Cantor sets and their algebraic differences

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• For $n \in \mathbb{N}$, $s \in \{0,1\}^n$ let P_s be an open interval concentric with I_s and such that $|P_s| = a_{n+1}|I_s|$.

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$$I_s \setminus P_s = I_{s0} \cup I_{s1}.$$

$$C_n(a) = \bigcup_{s \in \{0,1\}^n} I_s,$$

$$C(a) := \bigcap_{n \in \mathbb{N}} C_n(a).$$

$C(a)$ is called a central Cantor set.

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Every nonempty, compact, perfect and nowhere dense subset of $\mathbb{R}$ is called a *Cantor set*.

Definition

We say that a nonempty perfect set $E \subset \mathbb{R}$ is an M-Cantorval (a Cantorval) if it is not an interval and both endpoints of all gaps are accumulation points of other gaps and accumulation points of non-trivial components of E.

Theorem [Anisca, Ilie, 2001]

For any $a \in (0,1)^{\mathbb{N}}$, the set $C(a) - C(a)$ has one of the following forms:

1) a finite union of closed intervals;
2) a Cantor set;
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Theorem [Anisca, Ilie, 2001], [Sannami, 1992]

Let $a = (a_n) \in (0,1)^{\mathbb{N}}$. Then $C(a) - C(a)$ is:

(1) the interval $[-1, 1]$ if and only if $a_n \leqslant \frac{1}{3}$ for all $n \in \mathbb{N}$;

(2) a finite union of intervals if and only if the set $\{n \in \mathbb{N} \colon a_n > \frac{1}{3}\}$ is finite;

(3) a Cantor set if the set $\{n \in \mathbb{N} \colon a_n \leqslant \frac{1}{3}\}$ is finite.

Basic assumptions and additional notation

Assume that $a = (a_j)_{j\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ is a sequence such that: $a_n > \frac{1}{3}$ for infinitely many terms, $a_n \leqslant \frac{1}{3}$ for infinitely many terms, and $k_0 \in \mathbb{N}\cup\{0\}$ is such that $a_{k_0+1} < \frac{1}{3}$. Let (k_n) be the sequence of all indices greater than k_0, for which $a_{k_n} > \frac{1}{3}$. Denote

$$\delta_n := \min\{3d_i - d_{i-1} \colon i \in \{k_{n-1}+1, \ldots, k_n - 1\}\},$$

$$\Delta_n := \max\{3d_i - d_{i-1} \colon i \in \{k_{n-1}+1, \ldots, k_n - 1\}\},$$

where $\max\emptyset = -\infty, \min\emptyset = \infty$.

Theorem 1 [Filipczak, N., 2023]

Let $a \in (0,1)^{\mathbb{N}}$ satisfy basic assumptions. Put

$$m'_n := \min\{\delta_{n-1} - (d_{k_n-1} - d_{k_n}), 4d_{k_n} - \Delta_{n-1}, \delta_n\}$$

$$M'_n := \max\{\delta_{n-1} - (d_{k_n-1} - d_{k_n}), 4d_{k_n} - \Delta_{n-1}, \Delta_n\}$$

for $n \in \mathbb{N}$. If for any $n \in \mathbb{N}$ we have

$$m'_n = M'_n = \sum_{i=n}^{\infty} (d_{k_i-1} - d_{k_i}),$$

then the set $C(a) - C(a)$ is a Cantorval.
Moreover, if $k_0 = 0$, then

$$|C(a) - C(a)| = 2 - 2\sum_{n=1}^{\infty} 3^{n-1}\left(d_{k_n-1} - 3d_{k_n}\right).$$

Definition

Let $x = (x_j)_{j\in\mathbb{N}}$ be a nonincreasing sequence of positive numbers such that the series $\sum_{j=1}^{\infty} x_j$ is convergent. The set

$$E(x) := \left\{ \sum_{j\in A} x_j : A \subset \mathbb{N} \right\}$$

(where $\sum_{j\in\emptyset} x_j := 0$) of all subsums of $\sum_{j=1}^{\infty} x_j$ is called *the achievement set* of x.

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Notation

$S := \sum_{j=1}^{\infty} x_j$,
$r_n := \sum_{j=n+1}^{\infty} x_j$.

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Definition

If $x_n > r_n$ for $n \in \mathbb{N}$, then the series is called *fast convergent*.

Proposition

The following conditions hold.

1. If $a = (a_j)_{j\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$, $\lambda_n = \frac{1-a_n}{2}$ for $n \in \mathbb{N}$, then the series $\sum_{j=1}^{\infty} x_j$ given by the formula

$$x_1 = 1 - \lambda_1 \quad \text{and} \quad x_j = \lambda_1 \cdot \ldots \cdot \lambda_{j-1} \cdot (1 - \lambda_j) \text{ for } j > 1,$$

is fast convergent, $S = 1$ and $C(a) = E(x)$.

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1. If $a = (a_j)_{j\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$, $\lambda_n = \frac{1-a_n}{2}$ for $n \in \mathbb{N}$, then the series $\sum_{j=1}^{\infty} x_j$ given by the formula
$$x_1 = 1 - \lambda_1 \quad \text{and} \quad x_j = \lambda_1 \cdot \ldots \cdot \lambda_{j-1} \cdot (1-\lambda_j) \text{ for } j > 1,$$
is fast convergent, $S = 1$ and $C(a) = E(x)$.
2. If a series $\sum_{j=1}^{\infty} x_j$ is fast convergent and $a_j = \frac{x_j - r_j}{r_{j-1}}$ for $j \in \mathbb{N}$, then $(a_j)_{j\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ and $E(x) = S \cdot C(a)$.

Theorem 2 [Filipczak, N., 2023]

Let $(k_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers such that $k_1 > 1$ and the set $\mathbb{N} \setminus \{k_n : n \in \mathbb{N}\}$ is infinite. Put

$$x_j := \begin{cases} \frac{2}{3^{j-1}} & \text{if} \quad j \in \{k_n : n \in \mathbb{N}\} \\ \frac{1}{3^{j-1}} & \text{if} \quad j \notin \{k_n : n \in \mathbb{N}\} \end{cases},$$

$S := \sum_{j=1}^{\infty} x_j$, $r_n := \sum_{j=n+1}^{\infty} x_j$, and $a_n := \frac{x_n - r_n}{r_{n-1}}$ for $n \in \mathbb{N}$. The following conditions hold.

1. The sequence $a = (a_n)$ is the only sequence satisfying the assumptions of Theorem 1 with $k_0 = 0$ such that $a_n > \frac{1}{3}$ if and only if $n = k_i$ for some $i \in \mathbb{N}$.
2. The set $E(x) - E(x)$ is a Cantorval and $E(x) = S \cdot C(a)$.
3. $|E(x) - E(x)| = 3$.

Example 1

Assume that $k_n = 2n$ and the sequences x and a are defined as in Theorem 2. Then for $n \in \mathbb{N}$ $a_{2n-1} = \frac{1}{15}$, $a_{2n} = \frac{11}{21}$, $C(a) - C(a)$ is a Cantorval and $\left|C\left(a\right) - C\left(a\right)\right| = \frac{8}{5}$.

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Example 2

Assume that $k_n = 3n$ and the sequences x and a are defined as in Theorem 2. Then for $n \in \mathbb{N}$ $a_{3n-2} = \frac{5}{21}$, $a_{3n-1} = \frac{1}{12}$, $a_{3n} = \frac{19}{33}$, $C(a) - C(a)$ is a Cantorval and $|C(a) - C(a)| = \frac{13}{7}$.

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Example 2

Assume that $k_n = 3n$ and the sequences x and a are defined as in Theorem 2. Then for $n \in \mathbb{N}$ $a_{3n-2} = \frac{5}{21}$, $a_{3n-1} = \frac{1}{12}$, $a_{3n} = \frac{19}{33}$, $C(a) - C(a)$ is a Cantorval and $|C(a) - C(a)| = \frac{13}{7}$.

Example 3

Assume that $(k_n) = (2, 3, 5, 6, 8, 9, \ldots)$ and the sequences x and a are defined as in Theorem 2. Then for $n \in \mathbb{N}$ $a_{3n-2} = \frac{1}{51}$, $a_{3n-1} = \frac{29}{75}$, $a_{3n} = \frac{35}{69}$, $C(a) - C(a)$ is a Cantorval and $|C(a) - C(a)| = \frac{26}{17}$.

Theorem 3

Suppose that $a = (a_n) \in (0,1)^{\mathbb{N}}$ satisfies basic assumptions. Put

$$m_n := \min\{\delta_n - (d_{k_n-1} - d_{k_n}), 4d_{k_n} - \Delta_n\}$$

for $n \in \mathbb{N}$. If for any $n \in \mathbb{N}$ we have

$$m_n \geqslant 2 \cdot \sum_{i=n+1}^{\infty} (d_{k_i-1} - d_{k_i}),$$

then the set $C(a) - C(a)$ is a Cantorval. Moreover, if $k_0 = 0$, then

$$|C(a) - C(a)| = 2 - 2\sum_{n=1}^{\infty} 3^{n-1}\left(d_{k_n-1} - 3d_{k_n}\right).$$

Corollary 1

Let $a = (a_1, a_2, a_1, a_2, \dots)$, where $a_1 < \frac{1}{3}, a_2 > \frac{1}{3}$. If $a_1 \leqslant \frac{1}{35}$ and

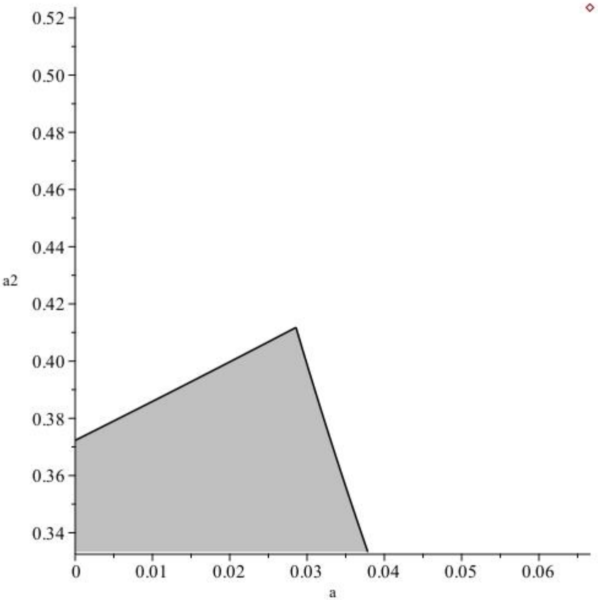
$$a_2 \leqslant \frac{-a_1 - 5 + \sqrt{a_1^2 + 34a_1 + 33}}{2 - 2a_1}$$

or $a_1 \in (\frac{1}{35}, \frac{6\sqrt{5}-13}{11})$ and

$$a_2 \leqslant \frac{3a_1 + 1 - 4\sqrt{a_1^2 + a_1}}{1 - a_1},$$

then the set $C(a) - C(a)$ is a Cantorval.

Cantorvals for sequences $(a_1, a_2, a_1, a_2, \dots)$



S-Cantor sets

Notation

$A \subset \mathbb{Z}$ - a finite set, $p \in \mathbb{N}$, $p \geqslant 2$.

$$A_p := \left\{ \sum_{i=1}^{\infty} \frac{x_i}{p^i} : x_i \in A \right\}.$$

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Definition

We say that a nonempty perfect set $E \subset \mathbb{R}$ is an L-Cantorval (R-Cantorval, respectively) if it is not an interval and the left (right) endpoints of all gaps are accumulation points of other gaps and accumulation points of nontrivial components of E and the right (left) endpoints of all gaps are also endpoints of nontrivial components of E.

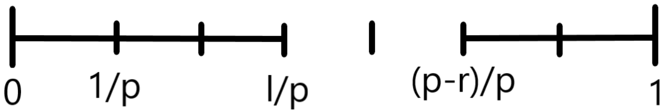
S-Cantor sets

Definition

Let $l, r, p \in \mathbb{N}$ be such that $p > 2$ and $l + r < p$. A set $C(l, r, p) := A(l, r, p)_p$, where

$$A(l, r, p) := \{0, 1, \dots, l-1\} \cup \{p-r, p-r+1, \dots, p-1\}$$

is called a special Cantor set (or an S-Cantor set).



$p, l_1, r_1, l_2, r_2 \in \mathbb{N},\ p > 2,\ l_1 + r_1 < p,\ l_2 + r_2 < p$

Conditions

$$(S1)\ \ l_1 + l_2 + r_2 \geqslant p \ \text{ or } \ l_1 + r_1 + r_2 \geqslant p;$$

$$(S1^*)\ \ l_1 + l_2 + r_2 > p \ \text{ or } \ l_1 + r_1 + r_2 > p;$$

$$(S2)\ \ l_1 + r_1 + l_2 \geqslant p \ \text{ or } \ r_1 + l_2 + r_2 \geqslant p;$$

$$(S2^*)\ \ l_1 + r_1 + l_2 > p \ \text{ or } \ r_1 + l_2 + r_2 > p;$$

$$(S3)\ \ l_1 + r_1 + l_2 + r_2 \leqslant p.$$

Theorem

Assume that $l_1, r_1, l_2, r_2, p \in \mathbb{N}$, $p > 2$, $l_1 + r_1 < p$ and $l_2 + r_2 < p$.

(1) $C(l_1, r_1, p) - C(l_2, r_2, p) = [-1, 1]$ if and only if $(S1)$ and $(S2)$ hold.

(2) $C(l_1, r_1, p) - C(l_2, r_2, p)$ is a Cantor set if and only if $(S3)$ holds.

(3) $C(l_1, r_1, p) - C(l_2, r_2, p)$ is an L-Cantorval if and only if $(S1^*)$, holds, but $(S2)$ does not hold.

(4) $C(l_1, r_1, p) - C(l_2, r_2, p)$ is an R-Cantorval if and only if $(S2^*)$ holds, but $(S1)$ does not hold.

(5) $C(l_1, r_1, p) - C(l_2, r_2, p)$ is an M-Cantorval if and only if $(S1^*)$, $(S2^*)$, $(S3)$ do not hold and at least one from $(S1)$, $(S2)$ also does not hold.

Corollary 1

Let $l, r, p \in \mathbb{N}$, $p > 2$, $l + r < p$. Then

(1) $C(l,r,p) - C(l,r,p) = [-1,1]$ if and only if

$$2l + r \geqslant p \quad \text{or} \quad l + 2r \geqslant p;$$

(2) $C(l,r,p) - C(l,r,p)$ is a Cantor set if and only if

$$2l + 2r \leqslant p;$$

(3) $C(l,r,p) - C(l,r,p)$ is an M-Cantorval if and only if

$$2l + r < p \quad \text{and} \quad l + 2r < p \text{ and } \quad 2l + 2r > p.$$

Example

Let $r_1 = 2,\ l_1 = 3,\ r_2 = 3,\ l_2 = 1,\ p = 7$. Then
$l_1 + r_1 < p, l_2 + r_2 < p,$

$$l_1 + r_1 + r_2 > p \Rightarrow (S1^*),$$

$$(l_1 + r_1 + l_2 < p \wedge r_1 + l_2 + r_2 < p) \Rightarrow \neg(S2).$$

In consequence,

$$C(l_1, r_1, p) - C(l_2, r_2, p) = \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 5, 6\}_7$$

is an L-Cantorval, and

$$C(l_2, r_2, p) - C(l_1, r_1, p) = \{-6, -5, -2, -1, 0, 1, 2, 3, 4, 5, 6\}_7$$

is an R-Cantorval. At the same time, $2l_1 + r_1 \geqslant p$ and $2r_2 + l_2 \geqslant p$, thus

$$C(l_1, r_1, p) - C(l_1, r_1, p) = C(l_2, r_2, p) - C(l_2, r_2, p) = [-1, 1].$$

$C(l, r, p)$ is symmetric with respect to $\frac{1}{2}$ if and only if $l = r$.

Definition

Any set $C(l, p) := C(l, l, p)$ is called a symmetric S-Cantor set.

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Definition

Any set $C(l, p) := C(l, l, p)$ is called a symmetric S-Cantor set.

Corollary 2

Let $l_1, l_2, p \in \mathbb{N}$, $p > 2$, $2l_1 < p$, $2l_2 < p$. Then

(1) $C(l_1, p) - C(l_2, p) = [-1, 1]$ if and only if

$$2l_1 + l_2 \geqslant p \text{ or } l_1 + 2l_2 \geqslant p;$$

(2) $C(l_1, p) - C(l_2, p)$ is a Cantor set if and only if

$$2l_1 + 2l_2 \leqslant p;$$

(3) $C(l_1, p) - C(l_2, p)$ is an M-Cantorval if and only if

$$2l_1 + l_2 < p \text{ and } l_1 + 2l_2 < p \text{ and } 2l_1 + 2l_2 > p.$$

Corollary 3

Let $l, p \in \mathbb{N}$, $p > 2$, $2l < p$. Then

(1) $C(l,p) - C(l,p) = [-1,1]$ if and only if
$$\frac{l}{p} \geqslant \frac{1}{3};$$

(2) $C(l,p) - C(l,p)$ is a Cantor set if and only if
$$\frac{l}{p} \leqslant \frac{1}{4};$$

(3) $C(l,p) - C(l,p)$ is an M-Cantorval if and only if
$$\frac{l}{p} \in \left(\frac{1}{4}, \frac{1}{3}\right).$$

Example

Let $l_1 = 2$, $l_2 = 1$, $p = 5$. Then $2l_1 + l_2 \geqslant p$, so $C(l_1, p) - C(l_2, p) = [-1, 1]$. Moreover, $\frac{l_1}{p} \geqslant \frac{1}{3}$ and $\frac{l_2}{p} < \frac{1}{4}$, therefore $C(l_1, p) - C(l_1, p) = [-1, 1]$, but $C(l_2, p) - C(l_2, p)$ is a Cantor set.

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Example

Let $l = 1, p = 3$. The set $C = C(l, p)$ is the classical Cantor ternary set. By Corollary 3, $C - C = [-1, 1]$.

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Example

Let $l = 2$, $p = 7$. Then $\frac{l}{p} \in (\frac{1}{4}, \frac{1}{3})$, and therefore

$$C(l, p) - C(l, p) = \{-6, -5, -4, -1, 0, 1, 4, 5, 6\}_7$$

is an M-Cantorval.

Bibliography

R. Anisca, M. Ilie, *A technique of studying sums of central Cantor sets*, Canad. Math. Bull. **44** (2001), 12—18.

T. Banakh, A. Bartoszewicz, M. Filipczak, E. Szymonik, *Topological and measure properties of some self-similar sets*, Topol. Methods Nonlinear Anal. **46** (2015), 1013–1028.

A. Bartoszewicz, S. Głąb, J. Marchwicki, *Recovering a purely atomic finite measure from its range*, J. Math. Anal. Appl. **467** (2018) 825–841.

T. Filipczak, P. Nowakowski, *Conditions for the difference set of a central Cantor set to be a Cantorval*, Results Math. **78** (2023), (art. 166).

Bibliography

- S. Kakeya, *On the partial sums of an infinite series*, Tôhoku Sci. Rep. **3** (1914), 159–164.
- R. L. Kraft, *What's the difference between Cantor sets?*, Amer. Math. Monthly 101 (1994), 640–650.
- P. Mendes, F. Oliveira, *On the topological structure of the arithmetic sum of two Cantor sets*, Nonlinearity **7** (1994), 329–343.
- P. Nowakowski, *Conditions for the difference set of a central Cantor set to be a Cantorval. Part II*, Indag. Math (2025), https://doi.org/10.1016/j.indag.2025.03.005.

Bibliography

P. Nowakowski, *The algebraic difference of central Cantor sets and self-similar Cantor sets*, Topol. Methods Nonlinear Anal. **64** (2024), 295–316.

F. Prus-Wiśniowski, F. Tulone, *The arithmetic decomposition of central Cantor sets*, J. Math. Anal. Appl. **467** (2018), 26–31.

M. Repický, *Sets of points of symmetric continuity*, Arch. Math. Logic **54** (2015), 803–824.

A. Sannami, *An example of a regular Cantor set whose difference set is a Cantor set with positive measure*, Hokkaido Math. J. **21** (1992), 7–24.