (countable) dense set **(the map φ is baptized as a Pompei derivative)**. Given one such φ, and supposing the existence of an integrable function $f : [0,1] \to \mathbb{R}$, with $f \neq 0$ almost everywhere, and the product $(f \circ \varphi)\varphi'$ integrable, we get a contradiction.

REFERENCES

[1] R. J. Bagby, *The substitution theorem for Riemann integrals*, Real Anal. Exchange, **27(1)** (2001/2002), 309-314.

[2] R. O. Davies, *An Elementary Proof of the Theorem on Change of Variable in Riemann Integration*, Math. Gaz., **45** (1961), 23–25, `http://www.jstor.org/stable/3614765`.

[3] O. R. B. de Oliveira, *Changes of Variable for the Riemann integral on the real line*, Real Anal. Exchange, **49(2)** (2024), 1-11, `https://doi.org/10.14321/realanalexch.49.2.1689708439`.

[4] O. R. B. de Oliveira, *Changes of Variable Formula integrals, do they have equal value?*, arXiv:2502.12679, `https://doi.org/10.48550/arXiv.2502.12679`.

[5] C. Goffman, *A bounded derivative which is not Riemann integrable*, Amer. Math. Monthly **84** (1977), 205-206.

[6] A. Russel Gordon, *A bounded derivative that is not Riemann Integrable*, Math. Mag. **89(5)** (2016), 364-370, `https://www.jstor.org/stable/10.4169/math.mag.89.5.364`.

[7] Y. Katznelson and K. Stromberg, *Everywhere differentiable, nowhere monotone functions*, Amer. Math. Monthly **81** (1974), 349-354.

[8] H. Kestelman, *Change of variable in Riemann integration*, Math. Gaz., **45(351)** (1961), 17-23, `http://www.jstor.org/stable/3614764`.

[9] A. Kuleshov, *A Remark on the Change of Variable Theorem for the Riemann integral*, Mathematics, 9, 1899 (2021), `https://doi.org/10.3390/math9161899`.

[10] D. Pompeiu, *Sur les fonctions derivatives*, Mathematische Annalen (French), **63 (3)** (1907), 326-332,

[11] R. L. Pouso, *Riemann integration via primitives for a new proof to the change of variable theorem*, arXiv:1107.1996v1[[math.CA].

[12] D. Preiss and J. Uher, *A remark on the substitution for the Riemann integral.* (Czech) Casopis Pest. Mat. **95**, 345–347 (1970).

[13] A. C. M. van Rooij and W. H. Schikhof, *A Second Course on Real Functions*, Cambridge University Press, 1982.

[14] W. Rudin, *Real & Complex Analysis*. 3rd ed., McGraw-Hill Inc, 1987.

[15] D. N. Sarkhel and R. Výborný, *A Change of Variables Theorem for the Riemann Integral*, Real Anal. Exch., **22**(1) (1996-97), 390-395.

[16] J. B. Serrin and D. E. Varberg, *A general chain rule for derivatives and the change of variables formula for the Lebesgue integral*, Amer. Math. Monthly **76** (1969), 514-520.

[17] H. Tandra, *A simple proof of the Change of Variable Theorem for the Riemann Integral*, The Teaching of Mathematics, Vol. **XVIII(1)** (2015), 25-28.

[18] B. S. Thomson, *The bounded Convergence Theorem*, Amer. Math. Monthly, **127(6)** (2020), 483-503, https://doi.org/10.1080/00029890.2020.1736470.

[19] A. Torchinsky, *The change of variable formulas for Riemann integrals*, Real Anal. Exch., **45**(1) (2020), 151-172, https://doi.org/10.14321/realanalexch.45.1.0151.

[20] A. Torchinsky, *A Modern View of the Riemann Integral*, Lecture Notes in Mathematics 2309, Springer, 2022.

[21] V. Volterra, *Sui principii del calcolo integrale*, Giorn. Mat. Battaglini, **19** (1881), 333–372.

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