RELATIONAL GENERALIZED ITERATED FUNCTION SYSTEMS

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Preliminaries and notation

For a metric space (X, d) and $m \in \mathbb{N}$, we consider:

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$$P_{cp}(X) = \{K \subseteq X \mid K \neq \emptyset \text{ and } K \text{ compact}\}$$

- the Hausdorff-Pompeiu metric $h : P_{cp}(X) \times P_{cp}(X) \to [0, \infty)$, given by

$$h(K_1, K_2) = \max\{\sup_{x \in K_1} d(x, K_2), \sup_{x \in K_2} d(x, K_1)\},$$

for every $K_1, K_2 \in P_{cp}(X)$

- the Cartesian product X^m endowed with the maximum metric $d_{\max}$, defined by

$$d_{\max}((x_1, ..., x_m), (y_1, ..., y_m)) = \max\{d(x_1, y_1), ..., d(x_m, y_m)\},$$

for all $x_1, ..., x_m, y_1, ..., y_m \in X$.

Generalized iterated function systems

A `generalized iterated function system` (of order $m \in \mathbb{N}$) - for short a GIFS - is a pair $\mathcal{S} = ((X, d), (f_i)_{i \in I})$, where:

- (X, d) is a complete metric space
- I is a finite set
- $f_i : X^m \to X$ is continuous for each $i \in I$.

The function $F_{\mathcal{S}} : (P_{cp}(X))^m \to P_{cp}(X)$, described by

$$F_{\mathcal{S}}(K_1, ..., K_m) = \bigcup_{i \in I} f_i(K_1 \times ... \times K_m),$$

for all $(K_1, ..., K_m) \in (P_{cp}(X))^m$, is called the `fractal operator` associated with $\mathcal{S}$.

$A \in P_{cp}(X)$ is called an `attractor` of $\mathcal{S}$ provided that

$$F_{\mathcal{S}}(A, ..., A) = A.$$

Each GIFS comprising contractions has attractor

Each GIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I})$ *(of order* m*) comprising contractions has a unique attractor denoted by* $A_{\mathcal{S}}$*.*

In addition, for every $K_1, ..., K_m \in P_{cp}(X)$*, the sequence* $(K_n)_{n\in\mathbb{N}}$ *defined by*

$$K_{n+m} = F_{\mathcal{S}}(K_{n+m-1}, K_{n+m-2}, ..., K_n),$$

for every $n \in \mathbb{N}$*, converges, with respect to* h*, to* $A_{\mathcal{S}}$*.*

A. Mihail and R. Miculescu, Generalized IFSs on Noncompact Spaces, Fixed Point Theory Appl. Volume 2010, Article ID 584215, 11 pages doi: 10.1155/2010/584215.

GIFSs consisting of contractions are extensions of IFSs

For $m = 1$ one obtains the classical theory of iterated function systems (for short IFSs).

This theory

- is one of the main ways to generate self-similar fractal sets

- has applications in various domains such as image compression, engineering sciences, medicine, forestry, economy, human anatomy, physics etc.

J. Hutchinson, Fractals and self similarity, Indiana Univ. Math. J., 30 (1981), 713-747.

M. Barnsley, Fractals everyhere, Academic Press, Boston, 1993.

GIFSs consisting of contractions are effective extensions of IFSs

F. Strobin proved that for any $m \in \mathbb{N}$, $m \geq 2$, there exists a Cantor subset of the plane having the following properties:

- it is the attractor of some GIFS, of order m, comprising contractions

- it is not the attractor of any GIFS, of order $m-1$, comprising contractions.

He also presented an example of a compact set which is not the attractor of any GIFS comprising contractions.

F. Strobin, Attractors of generalized IFSs that are not attractors of IFSs, J. Math. Anal. Appl., 422 (2015), 99-108.

Relational generalized iterated function systems

A `relational generalized iterated function system` (of order 2) - for short an RGIFS - is a triplet $\mathcal{S} = ((X,d),(f_i)_{i\in I},R)$, where:

i) (X,d) is a complete metric space

ii) I is a finite set

iii) R is a closed equivalence relation on X

iv) $f_i : X^2 \to X$ is continuous for every $i \in I$

v) there exists $c \in [0,1)$ such that

$$d(f_i(x_1,y_1),f_i(x_2,y_2)) \leq c\max(d(x_1,x_2),d(y_1,y_2)),$$

for every $x_1,x_2,y_1,y_2 \in X$ such that $x_1 R x_2$ and $y_1 R y_2$ and every $i \in I$

vi)

$$f_i([x]\times[x]) \subseteq [x],$$

for every $x \in X$ and every $i \in I$.

Notation

Given an RGIFS $\mathcal{S} = ((X, d), (f_i)_{i \in I}, R)$ and $K_1, K_2 \in P_{cp}(X)$, we use the following notation:

-
$$[K_1] = \{[x] \mid x \in K_1\}$$

-
$$K_1 \overset{R}{\sim} K_2 \text{ if } [K_1] = [K_2]$$

-
$$\mathcal{M}_R = \{(K_1, K_2) \in P_{cp}(X) \times P_{cp}(X) \mid K_1 \overset{R}{\sim} K_2\}.$$

Note that $K_1 \overset{R}{\sim} K_2$ if and only if $(K_1 \cap [x] \neq \emptyset \Longleftrightarrow K_2 \cap [x] \neq \emptyset)$ for every $x \in X$.

The fractal operator associated with an RGIFS

Given an RGIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I},R)$, we can consider:

- the function $F_{\mathcal{S}} : \mathcal{M}_R \to P_{cp}(X)$, described by

$$F_{\mathcal{S}}(K_1,K_2) = \bigcup_{i\in I, x\in X} f_i((K_1 \cap [x]) \times (K_2 \cap [x])),$$

for all $(K_1,K_2) \in \mathcal{M}_R$, which is called the `fractal operator` associated with $\mathcal{S}$

- the function $G_{\mathcal{S}} : P_{cp}(X) \to P_{cp}(X)$, described by

$$G_{\mathcal{S}}(K) = F_{\mathcal{S}}(K,K),$$

for all $K \in P_{cp}(X)$.

In addition, $A \in P_{cp}(X)$ is called an `attractor` of $\mathcal{S}$ provided that

$$F_{\mathcal{S}}(A,A) = A.$$

$G_{\mathcal{S}}$ is weakly Picard

Given an RGIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I},R)$, for any $K \in P_{cp}(X)$, there exists $A_K \in P_{cp}(X)$ such that

$$F_{\mathcal{S}}(A_K, A_K) = A_K,$$

i.e.

$$G_{\mathcal{S}}(A_K) = A_K$$

and

$$\lim_{n\to\infty} G_{\mathcal{S}}^{[n]}(K) = A_K,$$

where $G_{\mathcal{S}}^{[n]} = \underbrace{G_{\mathcal{S}} \circ ... \circ G_{\mathcal{S}}}_{n \text{ times}}$.

Hence A_K is an attractor of $\mathcal{S}$.

Moreover, we have

$$h(G_{\mathcal{S}}^{[n]}(K), A_K) \leq \frac{c^n}{1-c} diam(K \cup \bigcup_{i\in I} f_i(K \times K)),$$

for every $n \in \mathbb{N}$.

I. Abraham, R. Miculescu and A. Mihail, Relational generalized iterated function systems, Chaos, Solitons & Fractals, 182

About the proof of the previous result

The main idea of the proof is to show that $(G_{\mathcal{S}}^{[n]}(K))_{n\in\mathbb{N}}$ is convergent, via the following inequality:

$$h(G_{\mathcal{S}}^{[n]}(K), G_{\mathcal{S}}^{[n+1]}(K)) \leq c^n diam(K \cup \bigcup_{i\in I} f_i(K \times K)),$$

for every $n \in \mathbb{N}$.

The justification of the above inequality is very technical as it makes use of the machinery of the code space for GIFSs introduced by F. Strobin.

F. Strobin, J. Swaczyna, A code space for a generalized IFS, Fixed Point Theory, 17 (2016), 477-494.

The sequence generated by $F_{\mathcal{S}}$

Given an RGIFS $S = ((X,d),(f_i)_{i\in I},R)$, *for any* $K_1, K_2 \in P_{cp}(X)$ *such that* $K_1 \overset{R}{\sim} K_2$, *the sequence* $(K_n)_{n\in\mathbb{N}}$, *described by*

$$K_{n+2} = F_{\mathcal{S}}(K_{n+1}, K_n),$$

for every $n \in \mathbb{N}$, *is convergent and*

$$\lim_{n\to\infty} K_n = A_{K_1} = A_{K_2}.$$

I. Abraham, R. Miculescu and A. Mihail, Relational generalized iterated function systems, Chaos, Solitons & Fractals, 182 (2024), 114823.

A_K is determined by $A_{\{x\}}$, $x \in K$

Given an RGIFS $S = ((X, d), (f_i)_{i \in I}, R)$, *we have*

$$A_K = \bigcup_{x \in K} A_{\{x\}},$$

for any $K \in P_{cp}(X)$.

I. Abraham, R. Miculescu and A. Mihail, Relational generalized iterated function systems, Chaos, Solitons & Fractals, 182 (2024), 114823.

Supplementary properties

Given an RGIFS $S = ((X, d), (f_i)_{i \in I}, R)$, *we have:*

i) the function $f : P_{cp}(X) \to P_{cp}(X)$, *described by*

$$f(K) = A_K,$$

for any $K \in P_{cp}(X)$, *is continuous.*

ii)

$$[A_K] = [K],$$

for any $K \in P_{cp}(X)$.

iii)

$$A_{K_1} = A_{K_2} \text{ if and only if } [K_1] = [K_2],$$

for all $K_1, K_2 \in P_{cp}(X)$.

I. Abraham, R. Miculescu and A. Mihail, Relational generalized iterated function systems, Chaos, Solitons & Fractals, 182 (2024), 114823.

GIFSs are particular cases of RGIFSs

Let us consider the RGIFS $\mathcal{S} = ((X, d), (f_i)_{i \in I}, R)$, where

$$R = X \times X.$$

Then

$$[x] = X,$$

for every $x \in X$, so $\mathcal{S}$ turns out to be a GIFS.

RGIFSs are real extensions of GIFSs

Let us consider the RGIFS $\mathcal{S} = ((\mathbb{R}^2, \|.\|), (f_i)_{i\in\{1,2\}}, R)$, where:
- $f_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ are given as follows:

$$f_1((x_1, y_1), (x_2, y_2)) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{4})$$

and

$$f_2((x_1, y_1), (x_2, y_2)) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{4} + g(x_1, x_2)),$$

for any $x_1, x_2, y_{1,} y_2 \in \mathbb{R}$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function

-

$$R = \{((x, y_1), (x, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x, y_1, y_2 \in \mathbb{R}\},$$

i.e.

$$(x_1, y_1)R(x_2, y_2) \text{ if and only if } x_1 = x_2,$$

for any $x_1, x_2, y_{1,} y_2 \in \mathbb{R}$.

Indeed, we have $[(x, y)] = \{(x, u) \mid u \in \mathbb{R}\}$ for all $(x, y) \in \mathbb{R}^2$ and

$$\|f_i((x, y_1), (z, v_1)) - f_i((x, y_2), (z, v_2))\| \leq$$

$$\leq \frac{1}{2} d_{\max}(((x, y_1), (z, v_1)), ((x, y_2), (z, v_2))),$$

for all $x, z, y_1, y_2, v_1, v_2 \in \mathbb{R}$ and $i \in \{1, 2\}$.

$\mathcal{S}$ is not a GIFS, since f_1 is not a contraction because

$$\|f_1((1, 0), (2, 1)) - f_1((0, 0), (1, 1))\| = 1$$

and

$$\max\{\|(1, 0) - (0, 0)\|, \|(2, 1) - (1, 1)\|\} = \|(1, 0)\| = 1.$$

Note that

$$A_{K_1 \times K_2} = \bigcup_{x \in K_1} (\{x\} \times [0, 2g(x, x)]),$$

for any $K_1, K_2 \in P_{cp}(\mathbb{R})$.

t - relational generalized iterated function systems

A t - `relational generalized iterated function system` (of order 2) - for short a tRGIFS - is a quadruple $\mathcal{S} = ((X, d), (f_i)_{i \in I}, R, t)$, where:

i) (X, d) is a complete metric space
ii) I is a finite set
iii) R is a closed equivalence relation on X
iv) $f_i : X \times X \to X$ is continuous for every $i \in I$
v) there exists $c \in [0, 1)$ such that

$$d(f_i(x_1, y_1), f_i(x_2, y_2)) \leq c \max(d(x_1, x_2), d(y_1, y_2)),$$

for every $x_1, x_2, y_1, y_2 \in X$ such that $x_1 R x_2$ and $y_1 R y_2$ and every $i \in I$

vi) $t : X \times X \to X$ is continuous function such that

$$t(x, x) = x,$$

for every $x \in X$

vii)

$$f_i([x] \times [y]) \subseteq [t(x, y)],$$

for every $x, y \in X$ and every $i \in I$

viii)

$$\overline{\bigcup_{n \in \mathbb{N}} K_n} \in P_{cp}(X),$$

for every sequence $(K_n)_{n \in \mathbb{N}}$, given by

$$K_1 \in P_{cp}(X)$$

and

$$K_{n+1} = t(K_n \times K_n),$$

for every $n \in \mathbb{N}$.

Note that vi) and vii) imply vi) from the definition of an RGIFS (i.e. $f_i([x] \times [x]) \subseteq [x]$ for every $x \in X$ and every $i \in I$).

The fractal operator associated with a tRGIFS

Given a tRGIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I},R,t)$, we can consider:

- the `fractal operator` associated with $\mathcal{S}$, i.e. the function $F_{\mathcal{S}} : P_{cp}(X) \times P_{cp}(X) \to P_{cp}(X)$, described by

$$F_{\mathcal{S}}(K_1,K_2) = \underset{i\in I}{\cup} f_i(K_1 \times K_2),$$

for all $K_1, K_2 \in P_{cp}(X)$

- the function $G_{\mathcal{S}} : P_{cp}(X) \to P_{cp}(X)$, described by

$$G_{\mathcal{S}}(K) = F_{\mathcal{S}}(K,K),$$

for all $K \in P_{cp}(X)$.

In addition, $A \in P_{cp}(X)$ is called an `attractor` of $\mathcal{S}$ provided that

$$F_{\mathcal{S}}(A,A) = A.$$

The sequence $(H_n)_{n\in\mathbb{N}}$ associated with $K \in P_{cp}(X)$

Given a tRGIFS $\mathcal{S} = ((X, d), (f_i)_{i\in I}, R, t)$, for each $x \in X$, we can consider the GIFS

$$\mathcal{S}_x = (([x], d), (\widetilde{f}_i)_{i\in I}),$$

where $\widetilde{f}_i : [x] \times [x] \to [x]$ is given by $\widetilde{f}_i(u) = f_i(u)$ for all $u \in [x] \times [x]$ and $i \in I$, which has a unique attractor

$$A_{\mathcal{S}_x} \overset{not}{=} A_x.$$

For $K \in P_{cp}(X)$, the sequence $(H_n)_{n\in\mathbb{N}} \subseteq P_{cp}(X)$, given by

$$H_1 = \bigcup_{x\in \overline{\bigcup_{n\in\mathbb{N}} K_n}} A_x$$

and

$$H_{n+1} = H_n \cup G_{\mathcal{S}}(H_n),$$

for every $n \in \mathbb{N}$, where $K_1 = K$, turns out to be convergent and

$$\lim_{n\to\infty} H_n \overset{not}{=} A_K \in P_{cp}(X).$$

$G_{\mathcal{S}}$ is weakly Picard

For each tRGIFS $\mathcal{S} = ((X, d), (f_i)_{i \in I}, R, t)$ *and* $K \in P_{cp}(X)$, *we have*

$$F_{\mathcal{S}}(A_K, A_K) = A_K,$$

i.e.

$$G_{\mathcal{S}}(A_K) = A_K$$

and

$$\lim_{n \to \infty} G_{\mathcal{S}}^{[n]}(K) = A_K.$$

Hence A_K *is an attractor of* $\mathcal{S}$.

Supplementary properties

Given a tRGIFS $S = ((X, d), (f_i)_{i \in I}, R, t)$*:*

i)

$$A_{K_1} \subseteq A_{K_2},$$

for all $K_1, K_2 \in P_{cp}(X)$ *such that* $K_1 \subseteq K_2$.

ii)

$$A_{K_1} = A_{K_2},$$

for all $K_1, K_2 \in P_{cp}(X)$ *such that* $[K_1] = [K_2]$.

iii)

$$[A_K] = [\overline{\bigcup_{n \in \mathbb{N}} K_n}],$$

for any $K \in P_{cp}(X)$.

GIFSs are particular cases of tRGIFSs

Let us consider the tRGIFS $\mathcal{S} = ((X, d), (f_i)_{i \in I}, R, t)$, where

$$R = X \times X$$

and $t : X \times X \to X$ is given by

$$t(x, y) = x,$$

for every $x, y \in X$.

Then:

-

$$[x] = X,$$

for every $x \in X$

- for every $K \in P_{cp}(X)$, we have

$$K_n = K,$$

for every $n \in \mathbb{N}$, so $\mathcal{S}$ turns out to be a GIFS.

An example

Let us consider the tRGIFS $\mathcal{S} = ((\mathbb{R}^2, \|.\|), (f_i)_{i\in\{1,2\}}, R, t)$, where:
- $f_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ are given as follows:

$$f_1((x_1, y_1), (x_2, y_2)) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{4})$$

and

$$f_2((x_1, y_1), (x_2, y_2)) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{4} + g(x_1, x_2)),$$

for any $x_1, x_2, y_{1,} y_2 \in \mathbb{R}$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function

-
$$R = \{((x, y_1), (x, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x, y_1, y_2 \in \mathbb{R}\},$$

i.e.

$$(x_1, y_1)R(x_2, y_2) \text{ if and only if } x_1 = x_2,$$

for any $x_1, x_2, y_{1,} y_2 \in \mathbb{R}$.

- $t : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$t((x_1, y_1), (x_2, y_2)) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

for any $x_1, x_2, y_{1,} y_2 \in \mathbb{R}$.

Then

$$A_{(x,y)} = \{x\} \times [0, 2g(x, x)],$$

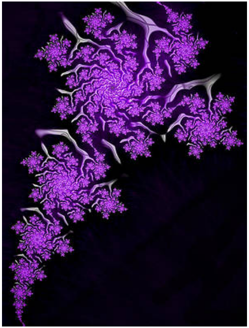
for all $(x, y) \in \mathbb{R}^2$.

For $K = \{(x_1, y_1), (x_2, y_2)\}$, where $x_1 < x_2$, we have:

-
$$\overline{\bigcup_{n \in \mathbb{N}} K_n} = [(x_1, y_1), (x_2, y_2)]$$

- A_K is the hypograph of a function $h : [x_1, x_2] \to [0, \infty)$.

THANK YOU FOR YOUR ATTENTION!



Purple and black fractal floral art

Matthias Hauser