Little Lipschitz constant of functions on the real line

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big Lip VS little lip

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Little Lipschitz constant – only pointwise:

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(i) There exists a Lipschitz $f : \mathbb{R} \to \mathbb{R}$ such that $A = \{x \in \mathbb{R} \colon \nexists f'(x)\}$;

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Note that in higher dimensions it is not so easy:
Preiss (1989?): There exists a Lebesgue null set $A \subseteq \mathbb{R}^2$ such that any Lipschitz function on $\mathbb{R}^2$ has a point of differentiability in A.

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Balogh–Csörnyei (2006): There exist a set $A \subset [0,1]$ of positive measure and a continuous function $f\colon [0,1] \to \mathbb{R}$ such that $\operatorname{lip} f(x) < \infty$ for all $x \in [0,1]$ and f has finite derivative at **no** point of A.

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Balogh–Csörnyei (2006): Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that $\operatorname{lip} f(x) < \infty$ for $x \in \mathbb{R}^d \setminus E$, where E has σ-finite $(d-1)$-dimensional Hausdorff measure.
Assume that $\operatorname{lip} f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $1 \leqslant p \leqslant \infty$.
Then $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$. If moreover $p > d$, then we have

$$\operatorname{lip} f(x) = \operatorname{Lip} f(x) = \|\nabla f(x)\| \quad \text{for a.e. } x \in \mathbb{R}^d.$$

points x where $\operatorname{lip} f(x) = \infty$

Definition

$$\operatorname{lip} f(x) = \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{r}.$$

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$L_f^\infty := \{x \in \mathbb{R}^d \colon \operatorname{Lip} f(x) = \infty\}$ and $l_f^\infty := \{x \in \mathbb{R}^d \colon \operatorname{lip} f(x) = \infty\}$.

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Easy:

(a) L_f^∞ is always G_δ;

(b) l_f^∞ is always $F_{\sigma\delta}$.

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Q: Is (b) optimal?

Descriptive quality of l_f^∞ in dimension 1

Definition (Little Lipschitz constant)

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Proof: The proof for F_σ consists of several steps.

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4. F countable union of nowhere dense sets;
5. General case: Remove the interior of F.

Optimal descriptive quality

Theorem (R., Zürcher)

Let $A \subseteq \mathbb{R}$ be $F_{\sigma\delta}$, $|A| = 0$. Then there exists a monotone, absolutely continuous $f : \mathbb{R} \to \mathbb{R}$ such that $l_f^\infty = A$.

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Conjecture (proof in progress)

For any $F_{\sigma\delta}$ set $F \subseteq \mathbb{R}$ there exists a function $f \in C([0, 1])$ such that $l_f^\infty = F$.

Lemma

Suppose $E, F \subseteq \mathbb{R}$ are closed, $A, M \subseteq \mathbb{R}$ are measurable, $|A| = 0$ and M has EPM. Let $\varepsilon > 0$ and H be the closed set from the previous lemma:

(1) $F \subseteq H \subseteq (F \cup M) \cap (F)_\varepsilon$;

(2) *H meets the middle third of every component of $\widehat{F}^c \cap (F)_\varepsilon$ in a set of positive measure.*

Then there is a nondecreasing absolutely continuous function $g\colon \mathbb{R} \to [0, \varepsilon]$ such that

(i) $\operatorname{spt}(g') \subseteq H$;

(ii) $\operatorname{Lip} g(x) < \infty$ *for every* $x \in \mathbb{R}$;

(iii) $g'(x) = 1$ *for every* $x \in A \cap F \cap E^c$;

(iv) $\operatorname{osc}(g, U) \leqslant \varepsilon |U|$ *whenever U is an interval meeting E*;

(v) $\|g'\|_1 < \varepsilon$.

Thank you for your attention.

Gracias por su atención.

Děkuji za pozornost.