On set–valued maps additive “modulo K”

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Introduction

Let X be a real linear space. A function $f\colon X \to \mathbb{R}$ is called:

- *additive*, if $f(x+y) = f(x) + f(y), \quad x, y \in X,$
- *subadditive*, if $f(x+y) \leq f(x) + f(y), \quad x, y \in X,$
- *midconvex*, if $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \quad x, y \in X,$

- *superadditive*, if $-f$ is subadditive,
- *midconcave*, if $-f$ is midconvex.

subadditivity	$\Longleftarrow$	additivity	$\Longrightarrow$	midconvexity
	$\not\Longrightarrow$		$\not\Longleftarrow$	
	$f(x) = \lvert x \rvert$		$f(x) = x - 2$	

	$\not\Longleftarrow$	
	$f(x) = x - 2$	
subadditivity		midconvexity
	$\not\Longrightarrow$	
	$\chi_{\mathbb{R}\setminus\mathbb{Q}}$	

Set–valued maps

Let X, Y be real vector spaces, and $n(Y) = 2^Y \setminus \{\emptyset\}$.

✓ D. Henney, *Set valued additive functions*, Math. Jap. 11 (1966), 117–120.

✓ W. Smajdor, *Subadditive and subquadratic set–valued functions*, Prace Nauk. Uniwersytetu Śląskiego w Katowicach 889, Katowice 1987.

A set–valued map $F: X \to n(Y)$ is called:

- *(Henney 1966)*
  additive, if $F(x_1) + F(x_2) = F(x_1 + x_2), \quad x_1, x_2 \in X$,
- *(Smajdor 1987)*
  subadditive, if $F(x_1) + F(x_2) \subset F(x_1 + x_2), \quad x_1, x_2 \in X$,
  superadditive, if $F(x_1) + F(x_2) \supset F(x_1 + x_2), \quad x_1, x_2 \in X$.

In fact, all three notions generalized the notion of additivity of single–valued maps!

Let $K \subset Y$ be a closed convex cone such that $K \cap (-K) = \{0\}$.

✓ K. Nikodem, *K–convex and K–concave set–valued functions*, Zeszyty Nauk. Politechniki Łódzkiej Mat. 559; Rozprawy Mat. 114, Łódź 1989.

✓ E. Jabłońska, K. Nikodem, *K–subadditive and K–superadditive set–valued functions bounded on "large" sets.* Aequationes Math. 95 (2021), 1221–1231.

- *(Nikodem 1989)*

  K–**midconvex**, if $\dfrac{F(x_1)+F(x_2)}{2} \subset F\left(\dfrac{x_1+x_2}{2}\right) + K, \quad x_1, x_2 \in X,$

  K–**midconcave**, if $F\left(\dfrac{x_1+x_2}{2}\right) \subset \dfrac{F(x_1)+F(x_2)}{2} + K, \quad x_1, x_2 \in X,$
- *(J.–Nikodem 2021)*

  K–**subadditive**, if $F(x_1)+F(x_2) \subset F(x_1+x_2) + K, \quad x_1, x_2 \in X,$

  K–**superadditive**, if $F(x_1+x_2) \subset F(x_1)+F(x_2) + K, \quad x_1, x_2 \in X.$

✓ E. Jabłońska, W. Jabłoński, *Properties of K-Additive Set-Valued Maps*, Results Math. 78 (2023), 221.

- *(J.–Jabłoński 2023)* K–**additive**, if it is simultaneously K–subadditive and K–superadditive, i.e.

$$\begin{cases} F(x_1) + F(x_2) \subset F(x_1 + x_2) + K, \\ F(x_1 + x_2) \subset F(x_1) + F(x_2) + K, \end{cases} \qquad x_1, x_2 \in X.$$

1. If $K = \{0\}$, then K–additivity of F means Henney's additivity.
2. If $K = [0, \infty)$, $Y = \mathbb{R}$, and $F(x) = [m(x), M(x)]$ for $x \in X$, where $m, M \colon X \to \mathbb{R}$ and $m \leq M$, then K–additivity of F means the classical additivity of m.

We introduce the equivalence relation $=_K$ in $n(Y)$ by

$$\boxed{\forall\, A, B \in n(Y) \quad [A =_K B \iff (A \subset B + K \ \wedge \ B \subset A + K)]}$$

$$F(x+y) =_K F(x) + F(y) \quad \text{for } \; x, y \in X.$$

$$\boxed{\forall\, A, B \in n(Y) \quad [A =_K B \iff A + K = B + K]}$$

This property allows us to find a minimal representative of any equivalence class.

Example 1

Let $Y = \mathbb{R}^2$ and $K = [0,\infty) \times [0,\infty)$. If S is the closed unit disk with center $(0,0)$, then $S =_K S_1$, where S_1 is the following part of the unit circle with the same center

$$S_1 := \left\{(x,y) \in \mathbb{R}^2 \colon x \le 0,\; y \le 0,\; x^2 + y^2 = 1\right\}.$$

Basic properties of K–additive s.v. maps

Let $F\colon X \to n(Y)$ be K–additive. Then,

- $(tF)(x) := tF(x)$ for $x \in X$, where $t \geq 0$,
- $(\mathrm{conv}\, F)(x) := \mathrm{conv}\, F(x)$ for $x \in X$,
- $(\mathrm{cl}\, F)(x) := \mathrm{cl}\, F(x)$ for $x \in X$,
  where $F(x)$ are relatively compact for $x \in X$,
- $(\mathrm{int}\, F)(x) := \mathrm{int}\, F(x)$ for $x \in X$,
  where $F(x)$ are convex and $\mathrm{int}\, F(x) \neq \emptyset$ for $x \in X$,

are also K-additive.

Let $F, G\colon X \to n(Y)$ be K–additive. Then,

- $(F + G)(x) := F(x) + G(x)$ for $x \in X$, is also K–additive,
- $(F \times G)(x) := F(x) \times G(x)$ for $x \in X$, is $(K \times K)$–additive.

K–Jensen v.s. K–additivity

Theorem: *A function $f : X \to \mathbb{R}$ is additive if and only if $f(0) = 0$ and f is Jensen, i.e.,*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \quad x, y \in X.$$

Theorem (J.–Jabłoński 2023)

If $F : X \to n(Y)$ is a K–Jensen midconvex–valued map, i.e.,

$$F\left(\frac{x+y}{2}\right) =_K \frac{1}{2}(F(x)+F(y)), \quad x, y \in X,$$

such that $F(0) =_K \{0\}$, then it is K–additive.

Conversely,

Theorem (J.–Jabłoński 2023)

If $F\colon X \to n(Y)$ is a K–additive midconvex–valued map, then it is K–Jensen. If, moreover, $F(0)$ **is a compact convex set**, *then $F(0) =_K \{0\}$.*

There are K–additive s.v. maps such that $F(0) \neq_K \{0\}$!

Example 2

Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, and $K = [0, \infty) \times \{0\}$. Let $F\colon \mathbb{R} \to n(\mathbb{R}^2)$,

$$F(x) := \{x\} \times \mathbb{R}, \quad x \in \mathbb{R}.$$

- $F(0) = \{0\} \times \mathbb{R}$, so it is **not compact**,
- $F(0) \neq_K \{(0, 0)\}$, because
$$F(0) + K = [0, \infty) \times \mathbb{R} \neq K = \{(0, 0)\} + K,$$
- F is convex–valued and K–additive, because
$$F(x + y) + K = [x + y, \infty) \times \mathbb{R} = F(x) + F(y) + K.$$

K–continuous K–additive s.v. maps

- $\mathcal{B}(Y)$ – the family of all nonempty bounded subsets of Y.
- $\mathcal{BC}(Y)$ – the family of all nonempty bounded convex subsets of Y.
- $\mathcal{CC}(Y)$ – the family of all nonempty compact convex subsets of Y.

A set–valued map $F\colon X \to n(Y)$ is called:

- ***K*–lower bounded** on a set $A \subset X$, if

$$\exists\, B \in \mathcal{B}(Y)\ \ \forall\, x \in A \quad F(x) \subset B + K,$$

- **weakly *K*–upper bounded** on a set $A \subset X$, if

$$\exists\, B \in \mathcal{B}(Y)\ \ \forall\, x \in A \quad F(x) \cap (B - K) \neq \emptyset,$$

- ***K*–continuous** at a point $x_0 \in X$, if

$$\forall\, W_0 \subset Y \quad \exists\, U_{x_0} \subset X \quad \forall\, x \in U_{x_0} \qquad \begin{cases} F(x_0) \subset F(x) + W + K, \\ F(x) \subset F(x_0) + W + K. \end{cases}$$

1. If $K = \{0\}$:
   - K–lower boundedness of $F \iff$ boundedness of F, i.e.,

$$\exists\, B \in \mathcal{B}(Y)\ \forall\, x \in A \quad F(x) \subset B,$$

   - weak K–upper boundedness of $F \iff$ weak boundedness of F, i.e.,

$$\exists\, B \in \mathcal{B}(Y)\ \forall\, x \in A \quad F(x) \cap B \neq \emptyset,$$

   - K–continuity of $F \iff$ continuity with respect to the Hausdorff metric.
2. If $K = [0, \infty)$, $Y = \mathbb{R}$, and $F(x) = [m(x), M(x)]$ for $x \in X$, where $m, M \colon X \to \mathbb{R}$ and $m \leq M$, then:
   - K–lower boundedness of $F \iff$ boundedness from below of m,
   - weak K–upper boundedness of $F \iff$ boundedness from above of m,
   - K–continuity of $F \iff$ the classical continuity of m.

Theorem: *Let $A \subset X$ be a "large" set, i.e.,*

- *A is of the positive Lebesgue measure provided $X = \mathbb{R}^n$ with $n \in \mathbb{N}$,*
- *A is non–meager with the Baire property.*

If $f : X \to \mathbb{R}$ is additive and bounded from above/below on A, then f is continuous.

Theorem (J.–Jabłoński 2023)

Let $A \subset X$ be a "large" set. If an s.v. map $F : X \to \mathcal{BC}(Y)$ is K–additive and weakly K–upper bounded or K–lower bounded on a set A, then F is K–continuous on X.

<u>Theorem:</u> *If $f: X \to \mathbb{R}$ is a continuous additive function, then f is homogenous, i.e., $f(tx) = tf(x)$ for $t \in \mathbb{R}$, $x \in X$, i.e.,*

$$\{t \in \mathbb{R} \colon f(tx) = tf(x) \text{ for } x \in X\} = \mathbb{R}.$$

Theorem (J.–Jabłoński 2023)

If an s.v. map $F \colon X \to \mathcal{CC}(Y)$ is K–continuous and K–additive, then F is K–homogenous, i.e.,

$$F(tx) =_K tF(x), \quad t \geq 0,\ x \in X.$$

Moreover, if F is not single–valued, then

$$\{t \in \mathbb{R} \colon F(tx) =_K tF(x) \text{ for } x \in X\} = [0, \infty).$$

Theorem: *If* $f: X \to \mathbb{R}$ *is an additive function, then the set*

$$H_f := \{t \in \mathbb{R} \colon f(tx) = tf(x) \text{ for } x \in X\}$$

is a subfield of $\mathbb{R}$ *(called the* **homogeneity field of** f*).*

Theorem (J.–Jabłoński 2023)

If an s.v. map $F \colon X \to \mathcal{CC}(Y)$ *is* K*–additive and not single–valued, then*

$$H_{F,K} := \{t \in \mathbb{R} \colon F(tx) =_K tF(x) \text{ for } x \in X\}$$

(called K**–homogeneity set of** F*) satisfies the following conditions:*

(i) $\{0, 1\} \subset H_{F,K} \subset [0, \infty)$,

(ii) $s + t \in H_{F,K}$ *for* $s, t \in H_{F,K}$,

(iii) $\frac{s}{t} \in H_{F,K}$ *for* $s, t \in H_{F,K}$ *with* $t \neq 0$.

Theorem: *For every field $L \subset \mathbb{R}$ there is an additive function $f\colon X \to \mathbb{R}$ such that $H_f = L$.*

Theorem (J.–Jabłoński 2023)

Let $S \subset \mathbb{R}$ be a set satisfying the following conditions:

(i) $\{0,1\} \subset S \subset [0,\infty)$,

(ii) $s+t \in S$ *for* $s,t \in S$,

(iii) $\frac{s}{t} \in S$ *for* $s,t \in S$ *with* $t \neq 0$,

(iv) $s-t \in S$ *for* $s,t \in S$ *with* $s-t \geq 0$.

Then there exists an s.v. map $F\colon X \to \mathcal{CC}(Y)$ such that $S = H_{F,K}$.

Problem: Is it true that

$$s-t \in H_{F,K} \quad \text{for } s,t \in H_{F,K} \text{ such that } s-t \geq 0$$

for a K–additive s.v. map $F\colon X \to \mathcal{CC}(Y)$ which is not single–valued?

Theorem: *A function* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and additive if and only if there is* $a \in \mathbb{R}$ *such that*

$$f(x) = ax \quad \text{for } x \in \mathbb{R}.$$

✓ E. Jabłońska, *Characterization of continuous additive set–valued maps "modulo K" on finite dimensional linear spaces*, Math. Slovaca 74 (2024), 1165–1172.

Theorem (J. 2024)

An s.v. map $F : \mathbb{R} \to \mathcal{CC}(Y)$ *is* K*–continuous and* K*–additive if and only if there are sets* $A, B \in \mathcal{CC}(Y)$ *such that* $0 \in A - B \subset K$ *and*

$$F(t) =_K \begin{cases} tA, & t \geq 0, \\ tB, & t < 0. \end{cases}$$

The above result is optimal, i.e., K–continuous and K–additive s.v. map needn't be given by $F(t) =_K tC$ for $t \in \mathbb{R}$ with some $C \in \mathcal{CC}(Y)$.

Example 3

Let $Y = \mathbb{R}$, $K = [0, \infty)$, $A = [0, 1]$ and $B = [-2, 0]$. Then,

- $0 \in A - B \subset K$, so F defined above is K–continuous and K–additive.
- $A \neq_K B$, because

$$A + K = [0, \infty) \neq [-2, \infty) = B + K.$$

Theorem (J. 2024)

Let $N \in \mathbb{N}$. An s.v. map $F\colon \mathbb{R}^N \to \mathcal{CC}(Y)$ is K-continuous and K-additive if and only if there exist sets $A_1, B_1, \ldots, A_N, B_N \in \mathcal{CC}(Y)$ such that

$$0 \in A_i - B_i \subset K, \qquad i \in \{1, \ldots, N\}$$

and

$$F(t_1, \ldots, t_N) =_K t_1 C_1 + \ldots + t_N C_N, \qquad (t_1, \ldots, t_N) \in \mathbb{R}^N,$$

where

$$tC_i = \begin{cases} tA_i, & t \geq 0, \\ tB_i, & t < 0, \end{cases} \qquad i \in \{1, \ldots, N\}.$$

i.e., for $N = 2$,

an s.v. map $F\colon \mathbb{R}^2 \to \mathcal{CC}(Y)$ is K-continuous and K-additive if and only if there exist sets $A_1, B_1, A_2, B_2 \in \mathcal{CC}(Y)$ such that

$$0 \in A_i - B_i \subset K, \qquad i \in \{1, 2\},$$

and

$$F(t_1, t_2) =_K \begin{cases} t_1 A_1 + t_2 A_2, & t_1 \geq 0, \ \ t_2 \geq 0, \\ t_1 A_1 + t_2 B_2, & t_1 \geq 0, \ \ t_2 < 0, \\ t_1 B_1 + t_2 A_2, & t_1 < 0, \ \ t_2 \geq 0, \\ t_1 B_1 + t_2 B_2, & t_1 < 0, \ \ t_2 < 0. \end{cases}$$

$K-$ continuous $K-$sublinear s.v. maps

A function $f : \mathbb{R} \to \mathbb{R}$ is called *sublinear* if it is subadditive and

$$f(nx) = nf(x) \quad \text{for} \ \ x \in X, \ \ n \in \mathbb{N}.$$

additivity	$\Longrightarrow$ $\not\Longleftarrow$ $f(x) = \lvert x \rvert$	sublinearity	$\Longrightarrow$ $\not\Longleftarrow$ $f(x) = \sqrt{\lvert x \rvert}$	subadditivity

<u>Theorem:</u> *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous and sublinear if and only if then there are reals $a \geq b$ such that*

$$f(x) = \begin{cases} ax, & x \geq 0, \\ bx, & x < 0. \end{cases}$$

✓ E. Jabłońska, *On K-sublinear set-valued maps.* Period. Math. Hungar. (accepted).

An s.v. map $F: \mathbb{R} \to n(Y)$ is called K–*sublinear* if it is K–subadditive and

$$F(nx) =_K nF(x) \quad \text{for } \; x \in X, \;\; n \in \mathbb{N}.$$

Theorem (J. 2024)

An s.v. map $F: \mathbb{R} \to \mathcal{CC}(Y)$ is K–continuous and K–sublinear if and only if there are sets $A, B \in \mathcal{CC}(Y)$ such that $A - B \subset K$ and

$$F(t) =_K \begin{cases} tA, & t \geq 0, \\ tB, & t < 0. \end{cases}$$

Thank you very much for your attention!