Nonadditive measure representation theorems for nonlinear functionals

Jun Kawabe

Shinshu University

47th Summer Symposium in Real Analysis
The Meninas Symposium

Universidad Complutense de Madrid

June 16-20, 2025

Nonadditive measure

To deal with various problems in game theory and expected utility theory, it may be needed to consider set functions that do not satisfy additivity due to interactions among sets of players or states of nature.

Definition 1 (nonadditive measure)

X: a non-empty set, $\mathcal{D}$: a collection of subsets of X with $\emptyset \in \mathcal{D}$. A set function $\mu : \mathcal{D} \to [0, \infty]$ is called a nonadditive measure if it satisfies:

- $\mu(\emptyset) = 0$
- $A, B \in \mathcal{D}, A \subset B \Rightarrow \mu(A) \leq \mu(B)$ (monotonicity)

It is widely used in theory and application, and has already appeared in many papers: Hausdorff dimension (Hausdorff 1918), lower/upper numerical probability (Koopman 1940), Maharam's submeasure problem (Maharam 1947), capacity (Choquet 1953/54), semivariation (Dunford-Schwartz 1955), quasimeasure (Alexiuk 1968), maxitive measure (Shilkret 1971), participation measure (Tsichritzis 1971), submeasure (Drewnowski 1972, Dobrakov 1974), fuzzy measure (Sugeno 1974), k-triangular set function (Agafanova-Klimkin 1974), game of characteristic function form, distorted measure (Aumann-Shapley 1974), belief/plausibility function (Shafer 1976), possibility measure (Zadeh 1978), pre-measure (Šipoš 1979), necessity measure (Dubois-Prade 1980), approximately additive (Kalton-Roberts 1983), decomposable measure (Weber 1984), Minkowski-Bouligrand dimension (Schroeder 1991), ...

Notation

Throughout this talk,

- X a nonempty set
- 2^X the power set of X, i.e., the collection of all subsets of X
- $\mathcal{D}$ a collection of subsets of X with $\emptyset \in \mathcal{D}$
- $\mathrm{F}(X)$ the collection of all functions $f\colon X \to \overline{\mathbb{R}}$
- $\mathrm{F}^+(X) := \left\{ f \in \mathrm{F}(X) \colon f \geq 0 \right\}$

For these nonnegative functions we introduce two types of measurability.

- $\mathrm{Lm}(\mathcal{D}) := \left\{ f \in \mathrm{F}^+(X) \colon \{f > t\} \in \mathcal{D} \text{ for any } t > 0 \right\}$ (lower $\mathcal{D}$-measurable)
- $\mathrm{Um}(\mathcal{D}) := \left\{ f \in \mathrm{F}^+(X) \colon \{f \geq t\} \in \mathcal{D} \text{ for any } t > 0 \right\}$ (upper $\mathcal{D}$-measurable)
- $\mathrm{Mf}(\mathcal{D}) := \mathrm{Lm}(\mathcal{D}) \cap \mathrm{Um}(\mathcal{D})$ ($\mathcal{D}$-measurable)

– Note that we do not impose any other structural conditions on the collection $\mathcal{D}$ of sets than $\emptyset \in \mathcal{D}$; we do not assume that $\mathcal{D}$ is a σ-field or $X \in \mathcal{D}$.

– To avoid that $\mathcal{D}$ always contains the whole set X, we need to assume $t > 0$.

The Choquet, Shilkret, and Sugeno integrals

Among the integral concepts based on a nonadditive measure $\mu\colon \mathcal{D} \to [0,\infty]$, the Choquet integral, the Shilkret intgtral, and the Sugeno integral of a function $f \in \mathrm{F}^+(X)$ are particularly popular, which are in turn defined as follows:

$$\mathrm{Ch}(\mu, f) := \begin{cases} \displaystyle\int_0^\infty \mu(\{f > t\})dt & \text{if } f \in \mathrm{Lm}(\mathcal{D}), \\ \displaystyle\int_0^\infty \mu(\{f \geq t\})dt & \text{if } f \in \mathrm{Um}(\mathcal{D}), \end{cases}$$

$$\mathrm{Sh}(\mu, f) := \begin{cases} \sup_{t>0} t \cdot \mu(\{f > t\}) & \text{if } f \in \mathrm{Lm}(\mathcal{D}), \\ \sup_{t>0} t \cdot \mu(\{f \geq t\}) & \text{if } f \in \mathrm{Um}(\mathcal{D}), \end{cases}$$

$$\mathrm{Su}(\mu, f) := \begin{cases} \sup_{t>0} t \wedge \mu(\{f > t\}) & \text{if } f \in \mathrm{Lm}(\mathcal{D}), \\ \sup_{t>0} t \wedge \mu(\{f \geq t\}) & \text{if } f \in \mathrm{Um}(\mathcal{D}). \end{cases}$$

- The Choquet and Sugeno integrals are used in the expected utility theory in economics, and the Choquet integral is also used in potential theory in mathematics.
- The Shilkret integral is needed not only to discuss maxitive measures but also to define weak L^1-space in real analysis.

The purpose of the study

- We will regard these nonlinear integrals as nonlinear functionals $L\colon \Phi \to [0,\infty]$ defined by

$$L(f) := \begin{cases} \mathrm{Ch}(\mu, f) & \text{Choquet functional} \\ \mathrm{Sh}(\mu, f) & \text{Shilkret functional} \\ \mathrm{Su}(\mu, f) & \text{Sugeno functional} \end{cases}$$

  on an appropriate space Φ of functions $f\colon X \to [0,\infty]$.

- Our purpose is to characterize a given nonlinear functional as these integrals!

- To this end, we begin with investigating what properties the Choquet, Shilkret, and Sugeno integrals have.

Recall that functions f and g are *comonotonic*, written $f \sim g$, if for any $x, y \in X$,

$$f(x) < f(y) \ \Rightarrow\ g(x) \leq g(y).$$

It is called "même tableau de variation" in Dellacherie and "similarly ordered" in Hardy–Littlewood-Pólya.

- C. Dellacherie, Quelques commentaires sur les prolongements de capacités, In: Séminaire de Probabilités, Strasbourg, 1969/1970, Lecture Notes in Math., 191, Springer, 1971, pp. 77-81.
- G.H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge Math. Lib., Cambridge Univ. Press, Cambridge, 1988.

Proposition 2 (Elementary properties of the Choquet integral)

Let $\mu\colon \mathcal{D} \to [0,\infty]$ be a nonadditive measure. Let $f, g \in \mathrm{Lm}(\mathcal{D})$, $D \in \mathcal{D}$, and $c > 0$.

- $\mathrm{Ch}(\mu, \chi_D) = \mu(D)$. (generativity)
- $\mathrm{Ch}(\mu, cf) = c\mathrm{Ch}(\mu, f)$. (positive homogeneity)
- If $f \leq g$, then $\mathrm{Ch}(\mu, f) \leq \mathrm{Ch}(\mu, g)$. (monotonicity)
- $\mathrm{Ch}(\mu, f) = \sup_{a>0} \mathrm{Ch}(\mu, (f-a)^+)$. (lower marginal continuity)
- $\mathrm{Ch}(\mu, f) = \sup_{b>0} \mathrm{Ch}(\mu, f \wedge b)$. (upper marginal continuity)
- $\mathrm{Ch}(\mu, f) = \mathrm{Ch}(\mu, f \wedge c) + \mathrm{Ch}(\mu, (f-c)^+)$. (horizontal additivity)
- $\mathrm{Ch}(\mu, f+g) = \mathrm{Ch}(\mu, f) + \mathrm{Ch}(\mu, g)$ whenever $f + g \in \mathrm{Lm}(\mathcal{D})$ and $f \sim g$. (comonotonic additivity)
- $\mathrm{Ch}(\mu, f) = \displaystyle\int_X f d\mu$ if μ is σ-additive.

The same results hold for $f, g \in \mathrm{Um}(\mathcal{D})$.

Horizontal additivity and marginal continuity

- The horizontal additivity

$$\mathrm{Ch}(\mu, f) = \mathrm{Ch}(\mu, f \wedge c) + \mathrm{Ch}(\mu, (f - c)^+)$$

corresponds to the fact that the Lebesgue integral can be split in the middle of an integral interval:

$$\int_0^\infty \mu(\{f > t\})dt = \int_0^c \mu(\{f > t\})dt + \int_c^\infty \mu(\{f > t\})dt$$

- The marginal continuity

$$\mathrm{Ch}(\mu, f) = \sup_{a>0} \mathrm{Ch}(\mu, (f - a)^+) = \sup_{b>0} \mathrm{Ch}(\mu, f \wedge b)$$

corresponds to the fact that the Lebesgue integral is continuous at lower and upper endpoints of the integral:

$$\int_0^\infty \mu(\{f > t\})dt = \sup_{a>0} \int_a^\infty \mu(\{f > t\})dt = \sup_{b>0} \int_0^b \mu(\{f > t\})dt$$

The Greco theorem (1982)

As a matter of fact, we already have the characterization of the Choquet integral as a nonlinear functional. Hereafter,

- X is a nonempty set.
- $\Phi \subset [0, \infty]^X$ with $0 \in \Phi$.
- Assume that $\Phi \subset \mathrm{Lm}(\mathcal{D})$ or $\Phi \subset \mathrm{Um}(\mathcal{D})$.
- $L \colon \Phi \to [0, \infty]$ be a monotone functional with $L(0) = 0$.

Define the nonadditive measures $\alpha, \beta \colon 2^X \to [0, \infty]$ by

$$\alpha(A) := \sup \{L(f) \colon f \in \Phi,\ f \leq \chi_A\}$$
$$\beta(A) := \inf \{L(f) \colon f \in \Phi,\ \chi_A \leq f\}$$

for every $A \in 2^X$, where $\inf \emptyset := \infty$. Then, it follows that

$$\alpha(A) \leq \beta(A)$$

for all $A \in 2^X$.

In this abstract setting, Professor Greco gave the following theorem in 1982.

Theorem 3 (A characterization of the Choquet integral)

Assume that $\Phi \subset [0,\infty]^X$ with $0 \in \Phi$ satisfies

(Φ_1) $cf, f \wedge c, (f-c)^+ \in \Phi$ for $\forall f \in \Phi$ and $\forall c > 0$.

ex. $\Phi = \mathrm{Lm}(\mathcal{D}), \mathrm{Um}(\mathcal{D}), C_b^+(X), C_{00}^+(X), C_0^+(X), LSC^+(X), USC^+(X)$

Assume that a monotone functional $L\colon \Phi \to [0,\infty]$ with $L(0) = 0$ satisfies

(L_1) $L(f) = L(f \wedge c) + L((f-c)^+)$ for $\forall f \in \Phi$ and $\forall c > 0$.

(L_2) $L(f) = \sup_{a>0} L((f-a)^+) = \sup_{b>0} L(f \wedge b)$ for $\forall f \in \Phi$.

Then, for any nonadditive measure $\lambda\colon \mathcal{D} \to [0,\infty]$, the following are equivalent.

(a) $\alpha(D) \leq \lambda(D) \leq \beta(D)$ for $\forall D \in \mathcal{D}$.

(b) $L(f) = \mathrm{Ch}(\lambda, f)$ for $\forall f \in \Phi$.

- G.H. Greco, Sulla rappresentazione di funzionali mediante integrali, Rend. Semin. Mat. Univ. Padova 66 (1982) 21–42.

Greco type theorem for the Shilkret integral

To obtain Greco type representation theorems for the Shilkret and Sugeno integrals, we collect elementary properties that those integrals have.

Proposition 4 (Elementary properties of the Shilkret integral)

Let $\mu\colon \mathcal{D} \to [0, \infty]$ be a nonadditive measure. Let $f, g \in \mathrm{Lm}(\mathcal{D})$, $D \in \mathcal{D}$, and $c > 0$.

- $\mathrm{Sh}(\mu, \chi_D) = \mu(D)$. (generativity)
- $\mathrm{Sh}(\mu, cf) = c\mathrm{Sh}(\mu, f)$. (positive homogeneity)
- If $f \leq g$, then $\mathrm{Sh}(\mu, f) \leq \mathrm{Sh}(\mu, g)$. (monotonicity)
- $\mathrm{Sh}(\mu, f) = \sup_{a>0} \mathrm{Sh}(\mu, (f - a)^+)$. (lower marginal continuity)
- $\mathrm{Sh}(\mu, f) = \sup_{b>0} \mathrm{Sh}(\mu, f \wedge b)$. (upper marginal continuity)
- $\mathrm{Sh}(\mu, f) = \mathrm{Sh}(\mu, f \wedge c) \vee \mathrm{Sh}(\mu, f\chi_{\{f>c\}})$. (horizontal maxitivity)
- $\mathrm{Sh}(\mu, f \vee g) = \mathrm{Sh}(\mu, f) \vee \mathrm{Sh}(\mu, g)$ whenever $f \sim g$. (comonotonic maxitivity)

The same results hold for $f, g \in \mathrm{Um}(\mathcal{D})$.

The horizontal additivity and comonotonic additivity do not hold for the Shilkret integral. The comonotonic minitivity does not hold either.

We can obtain a Greco type representation theorem for the Shilkret integral.

Theorem 5 (A characterization of the Shilkret integral)

Assume that $\Phi \subset [0,\infty]^X$ with $0 \in \Phi$ satisfies

(Φ_1) $cf, f \wedge c, (f-c)^+ \in \Phi$ for $\forall f \in \Phi$ and $\forall c > 0$.

(Φ_2) $f \vee g \in \Phi$ whenever $f, g \in \Phi$ and $f \sim g$. (comonotonic maxitivity)

Assume that a monotone functional $L\colon \Phi \to [0,\infty]$ with $L(0) = 0$ satisfies

(L_1) $L(cf) = cL(f)$ for $\forall f \in \Phi$ and $\forall c > 0$

(L_2) $L(f \vee g) = L(f) \vee L(g)$ whenever $f, g \in \Phi$ and $f \sim g$.

(L_3) $L(f) = \sup_{a>0} L((f-a)^+) = \sup_{b>0} L(f \wedge b)$ for $\forall f \in \Phi$.

(1) For any nonadditive measure $\lambda\colon \mathcal{D} \to [0,\infty]$, the following are equivalent.

   (a) $\alpha(D) \leq \lambda(D) \leq \beta(D)$ for $\forall D \in \mathcal{D}$.

   (b) $L(f) = \mathrm{Sh}(\lambda, f)$ for $\forall f \in \Phi$.

(2) Assume further that $\chi_D \in \Phi$ for all $D \in \mathcal{D}$. Then, $\alpha = \beta$ is the unique representing measure of L.

Proposition 6 (Elementary properties of the Sugeno integral)

Let $\mu\colon \mathcal{D} \to [0,\infty]$ be a nonadditive measure. Let $f, g \in \mathrm{Lm}(\mathcal{D})$, $D \in \mathcal{D}$, and $c > 0$.

- $\mathrm{Su}(\mu, c\chi_D) = c \wedge \mu(D)$. (generativity)
- If $f \leq g$, then $\mathrm{Su}(\mu, f) \leq \mathrm{Su}(\mu, g)$. (monotonicity)
- $\mathrm{Su}(\mu, f) = \sup_{a>0} \mathrm{Su}(\mu, (f-a)^+)$. (lower marginal continuity)
- $\mathrm{Su}(\mu, f) = \sup_{b>0} \mathrm{Su}(\mu, f \wedge b)$. (upper marginal continuity)
- $\mathrm{Su}(\mu, f \wedge c) = \mathrm{Su}(\mu, f) \wedge c$. (constant minitivity)
- $\mathrm{Su}(\mu, f) = \mathrm{Su}(\mu, f \wedge c) \vee \mathrm{Su}(\mu, f\chi_{\{f>c\}})$. (horizontal maxitivity)
- $\mathrm{Su}(\mu, f \vee g) = \mathrm{Su}(\mu, f) \vee \mathrm{Su}(\mu, g)$ whenever $f \sim g$. (comonotonic maxitivity)
- $\mathrm{Su}(\mu, f \wedge g) = \mathrm{Su}(\mu, f) \wedge \mathrm{Su}(\mu, g)$ whenever $f \sim g$. (comonotonic minitivity)

The same results hold for $f, g \in \mathrm{Um}(\mathcal{D})$.

The positive homogeniety, horizontal additivity, and comonotonic additivity do not hold for the Sugeno integral!

To formalize Greco type representation theorem for the Sugeno integral, we need to introduce new nonadditive measures α_e and β_e defined by the functional L.

- X is a nonempty set.
- $\Phi \subset [0, \infty]^X$ with $0 \in \Phi$.
- Assume that $\Phi \subset \mathrm{Lm}(\mathcal{D})$ or $\Phi \subset \mathrm{Um}(\mathcal{D})$.
- $0 < e \leq \infty$ is a constant and $\Phi_e := \{f \in \Phi \colon f \leq e\}$
- $L \colon \Phi \to [0, \infty]$ be a monotone functional with $L(0) = 0$.

Define the nonadditive measures $\alpha_e, \beta_e \colon 2^X \to [0, \infty]$ by

$$\alpha_e(A) := \sup\left\{L(f) \colon f \in \Phi_e,\ f \leq \chi_A^e\right\}$$

$$\beta_e(A) := \inf\left\{L(f) \colon f \in \Phi_e,\ \chi_A^e \leq f\right\}$$

for every $A \in 2^X$, where $\chi_A^e(x) := \begin{cases} e & (x \in A) \\ 0 & (x \notin A) \end{cases}$ and $\inf \emptyset := \infty$.

Then, it follows that

$$\alpha_e(A) \leq \beta_e(A)$$

for all $A \in 2^X$.

Theorem 7 (Greco type theorem for the Sugeno integral: uniformly bounded case)

Assume that $\Phi \subset [0,\infty]^X$ with $0 \in \Phi$ satisfies

(Φ_1) $cf, f \wedge c, (f-c)^+ \in \Phi$ for $\forall f \in \Phi$ and $\forall c > 0$.

(Φ_2) $f \vee g \in \Phi$ whenever $f, g \in \Phi$ and $f \sim g$. (comonotonic maxitivity)

Let $0 < e < \infty$. Assume that a monotone $L \colon \Phi \to [0,\infty]$ with $L(0) = 0$ satisfies

(L_1) $L(f \wedge c) = L(f) \wedge c$ for $\forall f \in \Phi_e$ and $\forall c > 0$,

(L_2) $L(f \vee g) = L(f) \vee L(g)$ whenever $f, g \in \Phi_e$ and $f \sim g$,

(L_3) $L(f) = \sup_{a>0} L((f-a)^+)$ for $\forall f \in \Phi_e$.

For any nonadditive measure $\lambda \colon \mathcal{D} \to [0,\infty]$, the following are equivalent.

(a) $\alpha_e(D) \leq \lambda(D) \wedge e \leq \beta_e(D)$ for $\forall D \in \mathcal{D}$.

(b) $L(f) = \mathrm{Su}(\lambda, f)$ for $\forall f \in \Phi_e$.

In particular, for any nonadditive measure $\lambda \colon \mathcal{D} \to [0,e]$, the following are equivalent.

(a) $\alpha_e(D) \leq \lambda(D) \leq \beta_e(D)$ for $\forall D \in \mathcal{D}$.

(b) $L(f) = \mathrm{Su}(\lambda, f)$ for $\forall f \in \Phi_e$.

By using the previous theorem, we can obtain the following theorem.

Theorem 8 (Greco type theorem for the Sugeno integral: general case)

Assume that $\Phi \subset [0,\infty]^X$ with $0 \in \Phi$ satisfies

(Φ_1) $cf, f \wedge c, (f-c)^+ \in \Phi$ for $\forall f \in \Phi$ and $\forall c > 0$,

(Φ_2) $f \vee g \in \Phi$ whenever $f, g \in \Phi$ and $f \sim g$. (comonotonic maxitivity)

Assume that a monotone functional $L\colon \Phi \to [0,\infty]$ with $L(0) = 0$ satisfies

(L_1) $L(f \wedge c) = L(f) \wedge c$ for $\forall f \in \Phi$ and $\forall c > 0$.

(L_2) $L(f \vee g) = L(f) \vee L(g)$ whenever $f, g \in \Phi$ and $f \sim g$,

(L_3) $L(f) = \sup_{a>0} L((f-a)^+)$ for $\forall f \in \Phi$.

Then, for any nonadditive measure $\lambda\colon \mathcal{D} \to [0,\infty]$, the following are equivalent.

(a) $\alpha_n(D) \leq \lambda(D) \wedge n \leq \beta_n(D)$ for $\forall D \in \mathcal{D}$ and $\forall n \in \mathbb{N}$.

(b) $L(f) = \mathrm{Su}(\lambda, f)$ for $\forall f \in \Phi$.

$$\alpha_n(A) := \sup\left\{L(f)\colon f \in \Phi_n,\ f \leq \chi_A^n\right\}, \quad \beta_n(A) := \inf\left\{L(f)\colon f \in \Phi_n,\ \chi_A^n \leq f\right\}$$

We are concerned that there really exists a nonadditive measure λ that satisfies

Condition (a): $\alpha_n(D) \le \lambda(D) \wedge n \le \beta_n(D)$ for $\forall D \in \mathcal{D}$ and $\forall n \in \mathbb{N}$,

but it exists.

Corollary 9 (Existence and uniqueness of the Sugeno representing measure)

(1) Let the nonadditive measure $\alpha_\infty \colon 2^X \to [0, \infty]$ by

$$\alpha_\infty(A) := \sup\left\{L(f) \colon f \in \Phi, f \le \chi_A^\infty\right\}, \text{ where } \chi_A^\infty(x) := \begin{cases} \infty & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

for every $A \in 2^X$. Then we have

$$\alpha_n = \alpha_\infty \wedge n$$

for all $n \in \mathbb{N}$, hence α_∞ is a Sugeno representing measure of L, that is,

$$L(f) = \mathrm{Su}(\alpha_\infty, f)$$

for every $f \in \Phi$.

(2) Assume further that $\chi_D^\infty \in \Phi$ for every $D \in \mathcal{D}$. Then, α_∞ is the unique Sugeno representing measure of L.

Example 10 (β_∞ is not a Sugeno representing measure)

Let $X := [0,1]$, $\mathcal{D} := 2^X$, and $\Phi := [0,\infty)^X$. Define the functional $L\colon \Phi \to [0,\infty)$ by $L(f) := f(0)$ for every $f \in \Phi$. Then, Φ and L satisfy the conditions assumed in Theorem 8. Since it follows that

$$\beta_\infty(D) := \inf\left\{I(f)\colon f \in \Phi, \chi_D^\infty \leq f\right\} = \begin{cases} \infty & \text{if } D \neq \emptyset, \\ 0 & \text{if } D = \emptyset \end{cases}$$

for any $D \in 2^X$, we conclude that

$$\alpha_1(\{1\}) = \beta_1(\{1\}) = 0 < 1 = \beta_\infty(\{1\}) \wedge 1.$$

So, it follows from Theorem 8 that β_∞ is not a Sugeno representing measure of L. In contrast, since it follows that

$$\alpha_\infty(D) = \begin{cases} \infty & \text{if } 0 \in D, \\ 0 & \text{if } 0 \notin D \end{cases}$$

for any $D \in 2^X$, we conclude that

$$\mathrm{Su}(\alpha_\infty, f) = f(0) = L(f)$$

for every $f \in \Phi$, that is, α_∞ is a Sugeno representing measure of L.

References

- A. Chateauneuf, A. Rico, A Sugeno integral representation under Stone conditions, Fuzzy Sets and Systems 151 (2005) 261–267.
- G. Choquet, Theory of capacities, Ann. Inst. Fourier Grenoble 5 (1953/54) 131–295.
- C. Dellacherie, Quelques commentaires sur les prolongements de capacités, in: Séminaire de Probabilités V, Lecture Notes in Math., Vol.191 (1970) 77–81.
- G.H. Greco, Sulla rappresentazione di funzionali mediante integrali, Rend. Sem. Mat. Univ. Padova 66 (1982) 21–42.
- J. Kawabe, The continuous Choquet integral representation theorems in an abstract setting, Fuzzy Sets and Systems 497 (2024) 109115 (20 pages).
- D. Ralescu, G. Adams, The fuzzy integral, J. Math. Anal. Appl. 75 (1980) 562–570.
- N. Shilkret, Maxitive measure and integration, Indag. Math. 33 (1971) 109–116.
- M. Sugeno, Theory of fuzzy integrals and its applications, Thesis, Tokyo Institute of Technology, 1974.

Thank you for your kind attention!