THE
HARDY–LITTLEWOOD MAXIMAL FUNCTION
ON SPACES OF HOMOGENEOUS TYPE

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THE SETTING FOR ANALYSIS.

SPACES OF HOMOGENEOUS TYPE (Ω, d, μ) —— define ——→ THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

quasi-metric
$$d(x,z) \leq \kappa(d(x,y)+d(y,z))$$

doubling Borel measure
$$\mu(B(x,r)) \leq A\mu(B(x,r/2))$$

$$Mf(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y)$$

boundedness?

QUASI-NORMED LATTICES
$X(\Omega, d, \mu) \subset L^0(\Omega, d, \mu)$

- $\|f+g\|_X \leq C(\|f\|_X + \|g\|_X)$
- $\|f\|_X \leq \|g\|_X$ whenever $|f| \leq |g|$

$\supset$ Quasi-Banach function spaces $\supset$

Variable Lebesgue spaces
$L^{p(\cdot)}(\Omega, d, \mu)$
Cruz-Uribe, Fiorenza, 2013

+

THEIR r-CONVEXIFICATIONS $(r > 0)$
$$X^{(r)}(\Omega, d, \mu) = \{\text{meas. } f \;:\; |f|^r \in X(\Omega, d, \mu)\}$$

$\|f\|_{X^{(r)}} := \| \, |f|^r \|_X^{1/r}$

$$(L^{p(\cdot)})^{(r)} = L^{rp(\cdot)}$$

The Setting for Analysis.

Spaces of Homogeneous Type (Ω, d, μ) —— define ——→ **The Hardy-Littlewood Maximal Operator**

- quasi-metric: $d(x,z) \leq \kappa(d(x,y)+d(y,z))$
- doubling Borel measure: $\mu(B(x,r)) \leq A\mu(B(x,r/2))$

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Quasi-Normed Lattices
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How *variable Lebesgue spaces* grew out of applications:

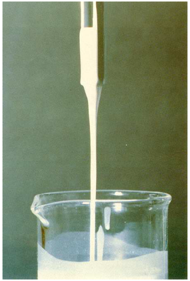
Electrorheological Fluids

change the viscosity—drastically, reversibly—when
an *electric field* is applied.

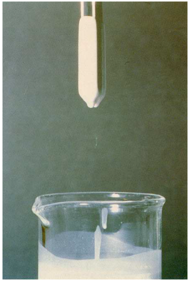
1996

Rajagopal, Růžička

On the Modeling of Electrorheological Materials



$E = 0$



$E \approx 4$ kV/mm

Simplified Model.

$\mathbf{v}$ velocity field

π pressure

$\mathbf{f}$ mechanical force

$\varepsilon(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$

$$\left.\begin{aligned} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \operatorname{div}\Big(|\varepsilon(\mathbf{v})|^{p(x,t)-2}\varepsilon(\mathbf{v})\Big) + \nabla\pi &= \mathbf{f} \\ \operatorname{div}\mathbf{v} &= 0 \end{aligned}\right\}$$

...... Dirichlet energy integral

$$\int |\varepsilon(\mathbf{v})|^{p(x,t)}\,dx$$

Self-Improving Boundedness of M over $\mathbb{R}^n$.

Variable Lebesgue Spaces $L^{p(\cdot)}(\Omega, d, \mu)$ —— when $p_- > 0$ ——→ = Quasi-Banach Lattices w.r.t. the Luxemburg–Nakano quasi-norm

Exponent = function $p(\cdot) : \Omega \to (0, \infty]$.

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 \;:\; \varrho_{p(\cdot)}(f/\lambda) \leq 1\}$$

All $f \in L^0(\Omega, d, \mu)$ s.t. for some $\lambda > 0$,

$$\varrho_{p(\cdot)}(f/\lambda) := \int_{\Omega \setminus \Omega_\infty} \left| \frac{f(x)}{\lambda} \right|^{p(x)} d\mu(x) + \operatorname*{ess\,sup}_{x \in \Omega_\infty} \left| \frac{f(x)}{\lambda} \right|.$$

+ satisfy the *Fatou property*

What do we get if M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$?

- *Boundedness of* pseudodifferential operators, Calderón–Zygmund singular integrals;
- *good properties of the space:* density of smooth functions in the Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ for Lipschitz domains $\Omega \subset \mathbb{R}^n$.

2005 *Diening*

$p_- := \operatorname{ess\,inf}_{x\in\Omega} p(x) > 1$,
$p_+ := \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty$

M bounded on $L^{p(\cdot)}(\mathbb{R}^n)$

$\Downarrow \Uparrow$

M bounded on $L^{p(\cdot)/s}(\mathbb{R}^n)$ for some $s > 1$

2010 *Lerner* *Ombrosi*

Extension to quasi-Banach function spaces $X(\mathbb{R}^n)$
+
simplified proof

What We Proved.

$\uparrow$ M bounded on $X^{(r)} \implies M$ bounded on $X^{(s)}$ $\forall s > r$. $\because$ *Hölder's inequality*

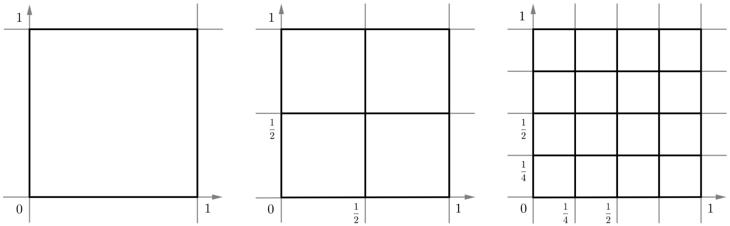
$\downarrow$ [Self-Improvement] Quasi-Banach lattices:
$X(\Omega, d, \mu)$ + Fatou property: $0 \le f_n \uparrow f$, $\sup \|f_n\|_X < \infty \implies \|f\|_X \le C_{\mathscr{F}} \sup \|f_n\|_X$

$\exists r_0 : \forall r_0 \le r < 1$ M bounded on $X^{(r)}$	$\Longleftarrow$	M bounded on X	$\Longleftarrow$	$\exists s_0 : \forall 1 < s < s_0$ M bounded on $X^{(s)}$ & $\lim_{s \to 1^+} (s-1)\|M\|_{X^{(s)} \to X^{(s)}} = 0$

Underlying machinery in $\mathbb{R}^n$: Dyadic Cubes.

Look:
Generations of one dyadic grid $\mathscr{D}^1$.

. . .



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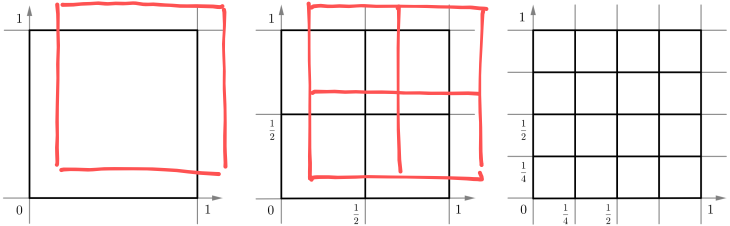

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Hytönen–Kairema dyadic system.

SHT allow construction of a system of **adjacent dyadic grids** $\mathscr{D} = \bigcup_{t=1}^{K} \mathscr{D}^t$ (!)

2012 *Hytönen, Kairema*

- Each grid $\mathscr{D}^t = \{Q_\alpha^{t,k} \,:\, \text{Gen } k \in \mathbb{Z},\ \alpha \in \mathscr{A}_k\}$ is modeled after the Euclidean dyadic decomposition:

$$\left.\begin{array}{ll} \text{(a)} & \Omega = \bigcup_{\alpha \in \mathscr{A}_k} Q_\alpha^{k,t} \text{ (disjoint);} \\ \text{(b)} & Q, P \in \mathscr{D}^t \implies Q \cap P \in \{\emptyset, Q, P\}; \\ \text{(c)} & B(z_\alpha^{k,t}, c_1\delta^k) \subset Q_\alpha^{k,t} \subset B(z_\alpha^{k,t}, C_1\delta^k); \\ \text{(d)} & Q_\beta^{k+1,t} \subset Q_\alpha^{k,t} \subset Q_\gamma^{k-1,t}. \end{array}\right\} \forall \text{ generation } k \in \mathbb{Z}$$

- [Nice Covering Property] Any ball $B = B(x,r)$ can be covered by a cube $Q_\alpha^{k-1,t} \in \mathscr{D}$, where $k \in \mathbb{Z}$ satisfies $\delta^{k+1} < r \leq \delta^k$.

Idea: prove the self-improvement theorem for the **"dyadic" maximal operator**

$$M^{\mathscr{D}} f(x) := \sup_{\substack{Q \ni x: \\ Q \in \mathscr{D}}} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y)$$

\+ use the equivalence

$$\frac{1}{c} M^{\mathscr{D}} f(x) \leq Mf(x) \leq c M^{\mathscr{D}} f(x)$$

Essentials of the Proof.

$A_1^{\mathscr{D}}$ Weights

$$[w]_{A_1^{\mathscr{D}}} := \operatorname*{ess\,sup}_{x\in\Omega} \frac{M^{\mathscr{D}} w(x)}{w(x)} < \infty$$

2017 *Anderson* *Hytönen* *Tapiola* — weak RHI for $A_\infty^{\mathscr{D}}$ $\Longrightarrow$ 2019 *Karlovich* — RHI for $A_1^{\mathscr{D}}$

Reverse-Hölder type inequality for dyadic weights:

$\forall w \in A_1^{\mathscr{D}}$ and $\forall Q \in \mathscr{D}$, for a certain range of $0 < \eta \leq \eta_0$,

$$\left(\frac{1}{\mu(Q)} \int_Q w^{1+\eta}(y)\, d\mu(y)\right)^{\frac{1}{1+\eta}} \leq C[w]_{A_1^{\mathscr{D}}} \frac{1}{\mu(Q)} \int_Q w(y)\, d\mu(y).$$

Principal Tool.

Rubio de Francia iteration algorithm: $\mathscr{R}_\varepsilon^{\mathscr{D}} f(x) := \sum_{k=0}^{\infty} \varepsilon^k (M^{\mathscr{D}})^k f(x) \in A_1^{\mathscr{D}}$.

Application: One Interpolation Lemma

for $X(\Omega,d,\mu)=L^{p(\cdot)}(\Omega,d,\mu)$ with $p(\cdot)\,:\,\Omega\to[1,\infty]$—*Banach case.*

Known 🗎 *2020* *Karlovich*	Duality $1<p_-\le p_+<\infty$	M bounded on $L^{p(\cdot)}(\Omega,d,\mu) \iff M$ bounded on $L^{p'(\cdot)}(\Omega,d,\mu)$, $\frac{1}{p(x)}+\frac{1}{p'(x)}=1, \quad x\in\Omega.$
New	Self-Improvement $p_->1$	M bounded on $L^{p(\cdot)}(\Omega,d,\mu) \Longrightarrow M$ bounded on $L^{p(\cdot)/s}(\Omega,d,\mu)$ for some $s\in(1,p_-)$.

Corollary—extension of 🗎 *2022* *Diening, Karlovych, Shargorodsky* from $\mathbb{R}^n$ to (Ω,d,μ).

A tool for *transferring properties like compactness* of linear operators from standard Lebesgue spaces to the variable ones.

The maximal operator M bounded on $L^{p(\cdot)}(\Omega,d,\mu)$ with $1<p_-\le p_+<\infty$

$\overset{\forall q\in(1,\infty)}{\Longleftrightarrow}$

the exponent $p(\cdot)$ can be represented as the **weighted harmonic mean**

$$\frac{1}{p(x)}=\frac{\theta}{q}+\frac{1-\theta}{r(x)}, \quad x\in\Omega,$$

s.t. M is bounded on $L^{r(\cdot)}(\Omega,d,\mu)$ for all sufficiently small $\theta>0$.

The Text Behind Today's Presentation:

A. Shalukhina. Self-improving boundedness of the maximal operator on quasi-Banach lattices over spaces of homogeneous type, *J. Math. Anal. Appl.* **548**(2):129419, 2025.

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