The canonical projection associated with a mixed possibly infinite iterated function system

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Introduction

An iterated function system is a pair

$$((X,d),(f_i)_{i\in I}) = \mathcal{S},$$

where (X,d) is a complete metric space, I is a finite set and $f_i : X \to X$, $i \in I$, are Banach contractions.

We can associate the operator $F_{\mathcal{S}} : P_{cp}(X) \to P_{cp}(X)$, given by

$$F_{\mathcal{S}}(K) = \underset{i\in I}{\cup} f_i(K),$$

for all $K \in P_{cp}(X) = \{C \mid C$ is a non empty compact subset of $X\}$.

Via the Banach contraction principle, J. Hutchinson [1] proved that $F_{\mathcal{S}}$ has a unique fixed point $A_{\mathcal{S}}$ (which is called the attractor of $\mathcal{S}$)

[1] J. Hutchinson, Fractals and self similarity, Indiana Univ. Math. J., **30** (1981), 713–747.

The above mentioned theory (which was initiated by J. Hutchinson and developed by M. Barnsley) has been extended in different directions. Two of them are of special interest from the point of view of the present paper:

i) the direction concentrating on systems involving not necessarily finite families of functions

ii) the direction focusing on systems involving functions from larger classes of contractions

We emphasize that the corresponding fractal operator is usually a Picard operator for the above mentioned generalizations of the concept of iterated function system.

In a recent work [2], we integrated the previous directions by considering possibly infinite iterated function systems enriched with orbital possibly infinite iterated function systems (called mixed possibly infinite iterated function systems) and we proved that the fractal operator associated with such a system is weakly Picard.

Our aim is to develop a canonical projection type theory for mixed possibly infinite iterated function systems, in order to obtain an alternative descriptions for the attractors.

[2] B. Anghelina, R. Miculescu, A. Mihail, On the fractal operator of a mixed possibly infinite iterated function system, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **119** (2025), 31.

The canonical projection associated with $\mathcal{S}$ is a continuous surjection π from the code space $\Lambda(I)$ onto the attractor $A_{\mathcal{S}}$.

The importance of this projection is emphasized by the following facts:

a) It provides, via the formula $\pi(\Lambda(I)) = A_{\mathcal{S}}$, an alternative description of the attractor of $\mathcal{S}$ and, consequently, it is an important device in the topological study of Hutchinson-Barnsley fractals.

b) It is involved in the alternative presentation of the Hutchinson measure associated with $\mathcal{S}$ as the push-forward measure of the Bernoulli measure on $\Lambda(I)$ through π.

Preliminaries

The Hausdorff-Pompeiu metric

Given a metric space (X, d), *we define a function* $h^* : P_b(X) \times P_b(X) \to [0, \infty)$, *described by*

$$h^*(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in A} \inf_{x \in B} d(x, y)\},$$

for every $A, B \in P_b(X)$

Given a metric space (X, d), *the restriction of* h^* *to* $P_{b,cl}(X) \times P_{b,cl}(X)$ *is a metric, denoted by* h, *called the Hausdorff-Pompeiu metric.*

Given a set I and $n \in \mathbb{N}$, we consider:

$$I^{\mathbb{N}} \overset{not}{=} \Lambda(I)$$

The elements of $\Lambda(I)$, which is called a code space, are written as infinite words $\alpha = \alpha_1\alpha_2 \ldots \alpha_n\alpha_{n+1} \ldots$ with letters from I

$$I^{\{1,\ldots,n\}} \overset{not}{=} \Lambda_n(I).$$

The elements of $\Lambda_n(I)$ are written as words $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$ having n letters from I and n, which is denoted by $|\alpha|$, is called the length of α. In the sequel we will use the following notation:

$$\bigcup_{n\in\mathbb{N}\cup\{0\}} \Lambda_n(I) \overset{not}{=} \Lambda^*(I),$$

where $\Lambda_0(I) = \{\lambda\}$ and λ designates the empty word.

If $\alpha = \alpha_1\alpha_2...\alpha_n\alpha_{n+1}.... \in \Lambda(I)$ or if $\alpha = \alpha_1\alpha_2...\alpha_n \in \Lambda_n(B)$ and $m, n \in \mathbb{N}$, $n \geq m$, then we will use the following notation:

$$\alpha_1\alpha_2...\alpha_m \overset{not}{=} [\alpha]_m.$$

By the concatenation of the words α and β, where $\alpha = \alpha_1\alpha_2...\alpha_n \in \Lambda_n(I)$ and $\beta = \beta_1\beta_2...\beta_m\beta_{m+1}... \in \Lambda(I)$, we mean the infinite word $\alpha_1...\alpha_n\beta_1...\beta_m\beta_{m+1}...$ which is denoted by $\alpha\beta$.

For $i \in I$, we introduce the function $\tau_i : \Lambda(I) \to \Lambda(I)$, given by

$$\tau_i(\alpha) = i\alpha,$$

for all $\alpha \in \Lambda(I)$.

$\Lambda(I)$ can be endowed with the metric $d_{\Lambda(I)}$, called the Baire metric, described by:

$$d_\Lambda(\alpha, \beta) = \begin{cases} 0, & \text{if } \alpha = \beta \\ \frac{1}{2^{\min\{k \in \mathbb{N} \mid \alpha_k \neq \beta_k\}}}, & \text{if } \alpha \neq \beta \end{cases},$$

for all $\alpha = \alpha_1\alpha_2 ... \alpha_n\alpha_{n+1} ... \in \Lambda(I)$ and $\beta = \beta_1\beta_2 ... \beta_n\beta_{n+1} ... \in \Lambda(I)$.

Given $f_i : X \to X$, $i \in I$, and $\alpha = \alpha_1\alpha_2...\alpha_n \in \Lambda_n(I)$, the following notation will be used in the sequel:

$$f_{\alpha_1} \circ f_{\alpha_2} \circ .... \circ f_{\alpha_n} \overset{not}{=} f_\alpha$$

and

$$Id_X \overset{not}{=} f_\lambda.$$

In particular if $I = \{i\}$, we have $f_{\underbrace{i...i}_{n \text{ times}}} = f_i^{[n]}$ for every $n \in \mathbb{N}$.

Definition *A possibly infinite iterated function system (for short IIFS) is a pair* $((X,d),(f_i)_{i\in I})\overset{not}{=}\mathcal{S}$*, where* (X,d) *is a complete metric space and* $f_i : X \to X$*,* $i \in I$*, are such that:*

a) f_i *is continuous for every* $i \in I$*;*

b) the family $(f_i)_{i\in I}$ *is bounded, i.e.*

$$\bigcup_{i\in I} f_i(B) \in P_b(X),$$

for every $B \in P_b(X)$.

The function $F_{\mathcal{S}} : P_{b,cl}(X) \to P_{b,cl}(X)$*, given by*

$$F_{\mathcal{S}}(B) = \overline{\bigcup_{i \in I} f_i(B)},$$

for every $B \in P_{b,cl}(X)$*, is called the fractal operator associated with* $\mathcal{S}$*.*

In the framework of the above definition, if the functions f_i are Lipschitz, then

$$lip(F_{\mathcal{S}}) \leq \sup_{i \in I} lip(f_i).$$

For each IIFS $\mathcal{S} = (X, (f_i)_{i \in I})$ such that the functions f_i are Lipschitz and

$$\sup_{i \in I} lip(f_i) < 1,$$

$F_{\mathcal{S}}$ is a Banach contraction with respect to h and its unique fixed point $A_{\mathcal{S}}$ is called the attractor of $\mathcal{S}$.

Theorem[3] *We have:*
a) For every $\alpha \in \Lambda(I)$*, the set* $\bigcap_{n\in\mathbb{N}} \overline{f_{[\alpha]_n}(A_{\mathcal{S}})}$ *has just one element, which is denoted by* a_α*.*
b)

$$\lim_{n\to\infty} f_{[\alpha]_n}(x) = a_\alpha,$$

for every $x \in X$ *and every* $\alpha \in \Lambda(I)$*.*

[3] A. Mihail, R. Miculescu, The shift space for an infinite iterated function system, Math. Rep. Bucur., **61** (2009), 21-32.

c) The continuous (with respect to the Baire metric) function $\pi : \Lambda(I) \to A_{\mathcal{S}}$*, defined by*

$$\pi(\alpha) = a_{\alpha},$$

for every $\alpha \in \Lambda(I)$*, is called the canonical projection associated with* $\mathcal{S}$ *and it has the following properties:*

i) $\pi \circ \tau_i = f_i \circ \pi$*, for every* $i \in I$;

ii) $\overline{\pi(\Lambda(I))} = A_{\mathcal{S}}$.

Mixed possibly infinite iterated function systems

Given a metric space (X,d), $x \in X$, $B \subseteq X$ and a family of functions $\mathcal{F} = (f_i)_{i \in I}$, where $f_i : X \to X$, we shall use the notation:

$$\bigcup_{n \in \mathbb{N} \cup \{0\}} \overline{\bigcup_{\alpha \in \Lambda_n(I)} f_\alpha(B)} \overset{not}{=} \mathcal{O}_{\mathcal{F}}(B)$$

and

$$\mathcal{O}_{\mathcal{F}}(\{x\}) \overset{not}{=} \mathcal{O}_{\mathcal{F}}(x).$$

In particular, given an IIFS $\mathcal{S} = ((X,d),(f_i)_{i \in I})$ and $B \in P_{b,cl}(X)$, we shall use the notation

$$\mathcal{O}_{(f_i)_{i \in I}}(B) \overset{not}{=} \mathcal{O}_{\mathcal{S}}(B).$$

Definition *A mixed possibly infinite iterated function system (briefly mIIFS) is an IIFS* $\mathcal{S}=((X,d),(f_i)_{i\in I\cup J})$, *where* I *and* J *are disjoint sets, such that:*

a)

$$lip(f_i)\leq 1,$$

for every $i\in I\cup J$;

b) there exists $a\in[0,1)$ *having the following properties:*

b 1)

$$lip(f_i)\leq a,$$

for every $i\in I$;

b 2)

$$lip(f_{i|\overline{\mathcal{O}_{(f_i)_{i\in J}}(x)}})\leq a,$$

for every $x\in X$ *and every* $i\in J$.

Theorem ([4])

For each mIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$ *and every* $B \in P_{b,cl}(X)$ *there exists* $A_B \in P_{b,cl}(X)$ *such that*

$$F_{\mathcal{S}}(A_B) = A_B$$

and

$$\lim_{n\to\infty} h(F_{\mathcal{S}}^{[n]}(B), A_B) = 0,$$

i.e. $F_{\mathcal{S}}$ *is weakly Picard.*

In the above mentioned framework, for a word $\alpha = \alpha_1\alpha_2 \cdots \in \Lambda(I\cup J)$, we will use the notation

$$n_I(\alpha) = \#\{\alpha_i | \alpha_i \in I\}.$$

[4] B. Anghelina, R. Miculescu, A. Mihail, On the fractal operator of a mixed possibly infinite iterated function system, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **119** (2025), 31.

Proposition

For each mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, $\alpha \in \Lambda(I\cup J)$ and $x \in X$, the sequence $(f_{[\alpha]_n}(x))_{n\in\mathbb{N}}$ is convergent.

For an mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$ and $\alpha \in \Lambda(I\cup J)$, based on the previous proposition, we can consider the function $a_\alpha : X \to X$, given by

$$a_\alpha(x) = \lim_{n\to\infty} f_{[\alpha]_n}(x),$$

for every $x \in X$.

Proof

The proof is split into two parts, for $\alpha \in \Lambda(I \cup J)$ we have
a) $n_I(\alpha)$ is infinite;
b) $n_I(\alpha)$ is finite.
It suffices to prove that in each case $(f_{[\alpha]_n}(x))_{n\in\mathbb{N}}$ is Cauchy.

Corollary *For each mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, $\alpha \in \Lambda(I\cup J)$ *and* $B \in P_b(X)$, *we have*

$$\lim_{n\to\infty} \sup_{x\in B} d(f_{[\alpha]_n}(x), a_\alpha(x)) = 0.$$

Proposition *Given an mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, *we have*

$$\lim_{n\to\infty} h^*(f_{[\alpha]_n}(B), a_\alpha(B)) = 0,$$

for all $B \in P_b(X)$ *and* $\alpha \in \Lambda(I\cup J)$.

Proposition *For each mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$ and $\alpha \in \Lambda(I\cup J)$, we have:*
a)

$$lip(a_\alpha) \leq 1;$$

b) a_α is constant for every $\alpha \in \Lambda(I\cup J)$ with $n_I(\alpha)=\infty$.

Proposition *Given an mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, we have*

$$a_{i\alpha} = f_i \circ a_\alpha,$$

for all $i \in I\cup J$ and $\alpha \in \Lambda(I\cup J)$.

Proposition *Let us consider an mIIFS* $\mathcal{S}=((X,d),(f_i)_{i\in I\cup J})$, $\alpha=\alpha_1\alpha_2...\alpha_n...\in\Lambda(I\cup J)$ *and* $B\in P_{b,cl}(X)$ *such that* $F_{\mathcal{S}}(B)\subseteq B$.
a) If $n_I(\alpha)=\infty$, *then*

$$\bigcap_{n\in\mathbb{N}}\overline{f_{[\alpha]_n}(B)}=Ima_\alpha$$

and

$$\lim_{n\to\infty}h(\overline{f_{[\alpha]_n}(B)},Ima_\alpha)=0.$$

b) If $n_I(\alpha)<\infty$, *then*

$$a_\alpha(B)=f_{\alpha_1\alpha_2...\alpha_{n^*}}(a_{\alpha_{n^*+1}...\alpha_m...}(B)),$$

and

$$\overline{a_{\alpha_{n^*+1}...\alpha_m...}(B)}=\bigcap_{p\in\mathbb{N}}\overline{f_{\alpha_{n^*+1}...\alpha_{n^*+p}}(B)}=\lim_{p\to\infty}\overline{f_{\alpha_{n^*+1}...\alpha_{n^*+p}}(B)},$$

Proposition *For an mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, $\alpha \in \Lambda(I\cup J)$ *and* $B \in P_{b,cl}(X)$, *we have*

$$\overline{a_\alpha(B)} \subseteq A_B.$$

Proposition *Given an mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, *we have*

$$A_B = \overline{\underset{x\in B}{\cup} A_x},$$

for all $B \in P_{b,cl}(X)$.

Lemma *Given an mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, *we have*

$$A_{f_i(x)} \subseteq A_x,$$

for all $x \in X$ *and* $i \in I \cup J$.

Lemma *For every mIIFS* $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, *we have*

$$A_{a_\alpha(x)} \subseteq A_x,$$

for all $x \in X$ *and* $\alpha \in \Lambda(I \cup J)$.

A Code Space for mIIFSs

$\Lambda_1(I \cup J)$ denotes the set of finite words with letters from $I \cup J$ ending with a letter from I

$\Lambda_2(I \cup J)$ denotes the set of finite words with letters from $I \cup J$ starting with a letter from I

$\Lambda_3(I \cup J) = \Lambda_1(I \cup J) \cap \Lambda_2(I \cup J)$ denotes the set of finite words with letters from $I \cup J$ starting and ending with letters from I

$\Sigma_0(I,J)$ denotes the set of finite words with letters from $I \cup J \cup \Lambda(J)$ having the form

$$\beta_0\gamma_1\beta_1 ... \gamma_n\beta_n,$$

with $n \in \mathbb{N}$, where

$$\beta_0 \in \{\lambda\} \cup \Lambda_1(I \cup J),$$

$$\beta_n \in \{\lambda\} \cup \Lambda_2(I \cup J),$$

$$\gamma_k \in \Lambda(J)$$

for every $k \in \{1,...,n\}$ and

$$\beta_k \in \Lambda_3(I \cup J),$$

for every $k \in \{1,...,n-1\}$ if $n \geq 2$

$$\Sigma_1(I,J) = \{\alpha \in \Lambda(I \cup J) \mid n_I(\alpha) = \infty\}$$

$$\Sigma(I,J) = \Sigma_0(I,J) \cup \Sigma_1(I,J).$$

The Canonical Projection associated with an mIIFS

For an mIIFS $\mathcal{S}=((X,d),(f_i)_{i\in I\cup J})$, $B\in P_{b,cl}(X)$ and $\alpha\in\Sigma(I,J)$, we introduce the function $\mathcal{A}_\alpha : X\to X$ as follows:

If $\alpha\in\Sigma_1(I,J)$ then

$$\mathcal{A}_\alpha = a_\alpha.$$

If $\alpha=\beta_0\gamma_1\beta_1...\gamma_n\beta_n\in\Sigma_0(I,J)$, $n\in\mathbb{N}$.

$$\beta_0\in\{\lambda\}\cup\Lambda_1(I\cup J),\qquad \beta_n\in\{\lambda\}\cup\Lambda_2(I\cup J),\qquad \beta_l\in\Lambda_3(I\cup J),$$
$$\gamma_k\in\Lambda(J)$$

for every $l\in\{1,...,n-1\}$ if $n\geq 2$ and $k\in\{1,\ldots,n\}$ then

$$\mathcal{A}_\alpha = f_{\beta_0}\circ a_{\gamma_1}\circ f_{\beta_1}\circ a_{\gamma_2}\circ\ldots\circ f_{\beta_{n-1}}\circ a_{\gamma_n}\circ f_{\beta_n}.$$

Theorem

For every mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$ *and* $B \in P_{b,cl}(X)$*, we have*

$$\overline{\{\mathcal{A}_\alpha(x) \mid \alpha \in \Sigma(I,J) \text{ and } x \in B\}} = A_B.$$

Definition

For an mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, *the function* $\pi : \Sigma(I,J)\times X \to X$, *given by*

$$\pi(\alpha,x) = \mathcal{A}_\alpha(x),$$

for every $(\alpha,x)\in\Sigma(I,J)\times X$, *is called the canonical projection associated with* $\mathcal{S}$.

Proposition

For each mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, *the set* $\pi(\alpha,X)$ *has just one element for all* $\alpha\in\Sigma_1(I,J)$.

Proposition

For each mIIFS $\mathcal{S} =((X,d),(f_i)_{i\in I\cup J})$, *we have*

$$\pi \circ \tau_i = f_i \circ \pi,$$

for every $i \in I \cup J$, *where* $\tau_i : \Sigma(I,J) \times X \to \Sigma(I,J) \times X$ *is defined by*

$$\tau_i(\alpha, x) = (i\alpha, x),$$

for every $(\alpha, x) \in \Sigma(I,J) \times X$, *i.e. the following diagram is commutative*

$$\begin{array}{ccc} \Sigma(I,J)\times X & \xrightarrow{\tau_i} & \Sigma(I,J)\times X \\ \pi\downarrow & & \downarrow\pi \\ X & \xrightarrow{f_i} & X \end{array}.$$

Theorem

Let $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$ *be a mIIFS and* $B \in P_{b,cl}(X)$. *Then*

$$\overline{\pi(\Sigma(I,J)\times B)} = A_B.$$

Question

Is there a suitable metric on $\Sigma(I, J) \times X$ that makes π continuous?

Thank you for your attention!

The Hausdorff-Pompeiu metric

Remark

i) Given a metric space (X, d) *and* A *and* B *subsets of* X*, we have*

$$D(A, B) = D(\overline{A}, \overline{B}).$$

ii) Given a metric space (X, d) *and a set* I*, we have*

$$D(\underset{i \in I}{\cup} A_i, \underset{i \in I}{\cup} B_i) \leq \sup_{i \in I} D(A_i, B_i),$$

for every A_i *and* B_i *subsets of* X*.*

Proposition *Given a metric space* (X, d), *the function* $h^* : P_b(X) \times P_b(X) \to [0, \infty)$, *described by*

$$h^*(A, B) = \max\{D(A, B), D(B, A)\},$$

for every $A, B \in P_b(X)$, *has the following properties:*
i)

$$h^*(\{x\}, \{y\}) = d(x, y),$$

for every $x, y \in X$;
ii)

$$h^*(A, B) = h^*(\overline{A}, \overline{B}),$$

for every $A, B \in P_b(X)$;
iii)

$$h^*(\underset{i \in I}{\cup} A_i, \underset{i \in I}{\cup} B_i) \leq \sup_{i \in I} h^*(A_i, B_i),$$

for all families $\{A_i\}_{i \in I}$ *and* $\{B_i\}_{i \in I}$ *of elements of* $P_b(X)$ *such that* $\underset{i \in I}{\cup} A_i \in P_b(X)$ *and* $\underset{i \in I}{\cup} B_i \in P_b(X)$.

Proposition *Given a complete metric space (X,d), the metric space $(P_{b,cl}(X),h)$ is complete and if $(A_n)_{n\in\mathbb{N}} \subseteq P_{b,cl}(X)$ is Cauchy, then*

$$\lim_{n\to\infty} A_n = \{x \in X \mid \textit{there exist an increasing sequence } (n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$$

$$\textit{and } x_{n_k} \in A_{n_k} \textit{ for every } k \in \mathbb{N} \textit{ such that } \lim_{k\to\infty} x_{n_k} = x\}.$$

Additionally, we consider the function $g : X \to P_{b,cl}(X)$, given by

$$g(x) = A_x,$$

for every $x \in X$.
Lemma *Given an mIIFS* $\mathcal{S} =((X, d), (f_i)_{i \in I \cup J})$, *we have*

$$lip(g) \leq 1.$$

Given an mIIFS $\mathcal{S} = ((X,d),(f_i)_{i\in I\cup J})$, $x\in X$, $B\in P_{b,cl}(X)$ and $\alpha=\alpha_1\alpha_2...\alpha_n...\in\Lambda(I\cup J)$ or $\alpha=\alpha_1\alpha_2...\alpha_n\in\Lambda^*(I\cup J)$, we shall use the following notation:

$$A_{\{x\}} \overset{not}{=} A_x$$

$$\mathcal{O}_{\mathcal{S}_J}(B) \overset{not}{=} \mathcal{O}_J(B)\in P_b(X)$$

$$\sup_{x\in B}\max\{diam(\{x\}\cup F_{\mathcal{S}}(\{x\})), diam(\mathcal{O}_{\mathfrak{S}}(x)), diam(\mathcal{O}_J(x))\} \overset{not}{=} N_B\in\mathbb{R}$$

$$card(\{l\in\mathbb{N}\mid \alpha_l\in I\}) \overset{not}{=} n_I(\alpha).$$