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A survey on the Riemann-Lebesgue integrability in non-additive setting

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The subject of this talk is in the field of integration in non-additive setting, which have applications, for instance, in

- statistics,
- computer science,
- control and game theories
- mathematical economics.

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Let S be a non-empty set with $\text{card}(C) \geq \aleph_0$ and $\mathcal{C}$ a σ-algebra of subsets of S.

The integrability we consider in this paper is related to the partitions of the whole space S.

A finite (countable) partition of S is a finite (countable) family of nonempty sets $P = \{E_i\}_{i=1}^n$ $(\{E_n\}_{n\in\mathbb{N}}$, resp.) $\subset \mathcal{C}$ such that $E_i \cap E_j = \emptyset, i \neq j$ and $\bigcup_{i=1}^n E_i = S$ $(\bigcup_{n\in\mathbb{N}} E_n = S,)$.

- If P and P' are two partitions of S, then P' is said to be *finer than* P,

$$P \leq P' \ \ (\text{or } P' \geq P), \text{ if every set of } P' \text{ is included in some set of } P. \tag{1}$$

- The *common refinement* of two finite (countable) partitions $P = \{E_i\}$ and $P' = \{G_j\}$ is the partition $P \wedge P' := \{E_i \cap G_j\}$.
- A countable tagged partition of S if a family $\{(E_n, s_n), n \in \mathbb{N}\}$ such that $(E_n)_n$ is a countable partition of S and $s_n \in E_n$ for every $n \in \mathbb{N}$.

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Let X be a Banach space over $\mathbb{R}$. Let $\nu : \mathcal{C} \to [0, \infty)$ be a set function, such that $\nu(\emptyset) = 0$.

Definition of $|RL|^1_\nu(X)$ the class of all X-valued function that are $|RL|$ integrable (on S) w.r.t. ν

A vector function $f : S \to X$ is called *absolutely (unconditionally* resp.) *Riemann-Lebesgue* ($|RL|$) (RL resp.) *ν-integrable* (on S) if $\exists\, a \in X$ such that for every $\varepsilon > 0$, $\exists\, P_\varepsilon$ of S, so that $\forall\, P = \{A_n\}_{n\in\mathbb{N}}$ of S with $P \geq P_\varepsilon$,

- f is bounded on every A_n, with $\nu(A_n) > 0$ and
- $\forall\, s_n \in A_n$, $n \in \mathbb{N}$, $\sum_{n=0}^{+\infty} f(s_n)\nu(A_n)$ is absolutely (unconditionally resp.) convergent and

$$\left\| \sum_{n=0}^{+\infty} f(s_n)\nu(A_n) - a \right\| < \varepsilon.$$

The vector a is called *the abs. (uncond.) Riemann-Lebesgue ν-integral of f on S* and it is denoted by $(|RL|)\int_S f \,\mathrm{d}\nu$ $\left((RL)\int_S f \,\mathrm{d}\nu \text{ resp.} \right)$.
In an analogous way we denote the class of all functions that are RL ν-integrable.

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- If a exists, then it is unique. Moreover, if $h \in |RL|^1_\nu(X) \Longrightarrow$ then $h \in RL^1_\nu(X)$
- if X is finite dimensional, then $|RL|^1_\nu(X) = RL^1_\nu(X)$.

In the countably additive case

- if $(S, \mathcal{C}, \nu)$ is a finite measure space, then $L^1_\nu(X) \subset |RL|^1_\nu(X) \subset RL^1_\nu(X) \subset P_\nu(X)$.[1]
- If X is a separable Banach space, then $L^1_\nu(X) = |RL|^1_\nu(X) \subset RL^1_\nu(X) = P_\nu(X.)$
- If $(S, \mathcal{C}, \nu)$ is σ-finite, then the Birkhoff integrability coincides with RL ν-integrability [2].
- If $h : [a, b] \to \mathbb{R}$ is Riemann integrable, then h is RL-integrable. The converse is not valid $h = \chi_{[0,1]\cap\mathbb{Q}}$

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[1] Kadets, V. M., Tseytlin, L. M., *On integration of non-integrable vector-valued functions*, Mat. Fiz. Anal. Geom. 7 (2000), 49-65.

[2] Potyrala, M., *Some remarks about Birkhoff and Riemann-Lebesgue integrability of vector valued functions*, Tatra Mt. Math. Publ. 35 (2007), 97–106.

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In the non additive case: assumption on $\nu : \mathcal{C} \to [0, \infty)$, $\nu(\emptyset) = 0$.

ν is said to be:

- *monotone* if $\nu(A) \le \nu(B)$, $\forall A, B \in \mathcal{C}$, with $A \subseteq B$;
- *subadditive* if $\nu(A \cup B) \le \nu(A) + \nu(B)$, $\forall A, B \in \mathcal{C}$, with $A \cap B = \emptyset$;
- *σ-subadditive* if $\nu(A) \le \sum_{n=0}^{+\infty} \nu(A_n)$, $\forall (A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$ with $A_i \cap A_j = \emptyset$, $i \neq j$ and $A = \bigcup_{n=0}^{+\infty} A_n$.
- *null-additive* if, for every $A, B \in \mathcal{C}$, $\nu(A \cup B) = \nu(A)$ when $\nu(B) = 0$.
- has the property $\boldsymbol{\sigma}$ if $\forall \{E_n\}_n \subset \mathcal{C}$ with $\nu(E_n) = 0\ \forall n \in \mathbb{N}$, we have $\nu(\cup_{n=0}^{\infty} E_n) = 0$;
- A set $A \in \mathcal{C}$ is an atom of ν if $\nu(A) > 0$ and $\forall B \in \mathcal{C}$, with $B \subseteq A$, $\Longrightarrow \nu(B) = 0$ or $\nu(A \backslash B) = 0$.

The variation of ν is $\overline{\nu} : \mathcal{P}(S) \to [0, +\infty]$ defined by
$\overline{\nu}(E) = \sup\{\sum_{i=1}^{n} \|\nu(A_i)\|, \quad \{A_i\}_{i=1}^{n} \subset \mathcal{C} : A_i \subseteq E, A_i \cap A_j = \emptyset, i \neq j, \forall i \in \{1, \ldots, n\}\}$, ν is said to be *of finite variation* (on $\mathcal{C}$) if $\overline{\nu}(S) < +\infty$.

Definition $Bs^1_\nu(X)$: the family of Birkhoff-simple functions on S.

$h : S \to X$ is called *Birkhoff simple ν-integrable (on S)* if $\exists\, b \in X$: $\forall\, \varepsilon > 0$, $\exists\, P_\varepsilon$ of S countable so that $\forall\, P = \{A_n\}_{n\in\mathbb{N}}$ of S, with $P \geq P_\varepsilon$ and $\forall\, s_n \in A_n, n \in \mathbb{N}$, $\limsup_{n\to+\infty} \left\| \sum_{k=0}^n h(s_k)\nu(A_k) - b \right\| < \varepsilon$.

Definition $G^1_\nu(X)$: the family of Gould integrable functions on S.

$h : S \to X$ is called *Gould ν-integrable*(on S) if $\exists\, a \in X$ such that $\forall\, \varepsilon > 0$, $\exists\, P_\varepsilon$ of S finite, so that $\forall\, P = \{E_i\}_{i=1}^n$ of S, with $P \geq P_\varepsilon$ and $\forall\, s_i \in E_i, i \in \{1, \dots, n\}$, we have $\left\| \sum_{i=1}^n h(s_i)\nu(E_i) - a \right\| < \varepsilon$.

- $RL^1_\nu(X) \subset Bs^1_\nu(X)$
- $RL^1_\nu(\mathbb{R}) = G^1_\nu(\mathbb{R})$ when $\overline{\nu}(S) < +\infty$, monotone and σ-subadditive (for bounded function). Without the σ-subadditivity of ν, $h = 1 \in RL^1_\nu(\mathbb{R}) \setminus G^1_\nu(\mathbb{R})$, when $S = \mathbb{N}, \mathcal{C} = \mathcal{P}(\mathbb{N})$ and $\nu(A) = 0$, if $\mathrm{card}(A) < +\infty, 1$, otherwise.
- $h : S \to \mathbb{R}$, $RL^1_\nu(A) \subset G^1_\nu(A)$ on each atom $A \in \mathcal{C}$ when ν is monotone, null additive and has (σ).
- $RL^1_\nu(X) = G^1_\nu(X)$ when ν is complete measure and $\overline{\nu}(S) < +\infty$;

Theorem $\nu : \mathcal{C} \to [0,\infty), \nu(\emptyset) = 0$

(a) If $h \in |RL|^1_\nu(X)$, then h is $|RL|$ ν-integrable on every $E \in \mathcal{C}$ ($\Longleftrightarrow h\chi_E$ is $|RL|^1_\nu(X)$)

$$(|RL|)\int_E h \, \mathrm{d}\nu = (|RL|)\int_S h\chi_E \, \mathrm{d}\nu.$$

Moreover, if $g, h \in |RL|^1_\nu(X)$ and $\alpha, \beta \in \mathbb{R}$. Then:

(b) $\alpha g + \beta h \in |RL|^1_\nu(X)$ and $(|RL|)\int_S (\alpha g + \beta h) \, \mathrm{d}\nu = \alpha \cdot (|RL|)\int_S g \, \mathrm{d}\nu + \beta \cdot (|RL|)\int_S h \, \mathrm{d}\nu,$

(c) $h \in |RL|^1_{\alpha\nu}(X)$ for $\alpha \in [0, +\infty)$ and $(|RL|)\int_S h \, \mathrm{d}(\alpha\nu) = \alpha (|RL|)\int_S h \, \mathrm{d}\nu.$

The results also hold for the RL ν-integrability.

Theorem (monotonicity)

Let $g, h \in RL^1_\nu(\mathbb{R})$ such that $g(s) \leq h(s)$, for every $s \in S$, then

$$(RL)\int_S g \,\mathrm{d}\nu \leq (RL)\int_S h \,\mathrm{d}\nu.$$

Let $\nu_1,\ \nu_2 : \mathcal{C} \to [0, +\infty)$ be set functions such that $\nu_1(A) \leq \nu_2(A)$, $\forall\, A \in \mathcal{C}$ and $h \in RL^1_{\nu_i}(\mathbb{R}^+_0)$ for $i = 1, 2$. Then

$$(RL)\int_S h \,\mathrm{d}\nu_1 \leq (RL)\int_S h \,\mathrm{d}\nu_2.$$

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Theorem $\overline{\nu}(S) < +\infty$

Let $\nu : S \to [0, \infty)$ be of finite variation and $h : S \to X$ be bounded

(a) then $h \in |RL|^1_\nu(X)$ and $\left\| (|RL|) \int_S h \, \mathrm{d}\nu \right\| \leq \sup_{s \in S} \|h(s)\| \cdot \overline{\nu}(S)$.

(b) If $h = 0$ ν-a.e.[a], then $h \in |RL|^1_\nu(X)$ and $(|RL|) \displaystyle\int_S h \, \mathrm{d}\nu = 0$.

Moreover let $g, h : S \to X$ be vector functions.

(c) If $\sup_{s \in S} \|g(s) - h(s)\| < +\infty$, $g \in |RL|^1_\nu(X)$ and $g = h$ ν-a.e., then $h \in |RL|^1_\nu(X)$ and

$$(|RL|) \int_S g \, \mathrm{d}\nu = (|RL|) \int_S h \, \mathrm{d}\nu.$$

(d) If $g, h \in |RL|^1_\nu(X)$ then $\left\| (|RL|) \displaystyle\int_S g \, \mathrm{d}\nu - (|RL|) \int_S h \, \mathrm{d}\nu \right\| \leq \sup_{s \in S} \|g(s) - h(s)\| \cdot \overline{\nu}(S)$.

[a] $h = 0$ holds ν-a.e. if there exists $\exists\, E \in \mathcal{C}$, with $\nu(E) = 0$ and $h = 0$ is valid on $S \setminus E$.

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Definition

For every $h : S \to X$ that is $|RL|$ (RL resp.) ν-integrable $\forall\, E \in \mathcal{C}$, $T_h : \mathcal{C} \to X$, defined by,

$$T_h(E) = (|RL|) \int_E h \, \mathrm{d}\nu \qquad \left(T_h(E) = (RL) \int_E h \, \mathrm{d}\nu \quad \text{resp.}\right), \quad \forall\, E \in \mathcal{C}$$

- *order-continuous* (shortly, *o-continuous*) if $\lim_{n\to+\infty} \nu(A_n) = 0$, $\forall\, (A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, with $A_n \searrow \emptyset$;
- *exhaustive* if $\lim_{n\to+\infty} \nu(A_n) = 0$, $\forall\ (A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, with $A_i \cap A_j = \emptyset$, $i \neq j$.

Theorem

Let $h \in |RL|^1_\nu(X)$. If h is bounded, and ν is of finite variation then

(a)
- T_h is of finite variation too;
- $\overline{T_h} \ll \overline{\nu}$ in the $\varepsilon - \delta$ sense;
- Moreover, if $\overline{\nu}$ is o-continuous (exhaustive resp.), then T_h is also o-continuous (exhaustive resp.).

(b) If $h : S \to [0, \infty)$, ν is monotone, then the same holds for T_h.

Convergence results

Theorem

Let $h, h_n : S \to X$, $\nu : \mathcal{C} \to [0, +\infty)$, $\overline{\nu}(S) < +\infty$. If $h, h_n \in |RL|^1_\nu(X)$ $\forall\, n \in \mathbb{N}$ and $h_n \rightrightarrows h$ then

$$\lim_{n\to\infty} (|RL|) \int_S h_n \, \mathrm{d}\nu = (|RL|) \int_S h \, \mathrm{d}\nu.$$

- ν satisfies the condition **(E)** if for every double sequence $(B_n^m)_{n,m\in\mathbb{N}^*} \subset \mathcal{C}$, such that for every $m \in \mathbb{N}^*$, $B_n^m \searrow B^m\, (n \to \infty)$ and $\nu(\cup_{m=1}^\infty B^m) = 0$, there exist two increasing sequences $(n_p)_p, (m_p)_p \subset \mathbb{N}$ such that $\lim\limits_{k\to\infty} \nu(\bigcup_{p=k}^\infty B_{n_p}^{m_p}) = 0$.
- the semivariation of ν is the set function $\widetilde{\nu} : \mathcal{P}(S) \to [0, +\infty]$ defined for every $A \subseteq S$, by

$$\widetilde{\nu}(A) = \inf\{\overline{\nu}(B);\ A \subseteq B,\ B \in \mathcal{C}\}.$$

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Theorem (scalar case)

Suppose $\nu : \mathcal{C} \to [0, +\infty)$ is a monotone set function with $\overline{\mu}(S) < +\infty$ and $\widetilde{\nu}$ satisfies **(E)**.

- $\forall\ n \in \mathbb{N}$, let (h_n) be uniformly bounded. Then

$$(RL) \int_S (\liminf_{n\to\infty} h_n)\, \mathrm{d}\nu \le \liminf_{n\to\infty} \left((RL) \int_S h_n \, \mathrm{d}\nu \right). \quad \text{(Fatou)}$$

- If $\exists\ h$ such that $\sup_{s \in S, n\in\mathbb{N}} \{h(s), h_n(s)\} < +\infty$ and $h_n \xrightarrow{\nu - ae} h$ or $h_n \xrightarrow{\widetilde{\nu}} h$ then

$$\lim_{n\to\infty} (RL) \int_S h_n \, \mathrm{d}\nu = (RL) \int_S h \, \mathrm{d}\nu.$$

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If $p > 0$ and $g : S \to \mathbb{R} : |g|^p \in RL^1_\nu(\mathbb{R})$ and measurable, then $\|g\|_p = \left((RL)\int_S |g|^p \mathrm{d}\nu\right)^{1/p}$

Theorem (Inequalities)

Let $\nu : \mathcal{C} \to [0, \infty)$ be suitable σ-subadditive set function and g, h be scalar measurable functions.

- Let $p, q \in (1, \infty)$, with $p^{-1} + q^{-1} = 1$.
  (a) If $|g \cdot h|, |g|^p, |h|^q \in RL^1_\nu(\mathbb{R})$, then $\|g \cdot h\|_1 \leq \|g\|_p \cdot \|h\|_q$. (Hölder Inequality)
  (b) Let $p \geq 1$. If $|g + h|^p, |g + h|^{q(p-1)}, |g|^p, |h|^p \in RL^1_\nu(\mathbb{R})$, then $\|g + h\|_p \leq \|g\|_p + \|h\|_p$ (Minkowski Inequality)
- Let $p, q \in (0, \infty)$, with $0 < p < 1$ and $p^{-1} + q^{-1} = 1$.
  (c) If $|g \cdot h|, |g|^p, |h|^q \in RL^1_\nu(\mathbb{R})$ and $0 < (RL)\int_S |h|^q \,\mathrm{d}\nu$, then $\|g \cdot h\|_1 \geq \|g\|_p \cdot \|h\|_q$ (Reverse Hölder Inequality).
  (d) If $|g + h|^p, |g + h|)^{(p-1)q}, |g|^p, |h|^p$ are RL-integrable, then $\left\| \, |g| + |h| \, \right\|_p \geq ||g||_p + ||h||_p$(Reverse Minkowski Inequality).

$ck(\mathbb{R})$ denotes the family of all non-empty, convex, compact subsets of $\mathbb{R}$, by convention, $\{0\} = [0, 0]$. We consider on $ck(\mathbb{R})$

- the Minkowski addition $A \oplus B := \{a + b \mid a \in A,\ b \in B\}, \qquad \forall A, B \in ck(\mathbb{R})$
- and the multiplication by scalars $\lambda A = \{\lambda a \mid a \in A\}, \qquad \forall \lambda \in \mathbb{R},\ \forall A \in ck(\mathbb{R})$.
- d_H denotes the Hausdorff distance $d_H([r, s], [x, y]) = \max\{|x - r|, |y - s|\}, \qquad \forall r, s, x, y \in \mathbb{R}$;
- $[r, s] \bullet [x, y] = [rx, sy]$;
- $[r, s] \preceq [x, y] \iff r \leq x$ and $s \leq y \quad ([r, s] \vee [x, y] = [\max\{r, x\}, \max\{s, y\}])$

Given $h_1, h_2 : S \to \mathbb{R}_0^+$ with $h_1(s) \leq h_2(s) \quad \forall s \in S$, let $H : S \to ck(\mathbb{R}_0^+)$ be the interval-valued multifunction defined by

$$H(s) := \big[h_1(s), h_2(s)\big], \qquad \forall s \in S. \tag{2}$$

For two set functions $\nu_1, \nu_2 : \mathcal{C} \to \mathbb{R}_0^+$ with $\nu_1(\emptyset) = \nu_2(\emptyset) = 0$ and $\nu_1(A) \leq \nu_2(A)\ \forall A \in \mathcal{C}$ let $\Gamma : \mathcal{C} \to ck(\mathbb{R}_0^+)$ be an interval-valued set function defined by

$$\Gamma(A) = \big[\nu_1(A), \nu_2(A)\big], \qquad \forall A \in \mathcal{C}. \tag{3}$$

Γ is an interval-valued multisubmeasure if

- $\Gamma(\emptyset) = \{0\}$; $\Gamma(A) \preceq \Gamma(B)$ for every $A, B \in \mathcal{C}$ with $A \subseteq B$ (monotonicity) and $\Gamma(A \cup B) \preceq \Gamma(A) \oplus \Gamma(B)$ for every disjoint sets $A, B \in \mathcal{C}$. (subadditivity).
- Γ is of finite variation $\Longleftrightarrow \overline{\nu}_2(S) < +\infty$.

$$\sum_{n=1}^{\infty} H(s_n) \cdot \Gamma(B_n) = \sum_{n=1}^{\infty} \left[h_1(s_n)\nu_1(B_n), h_2(s_n)\nu_2(B_n)\right] = \left\{ \sum_{n=1}^{\infty} y_n, y_n \in \left[h_1(s_n)\nu_1(B_n), h_2(s_n)\nu_2(B_n)\right], n \in \mathbb{N} \right\}.$$

Definition: Riemann-Lebesgue integrability with respect to Γ (on S)

A multifunction $H : S \to ck(\mathbb{R}_0^+)$ is RL integrable w.r.t. Γ (on S) if $\exists [c, d] \in ck(\mathbb{R}_0^+)$ such that $\forall \varepsilon > 0$, $\exists P_\varepsilon$ of S countable, so that $\forall P = \{(B_n, s_n)\}_{n \in \mathbb{N}}$ of S with $P \geq P_\varepsilon$,

- the series $\sum_{n=1}^{\infty} \left[h_1(s_n)\nu_1(B_n), h_2(s_n)\nu_2(B_n)\right]$ is convergent with respect to the Hausdorff distance d_H and
- $d_H(\sum_{n=1}^{\infty} \left[h_1(s_n)\nu_1(B_n), h_2(s_n)\nu_2(B_n)\right], [c, d]) < \varepsilon$.

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Theorem

An interval-valued multifunction $H = [h_1, h_2]$ is RL integrable w.r.t. Γ on $S \iff h_1$ and h_2 are RL integrable w.r.t. ν_1 and ν_2 respectively and

$$(RL)\int_S H \,\mathrm{d}\Gamma = \left[(RL)\int_S h_1 \,\mathrm{d}\nu_1, (RL)\int_S h_2 \,\mathrm{d}\nu_2\right].$$

Monotone Convergence

Suppose $\Gamma = [\nu_1, \nu_2]$ with $\nu_i,\ i \in \{1, 2\}$ non negative submeasures of finite variation. $\forall\, n \in \mathbb{N}$, let $H_n = [h_1^{(n)}, h_2^{(n)}]$ be such that $(h_2^{(n)})$ is uniformly bounded and $H_n \preceq H_{n+1}$ for every $n \in \mathbb{N}$. Then

$$(RL)\int_S \bigvee_n H_n \,\mathrm{d}\Gamma = \bigvee_n (RL)\int_S H_n \,\mathrm{d}\Gamma.$$

Convergence type theorem for varying multisubmeasures

Let $(H_n)_n := ([h_1^{(n)}, h_2^{(n)}])_n$ be a sequence of bounded multifunctions, and $(\Gamma_n)_n := ([\nu_1^{(n)}, \nu_2^{(n)}])_n$ a sequence of multisubmeasures. Suppose $\exists\, \Gamma := [\nu_1, \nu_2]$, with ν_2 of finite variation (interval-valued multisubmeasure) and a bounded multifunction $H := [h_1, h_2]$ such that:

(a) $H_n \preceq H_{n+1}$ for every $n \in \mathbb{N}$ and $d_H(H_n, H) \to 0$ uniformly on S,

(b) $\Gamma_n \preceq \Gamma_{n+1} \preceq \Gamma$ for every $n \in \mathbb{N}$ and $(\Gamma_n)_n$ setwise converges to Γ

Then

$$\lim_{n\to\infty} d_H\left((RL) \int_S H_n \, \mathrm{d}\Gamma_n, (RL) \int_S H \, \mathrm{d}\Gamma \right) = 0.$$

$\nu_n \xrightarrow{setwise} \nu$ if $\lim_n \overline{\nu_n - \nu}(A) = 0$, $\forall A \in \mathcal{C}$.

$(\Gamma_n)_n$ setwise converges to Γ (namely $\nu_i^{(n)} \xrightarrow{setwise} \nu_i, i = 1, 2 \implies \lim_n d_H(\Gamma_n(A), \Gamma(A)) = 0, \; \forall\, A \in \mathcal{C}$).

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and thanks for your attention!