Fractals and the Buffon Circle Problem

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Projections detect information about the size, geometric arrangement, and dimension of sets.

We focus on 1-dimensional sets of finite length and the shadows or projections of such sets.

Figure: continuous curve

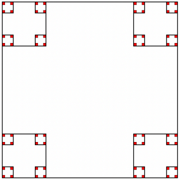
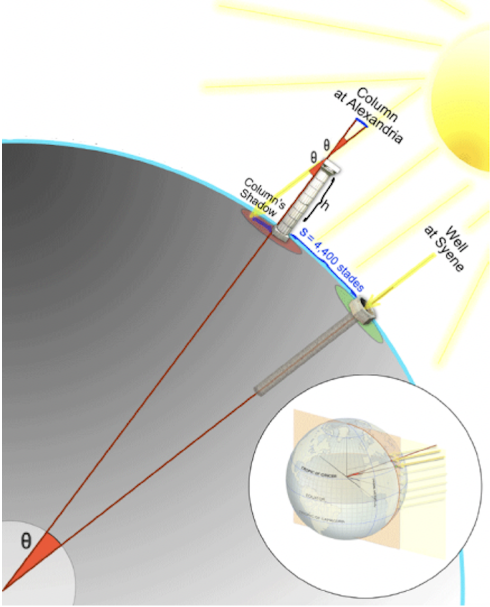


Figure: dust set

theme: highly scattered sets have no shadow in almost all directions.



- The importance of shadows and projections dates back to ancient times.
- The Greek mathematician, Eratosthenes of Alexandria, used angle measurements and the sun's shadows to compute the circumference of the Earth
- Fast forward over 2,000 years, what information can we obtain from the shadows of highly un-smooth objects?

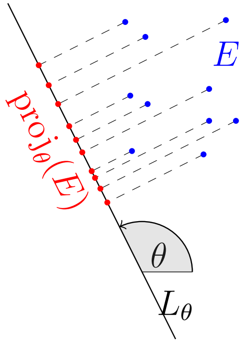
A single projection may not tell the full story

Theorem 1 (Marstrand)

Let $E \subset \mathbb{R}^2$ be a Borel set. Then for Lebesgue a.a. $\theta \in [0, \pi)$:

(i) $\dim_{\mathrm{H}} E \leq 1 \Longrightarrow \dim_{\mathrm{H}} (\mathrm{proj}_\theta E) = \dim_{\mathrm{H}} E$;

(ii) $\dim_{\mathrm{H}} E > 1 \Longrightarrow \mathcal{L}\mathrm{eb}\,(\mathrm{proj}_\theta E) > 0$.



- main tools: characterizations of Hausdorff dimension; push-forward measures
- The case when $s = 1$ falls outside the scope of Marstrand's theorem

Key tools: energies and pushforwards

There are two key tools in proving Marstrand's theorem

The *s*-**energy** of a measure μ is

$$I_s(\mu) = \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|^s}.$$

This measures *distribution* of the mass of μ.

If μ is a measure and f is a function, the **pushforward** measure is

$$f_\sharp\mu(A) = \mu(f^{-1}A).$$

Big idea: link the energy of a pushforward to the energy of the original set

Rectifiability is used to describe the structure (regularity) of a set or measure similar to the way that the degree of differentiability of charts is used to describe the *smoothness* of a map or manifold.

Figure: rectifiable

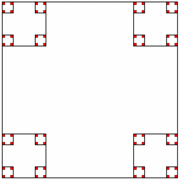


Figure: purely 1-unrectifiable

- Let E denote a set of positive and finite 1-dimensional Hausdorff measure:
  E is **purely unrectifiable** if $\mathcal{H}^1(E \cap \Gamma) = 0$ for all Lipschitz curves Γ.
- Hausdorff dimension cannot detect the difference. Average projection length can

Theorem 2 (Besicovitch)

If $E \subset \mathbb{R}^2$ has positive and finite length, then

$$|\mathrm{proj}_\theta(E)| = 0 \text{ a.e. } \theta \text{ if and only if } E \text{ is purely unrectifiable}.$$

Said differently,

$$\int_0^\pi |\mathrm{proj}_\theta(E)| d\theta = 0 \iff E \text{ is purely unrectifiable}.$$

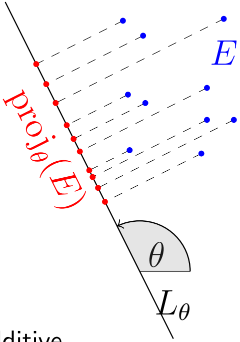
So, the key geometric property that projections detect is *rectifiability*.

Key idea in proof: purely unrectifiable sets have tangents almost nowhere

The *Favard length*

$$\mathrm{Fav}(E) = \frac{1}{\pi}\int_0^{\pi} |proj_\theta(E)|\, d\theta$$

is the average length of a sets orthogonal projections.



- it is translation invariant, rotation invariant, and subadditive,
- has $\mathrm{Fav}(\lambda E) = \lambda\, \mathrm{Fav}(E)$,
- has $\mathrm{Fav}(E) \leq \mathcal{H}^1(E)$ (so sets of dimension $s < 1$ have Favard length zero),
- but for unrectifiable sets, this inequality is strict.

Probabilistic interpretation: It is comparable to the **Buffon needle probability**: the probability that a thin needle dropped near the set intersects the set.

Favard length problem

The four corner Cantor set (or Garnett set), $\mathcal{K} = \cap K_n$, is an **example** of a purely unrectifiable set with positive and finite length.

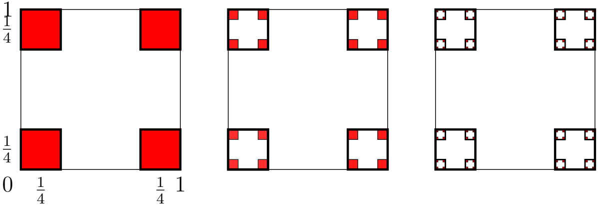


Figure: The first three generations in the construction, K_1, K_2, and K_3

- Consequence of Besi. projection theorem: $\text{Fav}(\mathcal{K}) = 0.$
- The *Favard length problem* asks for more quantitative information: determine the exact rate at which $Fav(K_n)$ decays.
- Peres and Solomyak proved preliminary upper bounds, and Mattila used energy techniques to show that $Fav(K_n) \gtrsim n^{-1}$.
- Subsequent work has achieved the tighter bounds

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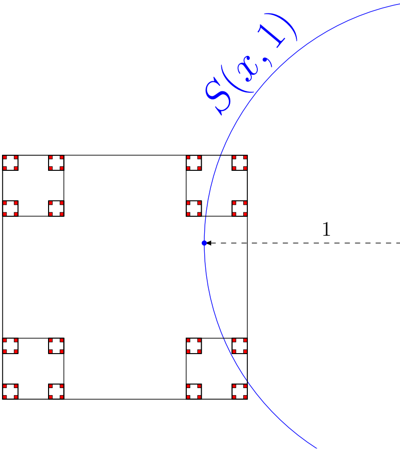
$$\frac{\log n}{n} \lesssim Fav(K_n) \lesssim \frac{1}{n^{1/6-\delta}},$$

where the lower bound is due to Bateman and Volberg, and the upper bound is due to Nazarov, Peres, and Volberg and holds for any $\delta > 0$.

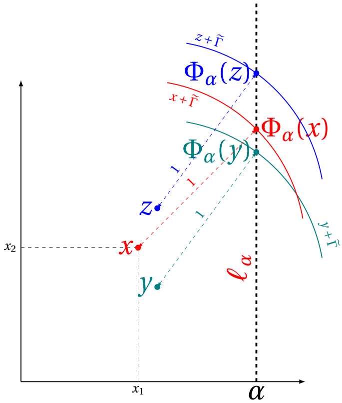
Additional works with four-corner replaced by:
1-dimensional Sierpinski gasket (Bond, Volberg 2010),
rational product Cantor sets (Bond, Łaba, Volberg 2014, Łaba, Marshall),
product Cantor sets with at least one projection of positive 1-dimensional Lebesgue measure (Łaba, Zhai 2010),
and random Cantor sets (Zhang 2018).
also see Łaba'a survey and December '24 Notices article.

Buffon Circle Problem

- Recall, the Favard length is equivalent to the Buffon needle probability. Let's replace needles (lines) by circles or other curves.
- The **Buffon circle problem** concerns the probability that a circle of fixed radius dropped near the four corner set intersects the four corner set.
- Recall that Besi's theorem implies that $\mathrm{Fav}(\mathcal{K}) = 0$
- In joint work with K. Simon, we prove that $Fav_\Gamma(K) = 0$.
- In this talk, we investigate the decay of $Fav_\Gamma(K_n)$.



Favard curve length



- Replace the family of orthogonal projections, $\{\mathrm{proj}_\theta\}$, with a family of curved projection-like maps, $\{\phi_\alpha\}$
- Now, foliate the plane by vertical lines ℓ_α
- Define the *curved* projection map by

$$\phi_\alpha(x) = (x + \Gamma) \cap \ell_\alpha$$

- The **Favard curve length** is defined by

$$\mathrm{Fav}_\Gamma(E) = \int |\phi_\alpha(E)| \, d\alpha.$$

- These maps also obey Marstrand and Besi's theorems

Best known results on Favard curve length

Let K_n denotes the n-th stage in the construction of the four-corner Cantor set.

- We obtain an upper bound in-line with NPV's result in the classic setting:

Theorem 3 (Cladek, Davey, T.)

$$Fav_{\Gamma}(K_n) \lesssim n^{\epsilon - 1/6}$$

- By generalizing the energy-techniques of Mattila, we obtain the lower bound:

Theorem 4 (Bongers, T.)

$$\frac{1}{n} \lesssim Fav_{\Gamma}(K_n)$$

- The proof of the latter theorem does not depend on self-similarity and can be generalized to much larger classes of transversal mappings.

- **proof of upper bound:** The key idea is to relate the circular projections to the classical projections on small collections of squares.
- **Set-up:** Let K_n denote the the n-th generation of the construction: made up of 4^n squares of side length $4^{-n} := \delta$
- We want to understand $\phi_\alpha(K_n)$. We show that for a small enough piece of K_n, there exists $\theta \in S^1$ so that $|\Phi_\alpha(\text{piece}))|$ is comparable to $|\mathrm{proj}_\theta(\text{piece})|$.
- Consider ancestor $K_{n/2}$: made up of $4^{n/2} = 2^n$ squares, $\{Q_j\}$ of side length $2^{-n} := \sqrt{\delta}$. Write $K_{n/2} = \bigsqcup_{j=1}^{4^{n/2}} Q_j$
- **Group squares together/ take advantage (but not too much advantage) of subadditivity:** Define $\tilde{Q}_j = K_n \cap Q_j$, a shifted and $\sqrt{\delta}-$rescaled copy of $K_{n/2}$.

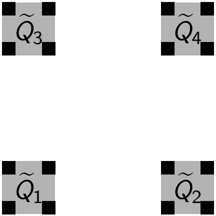
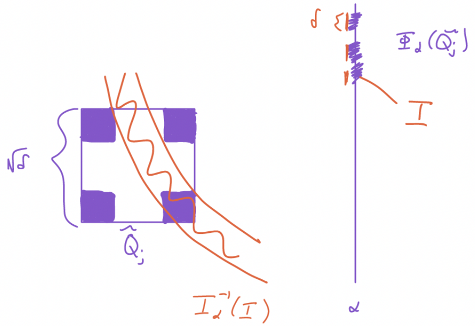


Figure: n=2: K_1 consists of 4 grey Q_j



- **Cover by tubes:** Cover the image of $\tilde{Q}_j$ by δ-intervals. (Let N denote the number of intervals needed). Now $|\Phi_\alpha(\tilde{Q}_j)| \sim N\delta,$ and it suffices to estimate N.
- **linearize:** The pre-image of each δ-interval is contained in a $\delta x \sqrt{\delta}$ tube in $\tilde{Q}_j$. (a Vitaly-covering argument assures tubes are essentially disjoint)
- **The angle comparison:** Choose an angle s.t. the orthogonal projection of the tubes in this carefully chosen angle is approximately the same size as that of the curved projection: $|\mathrm{proj}_{\theta(\alpha)}(\tilde{Q}_j)| \sim N\delta$.

- **use self-similarity:** Since $\tilde{Q}_j$ is a shifted and $\sqrt{\delta}-$rescaled copy of $K_{n/2}$, self-similarity yields $|\mathrm{proj}_{\theta(\alpha)}(K_{n/2})| \sim N\sqrt{\delta}$.
- Subadditivity combined with the above yields $|\Phi_\alpha(K_n)| \lesssim \sum_{k=1}^{4^{n/2}} |\Phi_\alpha(\tilde{Q}_j)| \lesssim N\delta^{1/2} \sim |\mathrm{proj}_{\theta(\alpha)}(K_{n/2})|$
- Integrating in α and applying the NPV bound completes the argument

NPV sketch: The counting (or stacking) function

- Fix θ and n.
- Recall K_n consists of 4^n squares, so that $Proj_\theta(K_n)$ is a union of 4^n intervals possibly with much overlap.
- Define $f_{n,\theta}$ as sum of characteristic functions of intervals:
  $f_{n,\theta}(x) =$ the number of squares of K_n which project onto x.
- A standard technique in analysis: upper bound on L^2 yields lower bound on the support via Cauchy-Schwarz

$$1 = \left(\int_{Proj_\theta(\mathcal{K}_n)} f_{n,\theta} \right)^2 \leq \left(\int_{\mathbb{R}} f_{n,\theta}^2 \right) |Proj_\theta(\mathcal{K}_n)|$$

  So, if θ satisfies $\int_{\mathbb{R}} f_{n,\theta}^2 \leq K$, then $\frac{1}{K} \leq |ProjR_\theta(\mathcal{K}_n)|$.
- A small L_2 norm, however, means that there is "not enough stacking". When seeking an **upper bound**, we need many θ to have a small projection.
- A Fourier analytic argument shows that the L^2 norm of $f_{n,\theta}$ is large for typical θ, and a combinatorial argument is used to handle the exceptional θ.

stacking: small projection: big L^2 norm

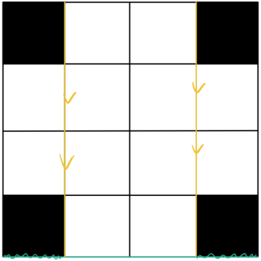


Figure: stacking

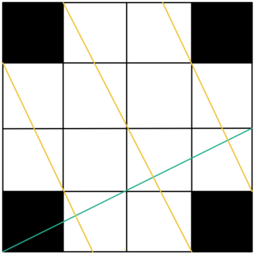


Figure: no stacking; $\theta = \arctan 1/2$

- $f_{n,\theta}(x) = 2^n$ for $x \in \pi_0(K_n)$
- $\pi_0(K_n)$ consists of 2^n intervals of length $\frac{1}{2^n}$
- $\int f_{n,\theta}^2 = \sum_{i=1}^{2^n} \int_I 2^{2n} dt = 4^n$

- $f_{n,\theta}(x) = 1$ for $x \in \pi_\theta(K_n)$
- $\pi_\theta(K_n)$ consists of 4^n intervals of length $\sim \frac{1}{2^n}$
- $\int f_{n,\theta}^2 = \sum_{i=1}^{2^n} \int_I 1 dt = 1$

Further work: Coverings and prescribed projections

- **Davies Theorem**: Let A be a compact set in $\mathbb{R}^2$. Then, there exists a collection of full lines $\mathcal{L}$ so that

$$A \subset \bigcup_{\ell \in \mathcal{L}} \ell$$

- proved using a Perron-tree like construction with rectangles. We cover the set by longer and skinnier rectangles with mass concentrating near A. Alternatively, Davies theorem follows from:
- **A prescribed projection theorem** (Falconer's sundial): For each $\theta \in [0, \pi)$, let $A_\theta \subset \mathbb{R}$ so that $\{(\theta, A_\theta) : \theta \in [0, \pi)\}$ is measurable. Then there exists a set $E \subset \mathbb{R}^2$ so that:
  (i) $\mathrm{proj}_\theta(E) \supset A_\theta$ and (ii) $|\mathrm{proj}_\theta(E)| = |A_\theta|$.
- In joint work with Chang and McDonald, we prove nonlinear versions for semi-circles (to appear in Analysis and PDE) stay tuned for the story for full circles...

Thank you for your attention

Figure: *boy stacking problem*

Summary & Further Research

- The Favard length averages the lengths of the orthogonal projections of a set. It is equivalent to the Buffon needle probability and can be used to detect rectifiability or lack thereof. Favard length can be formulated in higher dimensions and for more general families of projection operators
- The Favard curve length gives a nonlinear formulation for curve projections.
  $Fav_\Gamma(E) \sim |E + \Gamma| \sim$ (the Buffon curve probability of E)
  The best-known bounds for $Fav_\Gamma(K_n)$ agree with those for the classic $Fav(K_n)$ (up to a log)
- Our upper bound exploits subadditivity and self-similarity, and we linearize on small scales. Our lower bound uses Mattila's energy techniques and generalizes beyond curve projection operators to transversal families
- Beyond the four-corner set, one can study a quantitative Besicovitch projection theorem We are investigating: (1) a quantitative two projection theorem (with Z. Li); (2) upper bounds for Fav_Γ beyond the four corner set to include rotations in the construction of the set and more general mappings (A. McDonald and C. Marshall); (3) reverse or prescribed projection or covering problems