Four open problems in achievement set theory

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Introduction

Let $\sum a_n$ be a convergent series of real numbers. The *achievement set* (or the *set of subsums*) is defined by

$$E(x_n) \ := \ \left\{ y \in \mathbb{R} : \ \exists\, K \in \mathbb{N} \qquad y = \sum_{n \in K} x_n \right\}.$$

A set $B \subset \mathbb{R}$ is *achievable* if there is a series $\sum x_n$ such that $B = E(x_n)$. The n-th remainder

$$r_n := \sum_{i > n} x_i.$$

In particular, r_0 is the sum of the whole series.

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The set of n-initial subsums:

$$F_n(x_k) \; := \; \left\{ y \in \mathbb{R} : \; \exists \, K \in \{1, \ldots, n\} \qquad y = \sum_{n \in K} x_n \right\}.$$

The Gutrie-Nymann Classification Theorem

The achievement set of any absolutely convergent series is of one of the following four types:

(i) *the union of a finite family of closed and bounded intervals (a multi-interval set);*

(ii) *a Cantorval;*

(iii) *a Cantor set;*

(iv) *a finite set.*

J.A. Guthrie, J.E. Nymann, *The topological structure of the set of subsums of an infinite series*, Colloq. Math. 55(1988) 323-327

J.E. Nymann, R.A.Sáenz, *On the paper of Guthrie and Nymann on subsums of an infinite series*, Colloq. Math. 83(2000) 1-4

Theorem

A bounded set $A \subset \mathbb{R}$ is a Cantorval if and only if it is regularly closed and its boundary $\operatorname{Fr} A$ is a Cantor set.

Guthrie-Nymann Classification

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Observation

If two two series $\sum x_n$ and $\sum y_n$ differ only by finitely many terms, then $E(x_n)$ and $E(y_n)$ are of the same topological type.

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From now on: $x_n \geqslant x_{n+1} > 0$ and $\sum x_n < \infty$.

Kakeya conditions

Kakeya conditions and reversed Kakeya conditions:

$$K(x_n) \ := \ \{k \in \mathbb{N} : \ x_k > r_k\}, \qquad K^c(x_n) \ := \ \{k \in \mathbb{N} : \ x_k \leqslant r_k\}$$

Theorem

If $K(x_n) = \mathbb{N}$, then $E(x_n)$ is a Cantor set.

fast convergent series

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fast convergent series

Corollary

If $\operatorname{card} K^c(x_n) < \infty$, then $E(x_n)$ is a Cantor set.

S. Kakeya, On the partial sums of an infinite series, Tôhoku Sci. Rep.3(1914) 159-164

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Thinning out a sequence $=$ passing to an infinite subsequence by removing infinitely many terms

Observation

If $x_n < r_n$ *only for finitely many* n*, then* $E(x_n)$ *is either a multi-interval set or a Cantor set. Moreover, any thinning out of the sequence leads to a Cantor set.*

P. Nowakowski, FPW, *On the Lebesgue measure of boundaries of achievable Cantorvals*, it should appear on arXiv by the end of September 2025

Corollary

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Open Problem

Can every slowly convergent series $\sum x_n$ with $\mathrm{card}\{n : \; x_n < r_n\} = \infty$ be thinned out to Cantorval-yielding subseries?

Definition

A *multigeometric series* is a series of the form

$$w_1 q + w_2 q + \cdots + w_m q + w_1 q^2 + w_2 q^2 + \cdots + w_m q^2 + w_1 q^3 + \cdots$$

for some $q \in (0, 1)$ and some numbers $w_1 \geqslant w_2 \geqslant \ldots \geqslant w_m > 0$. It will be denoted by $\sum(w_1, w_2, \ldots, w_m : q)$.

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Theorem

$$E((x_n)_{n\in\mathbb{N}\setminus A}) \textit{ is a } \begin{cases} \textit{Cantorval} \quad \textit{iff} \quad \operatorname{card} A < \infty. \\ \textit{Cantor set} \quad \textit{iff} \quad \operatorname{card} A = \operatorname{card} A^c = \infty. \\ \textit{finite set} \quad \textit{iff} \quad \operatorname{card} A^c < \infty. \end{cases}$$

W. Bielas, S. Plewik, M. Walczyńska, *On the center of distances*, Eur. J. Math. 4 (2018), no. 2, 687–698.

Minimal representations

Let E be an achievable multi-interval set or an achievable Cantorval. A series $\sum x_n$ is a **minimal** representation for E if $E = E(x_n)$ and every thinning out $E((x_n)_{n\in\mathbb{N}\setminus A})$ is of thinner type than E.

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$$\text{the cardinal function: } \quad f_\mu(y) \ := \ \operatorname{card}\{A \subset \mathbb{N} : \ y = \sum_{n\in A} x_n\}$$

S. Głąb, J. Marchwicki, *Cardinal functions of atomic measures*, Results in Math. 75(4) (2020), article no. 141

points of uniqueness of E: $\ U(x_n) \ := \ \{x \in E(x_n) : \ f_\mu(x) = 1\}$

$\mathfrak{c}$-points of E: $\ \{x \in E(x_n) : \ f_\mu(x) = \mathfrak{c}\}$

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Proposition

If $U = U(x_n)$ is dense in $E(x_n)$, then $\sum x_n$ is minimal.
If $E(x_n)$ has no $\mathfrak{c}$-points, then $\sum x_n$ is minimal.

S. Głąb, J. Marchwicki, *Set of uniqueness for Cantorvals*, Results in Math. 78(9) (2023) DOI:10.1007/s00025-022-01777-3

Minimal representations

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Open Problem

Does the above characterization of minimality remain true for $E(x_n)$ being a Cantorval ?

points of uniqueness

Observation

If $U(x_n) = \{0, r_0\}$, *then* $E(x_n)$ *is an interval.*
If $E(x_n)$ *is a Cantroval, then* $U(x_n)$ *is infinite.*
If $U(x_n) = E(x_n)$, *then* $E(x_n)$ *is a Cantor set.*

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If $U(x_n) = E(x_n)$, *then* $E(x_n)$ *is a Cantor set.*

the achievement set of the k-th remainder:
$E_k = E_k(x_n) \ := \ E((x_n)_{n=k+1}^{\infty})$

Observation

$x \in U^c$ *if and only if* $\exists k \in \mathbb{N}\ \exists f, g \in F_k, f \neq g \quad x \in (f + E_k) \cap (g + E_k)$.

P. Nowakowski, FPW, *On the Lebesgue measure of boundaries of achievable Cantorvals*, it should appear on arXiv by the end of September 2025

Proposition

The following statements are equivalent:

(i) $\overline{U^c} = E$;

(ii) $\mathrm{int}_E U = \emptyset$;

(iii) 0 *is an accumulation point of* U^c;

(iv) $\forall k \in \mathbb{N} \quad E_k \cap U^c \neq \emptyset$.

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Lemma

Let $\mathcal{F}$ *be a countable family of closed proper subsets of* $[a, b]$ *such that* $\bigcup \mathcal{F} = [a, b]$. *Then there are two distinct elements of* $A, B \in \mathcal{F}$ *such that* $A \cap B \cap (a, b)$ *is non-empty.*

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Corollary

If E contains an interval, then U^c is dense in E.

S. Głąb, J. Marchwicki, *Set of uniqueness for Cantorvals*, Results in Math. 78(9) (2023) DOI:10.1007/s00025-022-01777-3

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$U(x_n)$ is a co-analytic subset of $\mathbb{R}$. In particular, it has the Baire property.

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Proposition

U is a $\mathcal{G}_\delta$-set always.

Corollary

If U is dense in E, then U^c is of the first category in E.

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If U is dense in E, then U^c is of the first category in E.

Open Problem

Does $f_\mu^{-1}(\mathfrak{c}) \neq \emptyset$ imply $U(x_n)$ is not dense/meager ?

the 1914 Kakeya hypothesis

$$K(x_n) \;:=\; \{k \in \mathbb{N} :\; x_k > r_k\}, \qquad K^c(x_n) \;:=\; \{k \in \mathbb{N} :\; x_k \leqslant r_k\}$$

Corollary

If card $K^c(x_n) < \infty$, *then* $E(x_n)$ *is a Cantor set.*

Corollary

card $K(x_n) < \infty$ *if and only if* $E(x_n)$ *is a multi-interval set.*

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Kakeya wrote sincerely: *that the relation* $x_n \leqslant r_n$ *fails only for an infinite number of values of* n*, seems to be the necessary and sufficient condition that the set* $E(x_n)$ *should be nowhere dense; but I have no proof of it.*

S. Kakeya, *On the set of partial sums of an infinite series*, Proc. Tokyo Math.-Phys. Soc., 2nd series, 7(1914), 250-251

Theorem

For any $S \subset \mathbb{N}$ such that $\operatorname{card} S = \operatorname{card} S^c = \infty$, there is a series $\sum x_n$ such that $K(x_n) = S$ and $E(x_n)$ is a Cantor set.

J. Marchwicki, J. Miska, *On Kakeya conditions for achievement sets*, Result in Math. (2021) 76:181; //doi.org./10.1007/s00025-021-01479-2

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Proposition

For any $0 < \alpha \leqslant \beta \leqslant 1$ there is $\sum x_n$ such that $E(x_n)$ is a Cantorval and $\underline{d}K^c(x_n) = \alpha$ and $\overline{d}K^c(x_n) = \beta$.

Proposition

For every sequence (m_n) of positive integers convergent to ∞ there is an achievable Cantorval $E(x_n)$ such that

$$\lim_{n\to\infty} \frac{card\{i \leqslant n : \ x_i \leqslant r_i\}}{m_n} = 0.$$

In particular, choosing $m_n := n$, we get the classic asymptotic density and the missing part of the previous Proposition.

FPW, Ptak J.*Achievable Cantorvals almost without reversed Kakeya conditions*, arXiv:2412.08768 [math CA] (2024)

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Open Problem

Does for every $S \subset \mathbb{N}$ such that $\mathrm{card}\, S = \mathrm{card}\, S^c = \infty$ *exist a series $\sum x_n$ such that $K^c(x_n) = S$ and $E(x_n)$ is a Cantorval ?*

measure of the boundary

the Ferens Cantorvals: $E(m+k-1,\ m+k-2,\ \ldots,\ m+1,\ m; q)$ where $m, k \in \mathbb{N},\ k > m$

$$s \ := \ \sum_{i=1}^{k-1}(m+i)$$

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Theorem

If $\frac{1}{s-m+1} \leqslant q < \frac{m}{s}$, then $E(m+k-1, m+k-2, \ldots, m+1, m; q)$ is a Cantorval. Moreover, its Lebesgue measure is $\frac{(s-2m)q}{1-3q}$ and equals to the sum of lengths of all E-intervals.

Banakiewicz, M. , Prus-Wiśniowski, F., *M-Cantorvals of Ferens type*, Math. Slovaca 67(4)(2017), 1-12

Theorem

$$\mu(\operatorname{Fr} E(3,2;\frac{1}{4})) \;=\; 0$$

W. Bielas, S. Plewik, M. Walczyńska, *On the center of distances*, Eur. J. Math. 4 (2018), no. 2, 687–698.

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Theorem

$$\mu(\mathrm{Fr}\, E(3,\underbrace{2,2,\ldots,2}_{m\text{-times}};\frac{1}{2m+2})) \;=\; 0$$

M. Banakiewicz, *The Lebesgue measure of some M-Cantorval*, J. Math. Anal. Appl. 471(2019) 170–179

$m = (m_n)$ – a sequence of postitive integers $\geqslant 2$
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the generalized Ferens series $\sum_{i=1}^{\infty} x_i(k, m, q)$:

$\cdots + (m_n + k_n - 1)q_n + (m_n + k_n - 2)q_n + \ldots + (m_n + 1)q_n + m_n q_n + \cdots$

$$s_n := \sum_{i=1}^{k_n - 1} (m_n + i)$$

measure of the boundary

Theorem

If $\sum x_n(m,k,q)$ *is a convergent GF series satisfying the conditions*

$$\forall n \in \mathbb{N} \qquad q_n \leqslant (s_{n+1} - m_{n+1} + 1)q_{n+1} \tag{GF$_1$}$$

and

$$\forall n \in \mathbb{N} \qquad m_n q_n > \sum_{i>n} (s_i + m_i) q_i, \tag{GF$_2$}$$

and then $E(x_n)$ *is a Cantorval.*

Theorem

The non-multigeometric generalized Ferens Cantorvals all have boundaries of measure zero.

J. Marchwicki, P. Nowakowski, FPW, *Algebraic sums of achievable sets involving Cantorvals*, arXiv: 2309.01589v1[math. CA], 4 Sep 2023

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Yet another family of special non-multigeometric Cantorvals used in the proof of the main result in

Prus-Wiśniowski, F., Ptak J.*Achievable Cantorvals almost without reversed Kakeya conditions*, arXiv:2412.08768 [math CA] (2024)

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Theorem

The boundary of every multigeometric Cantorval $E(k_1, \ldots, k_m; q)$ is of Lebesgue measure zero.

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Open Problem

Do there exist achievable Cantorvals with boundary of positive Lebesgue measure?