A general form of Gronwall inequality with Stieltjes integrals

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Gronwall inequality – classical version

If $u, K, L\colon [t_0, t_0 + T] \to [0, \infty)$ are continuous functions satisfying the integral inequality

$$u(t) \le K(t) + \int_{t_0}^{t} L(s)u(s)\,ds, \quad t \in [t_0, t_0 + T],$$

then we have the a priori bound

$$u(t) \le K(t) + \int_{t_0}^{t} K(s)L(s)\exp\left(\int_{s}^{t} L(\tau)\,d\tau\right)\,ds, \quad t \in [t_0, t_0 + T].$$

Special cases were first obtained by T. H. Gronwall (1919) and R. Bellman (1953).

If equality holds in the first relation, then it also holds in the second relation. If K is a constant function, we get the simpler estimate

$$u(t) \le K\exp\left(\int_{t_0}^{t} L(\tau)\,d\tau\right).$$

The goal

We are interested in Gronwall-type results with ordinary integrals replaced by Stieltjes integrals. A natural starting point might be to assume

$$u(t) \le K(t) + \int_{t_0}^{t} L(s)u(s)\,dg(s), \quad t \in [t_0, t_0 + T],$$

where g is a nondecreasing function.

But the integral on the right-hand side can be rewritten as $\int_{t_0}^{t} u(s)\,dP(s)$, where $P(s) = \int_{t_0}^{s} L(\tau)\,dg(\tau)$. Hence, it suffices to study the simpler integral inequality

$$u(t) \le K(t) + \int_{t_0}^{t} u(s)\,dP(s), \quad t \in [t_0, t_0 + T],$$

where P is a nondecreasing function. Our goal is to obtain a priori estimates for u.

Earlier results

Gronwall-type inequalities for Stieltjes integrals were investigated by several authors (Groh, Kurzweil, Schwabik, Márquez Albés). It is usually assumed that K is a constant function, or that P is a left-continuous or right-continuous function. We will show that these assumptions are not necessary, and offer a short proof based on the integral version of the quotient rule for Stieltjes integrals.

All Stieltjes integrals will be understood in the Kurzweil–Stieltjes sense, but they also exist in the Lebesgue–Stieltjes sense.

Kurzweil-Stieltjes integral

A function $f : [a, b] \to \mathbb{R}$ is Kurzweil-Stieltjes-integrable with respect to $g : [a, b] \to \mathbb{R}$ if there exists a number $I \in \mathbb{R}$ such that given an $\varepsilon > 0$, there is a gauge $\delta : [a, b] \to (0, \infty)$ such that

$$\left| \sum_{j=1}^{k} f(t_j)(g(\alpha_j) - g(\alpha_{j-1})) - I \right| < \varepsilon$$

for every partition with division points

$$a = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{k-1} \leq \alpha_k = b$$

and tags $t_j \in [\alpha_{j-1}, \alpha_j]$, $j \in \{1, \ldots, k\}$, such that

$$[\alpha_{j-1}, \alpha_j] \subset (t_j - \delta(t_j), t_j + \delta(t_j)), \quad j \in \{1, \ldots, k\}.$$

Preliminaries: substitution & the quotient rule

Theorem (McLeod, 1980)

Assume that $g, h\colon [a,b] \to \mathbb{R}$ are such that $\int_a^b g\,dh$ exists. Then for each function $f\colon [a,b] \to \mathbb{R}$, we have

$$\int_a^b f(x)\,d\left(\int_a^x g(z)\,dh(z)\right) = \int_a^b f(x)g(x)\,dh(x),$$

whenever either side of the equation exists.

Theorem (Márquez Albés & Slavík, 2023)

If $f, g : [a,b] \to \mathbb{R}$ have bounded variation and for each $t \in [a,b]$, we have $g(t) \neq 0$, $g(t-) \neq 0$, and $g(t+) \neq 0$, then

$$\frac{f(b)}{g(b)} - \frac{f(a)}{g(a)} = \int_a^b \frac{df(t)}{g(t+)} - \int_a^b \frac{f(t-)\,dg(t)}{g(t-)g(t+)}.$$

Preliminaries: generalized exponential function

If $P\colon [a,b] \to \mathbb{R}$ has bounded variation and satisfies $1+\Delta^+ P(t) \neq 0$ for all $t \in [a,t_0)$ and $1-\Delta^- P(t) \neq 0$ for all $t \in (t_0,b]$, then the linear integral equation

$$x(t) = 1 + \int_{t_0}^{t} x(s)\, dP(s), \quad t \in [a,b],$$

has a unique solution, which is known as the generalized exponential function, and is denoted by $t \mapsto e_{dP}(t,t_0)$.

Generalized exponential function: explicit formula

If P is continuous, then $e_{dP}(t,t_0)=e^{P(t)-P(t_0)}$.

$$e_{dP}(t,t_0)=\begin{cases}1, & t=t_0,\\[2ex] \dfrac{e^{P(t-)-P(t_0+)}}{e^{\sum_{s\in(t_0,t)}\Delta P(s)}}\;\dfrac{\prod\limits_{s\in[t_0,t)}(1+\Delta^+P(s))}{\prod\limits_{s\in(t_0,t]}(1-\Delta^-P(s))}, & t>t_0,\\[2ex] \dfrac{e^{\sum_{s\in(t,t_0)}\Delta P(s)}}{e^{P(t_0-)-P(t+)}}\;\dfrac{\prod\limits_{s\in(t,t_0]}(1-\Delta^-P(s))}{\prod\limits_{s\in[t,t_0)}(1+\Delta^+P(s))}, & t<t_0.\end{cases}$$

Corollary: If P is left-continuous and nondecreasing, then

$$e_{dP}(t,t_0)\le e^{P(t)-P(t_0)},\quad t\ge t_0.$$

Main result

Let $P\colon [t_0, t_0 + T] \to \mathbb{R}$ be a nondecreasing function such that $1 - \Delta^- P(s) > 0$ for all $s \in (t_0, t_0 + T]$.
If $K\colon [t_0, t_0 + T] \to [0, +\infty)$ is such that $\int_{t_0}^{t_0+T} K(s)\, dP(s)$ exists and $u\colon [t_0, t_0 + T] \to \mathbb{R}$ satisfies

$$u(t) \le K(t) + \int_{t_0}^{t} u(s)\, dP(s), \qquad t \in [t_0, t_0 + T], \tag{1}$$

then

$$u(t) \le K(t) + \int_{t_0}^{t} \frac{K(s) e_{dP}(t, s)}{(1 + \Delta^+ P(s))(1 - \Delta^- P(s))}\, dP(s), \quad t \in [t_0, t_0 + T], \tag{2}$$

If equality holds in (1), then it also holds in (2).
Finally, if K is bounded on $[t_0, t] \subset [t_0, t_0 + T]$, then

$$u(t) \le \left(\sup_{\xi \in [t_0, t]} K(\xi) \right) e_{dP}(t, t_0).$$

Proof outline (1)

We have

$$u(t) \le K(t) + U(t), \quad t \in [t_0, t_0 + T],$$

where

$$U(t) = \int_{t_0}^{t} u(s)\, dP(s), \quad t \in [t_0, t_0 + T].$$

Use quotient rule, substitution theorem and the properties of the generalized exponential function to get

$$\frac{U(t)}{e_{dP}(t, t_0)} = \int_{t_0}^{t} \frac{dU(s)}{e_{dP}(s+, t_0)} - \int_{t_0}^{t} \frac{U(s-)}{e_{dP}(s-, t_0) e_{dP}(s+, t_0)}\, d(e_{dP}(s, t_0))$$

$$= \int_{t_0}^{t} \frac{u(s)\, dP(s)}{(1 + \Delta^+ P(s)) e_{dP}(s, t_0)} - \int_{t_0}^{t} \frac{U(s-)\, dP(s)}{(1 - \Delta^- P(s))(1 + \Delta^+ P(s)) e_{dP}(s, t_0)}$$

for all $t \in [t_0, t_0 + T]$.

For the next step, observe that $U(s-) = U(s) - u(s)\Delta^- P(s)$.

Proof outline (2)

$$\frac{U(t)}{e_{dP}(t,t_0)} = \int_{t_0}^{t} \frac{u(s)}{(1+\Delta^+P(s))e_{dP}(s,t_0)}\,dP(s)$$
$$+\int_{t_0}^{t} \frac{(-U(s)+u(s)\Delta^-P(s))}{(1-\Delta^-P(s))(1+\Delta^+P(s))e_{dP}(s,t_0)}\,dP(s)$$
$$=\int_{t_0}^{t} \frac{1}{(1+\Delta^+P(s))e_{dP}(s,t_0)}\left(\frac{-U(s)+u(s)\Delta^-P(s)}{(1-\Delta^-P(s))}+u(s)\right)dP(s)$$
$$=\int_{t_0}^{t} \frac{1}{(1+\Delta^+P(s))e_{dP}(s,t_0)}\cdot\frac{u(s)-U(s)}{1-\Delta^-P(s)}\,dP(s).$$

Recall that $u(s)\le K(s)+U(s)$ and $1-\Delta^-P(s)>0$. Therefore,

$$\frac{U(t)}{e_{dP}(t,t_0)} \le \int_{t_0}^{t} \frac{K(s)}{(1+\Delta^+P(s))(1-\Delta^-P(s))e_{dP}(s,t_0)}dP(s).$$

Proof outline (3)

$$\frac{U(t)}{e_{dP}(t,t_0)} \le \int_{t_0}^{t} \frac{K(s)}{(1+\Delta^+P(s))(1-\Delta^-P(s))e_{dP}(s,t_0)}\,dP(s)$$

Multiply previous inequality by $e_{dP}(t,t_0)$ and observe that $e_{dP}(t,t_0)/e_{dP}(s,t_0) = e_{dP}(t,s)$ to get

$$u(t) \le K(t) + U(t) \le K(t) + \int_{t_0}^{t} \frac{e_{dP}(t,s)K(s)}{(1+\Delta^+P(s))(1-\Delta^-P(s))}\,dP(s).$$

If equality holds in the assumption (1), then it also holds in the conclusion (2).
If K is bounded on $[t_0,t]$, then

$$u(t) \le \left(\sup_{\xi\in[t_0,t]} K(\xi)\right)\left(1 + \int_{t_0}^{t} \frac{e_{dP}(t,s)}{(1+\Delta^+P(s))(1-\Delta^-P(s))}\,dP(s)\right).$$

It remains to show that the last term is $e_{dP}(t,t_0)$; use the quotient rule.

A comparison with existing results

- Schwabik (1992) as well as Monteiro & Slavík & Tvrdý (2019) assume that K is a constant, P is left-continuous and nondecreasing, and show that $u(t) \leq Ke^{P(t)-P(t_0)}$. But $e_{dP}(t, t_0) \leq e^{P(t)-P(t_0)}$, and the inequality is strict if P is not right-continuous.
- Kurzweil (2012) assumes that K is a constant and P is nondecreasing with $1 - \Delta^- P(t) > 0$ for $t \in (t_0, t_0 + T]$, and $1 - \Delta^+ P(t) > 0$ for $t \in [t_0, t_0 + T)$. The conclusion is that $u(t) \leq Ke_{dP}(t, t_0)$ for $t \in [t_0, t_0 + T]$.
- Márquez Albés (2021) deals with Lebesgue-Stieltjes integrals, P is left-continuous. In the second part, we assume that K is bounded instead of requiring that $t \mapsto K(t)(1 + \Delta^+ P(t))$ is nondecreasing.
- Groh (1980) assumes that K is a constant function and u has bounded variation. The assumptions on P are the same as ours.

Application: a uniqueness result for measure DEs

Consider the measure differential equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, dg(s), \qquad t \in [t_0, t_0 + T], \qquad (3)$$

where $f\colon [t_0, t_0 + T] \times \mathbb{R}^n \to \mathbb{R}^n$, and $g\colon [t_0, t_0 + T] \to \mathbb{R}$ has bounded variation.

Suppose there exists a function $L\colon [t_0, t_0 + T] \to [0, +\infty)$ such that $1 - L(s)|\Delta^- g(s)| > 0$ for all $s \in (t_0, t_0 + T]$, and

$$\left\| \int_c^d [f(s, x(s)) - f(s, y(s))]\, dg(s) \right\| \le \int_c^d L(s)\|x(s) - y(s)\|\, d(\mathrm{var}_{t_0}^s g)$$

for all $[c, d] \subseteq [t_0, t_0 + T]$ and all regulated functions $x, y\colon [t_0, t_0 + T] \to \mathbb{R}^n$. Then Eq. (3) has at most one solution.

Corollary: a uniqueness result for nabla dynamic equations (1)

Let us focus on nabla dynamic equations of the form

$$x^{\nabla}(t) = f(t, x(t)), \quad t \in [t_0, t_0 + T]_{\mathbb{T}},$$

where $f\colon [t_0, t_0 + T]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$.
If ρ is the backward jump operator given by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ and ν is the backward graininess given by $\nu(t) = t - \rho(t)$, then the nabla derivative $x^{\nabla}(t)$ is

$$x^{\nabla}(t) = \begin{cases} \frac{x(t) - x(\rho(t))}{\nu(t)} & \text{if } \nu(t) > 0, \\ x'(t) & \text{if } \nu(t) = 0. \end{cases}$$

Nabla dynamic equations are implicit in the sense that if $\nu(t) > 0$ and if we know the value $x(\rho(t))$, then finding the value $x(t)$ requires solving the equation

$$x(t) = x(\rho(t)) + f(t, x(t))\nu(t).$$

Corollary: a uniqueness result for nabla dynamic equations (2)

Consider the nabla dynamic equation in the integral form

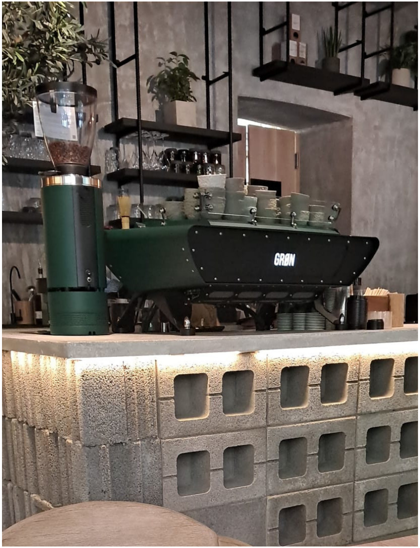
$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, \nabla s, \qquad t \in [t_0, t_0 + T]_{\mathbb{T}}, \qquad (4)$$

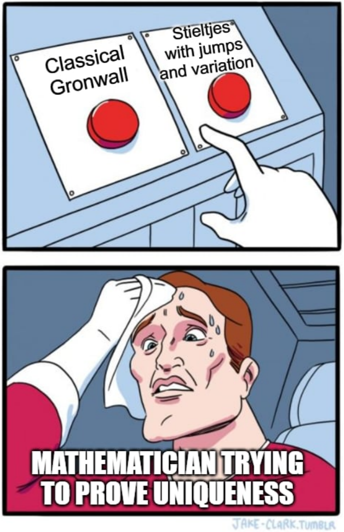
where $f\colon [t_0, t_0 + T]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$.
Suppose there exists a function $L\colon [t_0, t_0 + T]_{\mathbb{T}} \to [0, +\infty)$ such that $1 - L(s)\nu(s) > 0$ for all $s \in (t_0, t_0 + T]_{\mathbb{T}}$, and

$$\left\| \int_c^d [f(s, x(s)) - f(s, y(s))] \, \nabla s \right\| \le \int_c^d L(s) \|x(s) - y(s)\| \, \nabla s$$

for all $[c, d] \subseteq [t_0, t_0 + T]_{\mathbb{T}}$ and all regulated functions $x, y\colon [t_0, t_0 + T]_{\mathbb{T}} \to \mathbb{R}^n$. Then Eq. (4) has at most one solution.

Grøn wall & proving uniqueness





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