# On the almost-everywhere convergence of two-parameter ergodic averages along directional rectangles

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# Introduction

Let $T : X \to X$ be a measure-preserving transformation on a probability space $(X, \mathcal{A}, \mu)$.

**Theorem - Birkhoff, 1931**

If $f \in L^1(X)$, then the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$$

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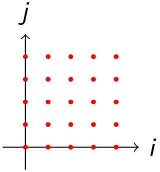
Let $S, T : X \to X$ be a pair of commuting, invertible, measure-preserving, ergodic transformations on a probability space $(X, \mathcal{A}, \mu)$.




**Theorem**

If $f \in L^1(X)$, then the limit

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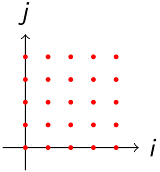


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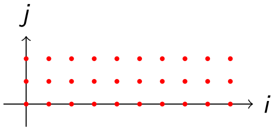


**Theorem - Dunford and Zygmund (independently), 1951**

If $f \in L\log L(X)$, then the limit

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(S^i T^j x)$$

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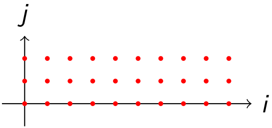


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**Remark :** There appears to be a strong analogy between ergodic averages and the differentiation of integrals.

**Theorem - Jessen, Marcinkiewicz, and Zygmund, 1935**

Let $\mathcal{I}$ denote the set of all rectangles in the plane whose sides are parallel to the coordinate axes. If $f \in L\log L(\mathbb{R}^2)$, then for almost every $x \in \mathbb{R}^2$, if $\{R_k(x)\}_{k\in\mathbb{N}} \subset \mathcal{I}$ is a sequence of rectangles containing $x$ and such that $\lim_{k\to\infty} \operatorname{diam}(R_k) = 0$, one has :

$$\lim_{k\to\infty} \frac{1}{|R_k(x)|} \int_{R_k(x)} f = f(x).$$

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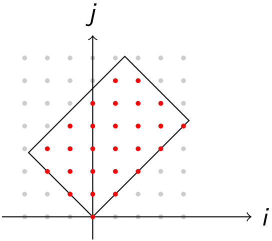


## Definition

- If $R \subset \mathbb{R}^2$ is a rectangle let $l(R)$ be the length of its shortest side.
- An *averaging process* is a sequence of rectangles $\{R_n\}_{n \in \mathbb{N}}$ containing the origin $(0,0)$ and such that $l(R_n) \to \infty$ as $n \to \infty$.

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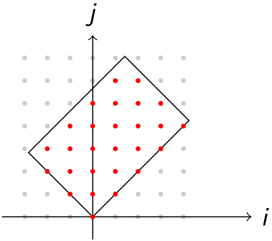


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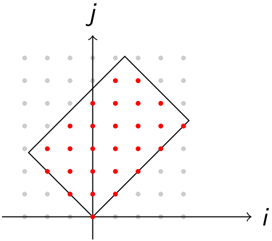


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# The Maximal Operator

Let $\{R_n\}_{n\in\mathbb{N}}$ be an averaging process, and define for $n \in \mathbb{N}$

$$M_n : f \mapsto \frac{1}{\#(R_n \cap \mathbb{Z}^2)} \sum_{(i,j)\in R_n} f(S^i T^j).$$

**Definition**

We define the maximal operator associated to the averaging process $\{R_n\}_{n\in\mathbb{N}}$ as

$$M^* : f \mapsto M^* f := \sup_{n\in\mathbb{N}} M_n|f|.$$

The maximal operator $M^*$ is of weak-type $(p, p)$, $1 \leqslant p < \infty$, if for any $f \in L^p(X)$ and any $\lambda > 0$ one has

$$\mu(\{x \in X : M^* f(x) > \lambda\}) \lesssim \left(\frac{\|f\|_{L^p(X)}}{\lambda}\right)^p.$$

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# A Sawyer-Stein principle

## Sawyer-Stein principle - Oniani, 2023

Let $\{R_n\}_{n\in\mathbb{N}}$ be an averaging process and $1 \leqslant p < \infty$. The following are equivalent.

1. Given any $f \in L^p(X)$, the limit

$$\lim_{n\to\infty} M_n f(x) = \lim_{n\to\infty} \frac{1}{\#(R_n \cap \mathbb{Z}^2)} \sum_{(i,j)\in R_n} f(S^i T^j x)$$

   exists for almost every $x \in X$.
2. The maximal operator $M^*$ is of weak-type $(p, p)$.

# A result in the theory of differentiation of integrals

## Definition

We define the slope of a rectangle as the tangent of the angle formed between its longest side and the horizontal axis.

## Definition

Let $\Omega = \{u_k^{-1}\}_{k\in\mathbb{N}^*}$, where $\{u_k\}_{k\in\mathbb{N}^*}$ is an increasing sequence of nonnegative real numbers. We say that $\Omega$ satisfies the property $(P)$ if the following two properties holds

$$\forall\, k \in \mathbb{N}^*, \quad 1 + u_{k-1}^2 \geqslant c(u_k - u_{k-1})^2; \tag{1}$$

$$\sup_{k\in\mathbb{N}} \sup_{1\leqslant l\leqslant k} \left( \frac{u_{k+2l} - u_{k+l}}{u_{k+l} - u_k} + \frac{u_{k+l} - u_k}{u_{k+2l} - u_{k+l}} \right) < \infty; \tag{2}$$

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# A result in the theory of differentiation of integrals

**Exemple**

The set of slopes $\Omega = \{k^{-s}\}_{k\in\mathbb{N}^*}$ for $s > 0$ satisfies the property $(P)$.

**Theorem - D'Aniello, Gauvan, Moonens, Rosenblatt, 2023**

Let $\Omega = \{u_k^{-1}\}_{k\in\mathbb{N}^*}$ be a sequence satisfying the property $(P)$. Then, there exists a sequence of rectangles $\{R_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^2$ such that :

(*i*) for each $k \in \mathbb{N}^*$, the slope of the rectangle $R_k$ is $\frac{1}{u_k}$ ;

(*ii*) $\mathrm{diam}(R_k) \to 0$ as $k \to \infty$ ;

(*iii*) there exists a function $f \in L^\infty(\mathbb{R}^2)$ such that

$$\frac{1}{|R_k|}\int_{x+R_k} f$$

fails to converge for almost every $x \in \mathbb{R}^2$ when $k \to \infty$.

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# Transfer lemma

Let $\{R_n\}_{n\in\mathbb{N}}$ be an averaging process. For $\varphi \in \ell^\infty(\mathbb{Z}^2)$, we define

$$A_n\varphi(k,l) := \frac{1}{\#(R_n \cap \mathbb{Z}^2)} \sum_{(i,j)\in R_n} \varphi(k+i, l+j)$$

and

$$A^*\varphi(k,l) := \sup_{n\in\mathbb{N}} A_n|\varphi|(k,l).$$

**Lemma**

Let $1 \leqslant p < \infty$. The following are equivalent.

1. The maximal operator $M^*$ is of weak-type $(p,p)$.
2. The maximal operator $A^*$ is of weak-type $(p,p)$ in $\ell^p(\mathbb{Z}^2)$.

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# A negative result

**Theorem**

Let $\Omega = \{u_k^{-1}\}_{k\in\mathbb{N}^*}$ be a sequence satisfying property $(P)$. Then, there exists an averaging process $\{R_k\}_{k\in\mathbb{N}}$ such that

(*i*) for each $k \in \mathbb{N}^*$, the slope of the rectangle $R_k$ is $\frac{1}{u_k}$ ;

(*ii*) for all $1 \leqslant p < \infty$, there exists a function $f \in L^p(X)$ such that the averages

$$\frac{1}{\#(R_k \cap \mathbb{Z}^2)} \sum_{(i,j)\in R_k} f(S^i T^j)$$

fail to converge almost everywhere.

# A positive result

Is it possible to have some convergence for all $f \in L^p(X)$ ?

## Definition

A sequence of nonnegative numbers $\{u_k\}_{k\in\mathbb{N}}$ is lacunary if there exists $\lambda \in (0,1)$ such that

$$\forall\, k \in \mathbb{N}, \quad u_{k+1} \leqslant \lambda u_k.$$

## Theorem

Let $\Omega \subset (0,1)$ be a lacunary sequence converging to 0 and let $\{R_n\}_{n\in\mathbb{N}}$ be an averaging process such that for any $n \in \mathbb{N}$, the slope of $R_n$ is in $\Omega$. Then $M^*$ is of weak-type $(p,p)$ for any $1 < p < \infty$.

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# A positive result

## Corollary

Let $\Omega \subset (0,1)$ be a lacunary sequence converging to 0 and let $\{R_n\}_{n\in\mathbb{N}}$ be a averaging process such that for any $n \in \mathbb{N}$, the slope of $R_n$ is in $\Omega$. Then for any $p \in (1,\infty]$, if $f \in L^p(X)$, the averages

$$\frac{1}{\#(R_n \cap \mathbb{Z}^2)} \sum_{(i,j)\in R_n} f(S^i T^j)$$

converge almost everywhere.

# Main references


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Thanks for your attention !