Radon-Nikodýmification of Integral Geometric Measures

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Thierry De Pauw

Westlake University

Papers

- DP, On SBV Dual, *Indiana Math. J.*, 1998
- Bouafia - DP, Radon-Nikodýmification of arbitrary measure spaces, *Extracta Math.*, 2023
- DP, Undecidably semilocalizable metric measure spaces, *Commun. Contemp. Math.*, 2024
- Bouafia - DP, A representation formula for members of SBV dual, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 2024

Dual of $\mathrm{L}_1(X, \mathscr{A}, \mu)$

Dual of $\mathbf{L}_1$

Let $(X, \mathscr{A}, \mu)$ be an *arbitrary* measure space. There is a canonical embedding:

$$\Upsilon : \mathbf{L}_\infty(X, \mathscr{A}, \mu) \to \mathbf{L}_1(X, \mathscr{A}, \mu)^*.$$

Theorem. Υ *is a (bijective) isometric isomorphism in case* $(X, \mathscr{A}, \mu)$ *is* σ*-finite.*

In general, Υ is neither injective nor surjective.

Injectivity of Υ

Let $(X, \mathscr{A}, \mu)$ be an *arbitrary* measure space.

$$\Upsilon : \mathbf{L}_\infty(X, \mathscr{A}, \mu) \to \mathbf{L}_1(X, \mathscr{A}, \mu)^*.$$

Theorem. *Υ is injective if and only if $(X, \mathscr{A}, \mu)$ is semi-finite.*

A measure space $(X, \mathscr{A}, \mu)$ is semi-finite, by definition, if for every $A \in \mathscr{A}$ such that $\mu(A) = \infty$ there exists $B \in \mathscr{A}$ such that $B \subseteq A$ and $0 < \mu(B) < \infty$. Obviously, σ-finite measure spaces are semi-finite.

Semi-finite measure spaces

Let $(X, \mathscr{A}, \mu)$ be an *arbitrary* measure space and define

$$\begin{aligned} \mathscr{N}_\mu &= \mathscr{A} \cap \{N : \mu(N) = 0\} \\ \mathscr{A}^f &= \mathscr{A} \cap \{A : \mu(A) < \infty\} \\ \mathscr{N}_{\mu,\mathrm{loc}} &= \mathscr{A} \cap \{A : A \cap F \in \mathscr{N}_\mu \text{ for all } F \in \mathscr{A}^f\}. \end{aligned}$$

It is easy to observe that the following are equivalent:

- $(X, \mathscr{A}, \mu)$ is semi-finite
- Every $A \in \mathscr{A} \setminus \mathscr{N}_\mu$ contains some $F \in \mathscr{A}^f \setminus \mathscr{N}_\mu$
- $\mathscr{N}_{\mu,\mathrm{loc}} = \mathscr{N}_\mu$.

Semi-finite version

Let $(X, \mathscr{A}, \mu)$ be an *arbitrary* measure space. It is not difficult to modify slightly the measure μ, leaving the underlying measurable space $(X, \mathscr{A})$ untouched, in order to make it semi-finite. Specifically,

$$\mu_{\mathrm{sf}}(A) = \sup\{\mu(A \cap F) : F \in \mathscr{A}^f\},$$

$A \in \mathscr{A}$. One checks that $(X, \mathscr{A}, \mu_{\mathrm{sf}})$ is semi-finite and $\mathscr{N}_{\mu_{\mathrm{sf}}} = \mathscr{N}_{\mu,\mathrm{loc}}$.

Surjectivity of Υ

Given $\alpha \in \mathbf{L}_1(X, \mathscr{A}, \mu)^*$ and $A \in \mathscr{A}^f$ we consider

$$\mathbf{L}_1(A, \mathscr{A}_A, \mu_A) \xrightarrow{\iota_A} \mathbf{L}_1(X, \mathscr{A}, \mu) \xrightarrow{\alpha} \mathbb{R}$$

There exists $g_A \in L_\infty(A, \mathscr{A}_A, \mu_A)$ representing $\alpha \circ \iota_A$.
The family $\langle g_A \rangle_{A \in \mathscr{A}^f}$ is *compatible*: for all $A, A' \in \mathscr{A}^f$

$$A \cap A' \cap \{g_A \neq g_{A'}\} \in \mathscr{N}_\mu.$$

A *gluing* of $\langle g_A \rangle_{A \in \mathscr{A}^f}$ is an $\mathscr{A}$-measurable function $g : X \to \mathbb{R}$ such that for all $A \in \mathscr{A}^f$

$$A \cap \{g_A \neq g\} \in \mathscr{N}_\mu.$$

MSN

- A *Measurable Space with Negligibles* $(X, \mathscr{A}, \mathscr{N})$ is a measurable space $(X, \mathscr{A})$ and a σ-ideal $\mathscr{N} \subseteq \mathscr{A}$.
- $\mathbf{L}_\infty(X, \mathscr{A}, \mathscr{N})$ makes sense.
- Given $\mathscr{E} \subseteq \mathscr{A}$, a family $\langle g_E \rangle_{E \in \mathscr{E}}$ of $\mathscr{A}_E$-measurable functions $g_E : E \to \mathbb{R}$ is *compatible* if, by definition, $E \cap E' \cap \{g_E \neq g_{E'}\} \in \mathscr{N}$ for all $E, E' \in \mathscr{E}$.
- A *gluing* of a compatible family $\langle g_E \rangle_{E \in \mathscr{E}}$ is an $\mathscr{A}$-measurable function $g : X \to \mathbb{R}$ such that $E \cap \{g_E \neq g\} \in \mathscr{N}$ for all $E \in \mathscr{E}$.

Essential supremum

Let $(X, \mathscr{A}, \mathscr{N})$ be an MSN and $\mathscr{E} \subseteq \mathscr{A}$. An *essential supremum of* $\mathscr{E}$ is an $A \in \mathscr{A}$ such that

- $\forall E \in \mathscr{E} : E \setminus A \in \mathscr{N}$
- $\forall B \in \mathscr{A} : [\forall E \in \mathscr{E} : E \setminus B \in \mathscr{N}] \Rightarrow A \setminus B \in \mathscr{N}$

Theorem. *Let* $(X, \mathscr{A}, \mathscr{N})$ *be an MSN. TFAE*

- *The Boolean algebra* $\mathscr{A} / \mathscr{N}$ *is order complete*
- *Each* $\mathscr{E} \subseteq \mathscr{A}$ *admits an essential supremum*
- *Each compatible family* $\langle g_E \rangle_{E \in \mathscr{E}}$ *admits a gluing.*

We say these MSNs are *localizable*.

Examples

- (Tarski, 1937) If $(X, \mathscr{A}, \mu)$ is a (σ-)finite measure space then $(X, \mathscr{A}, \mathscr{N}_\mu)$ is localizable.
- In $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathscr{N}_{\mathscr{L}^1})$ an essential supremum of $\{\{x\} : x \in \mathbb{R}\}$ is $\emptyset$.
- In $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \{\emptyset\})$ an essential supremum of $\{\{x\} : x \in \mathbb{R}\}$ is $\mathbb{R}$.

Surjectivity of Υ

Theorem (DP, 2024)**.** *Let* $(X, \mathscr{A}, \mu)$ *be an arbitrary measure space. TFAE*

- Υ *is surjective*
- $(X, \mathscr{A}, \mathscr{N}_{\mu,\mathrm{loc}})$ *is localizable.*

Theorem. *Let* $(X, \mathscr{A}, \mu)$ *be an arbitrary measure space. TFAE*

- Υ *is an isometric isomorphism*
- $(X, \mathscr{A}, \mu)$ *is semi-finite and* $(X, \mathscr{A}, \mathscr{N}_{\mu})$ *is localizable.*

Hausdorff measures: Injectivity

- If X is a complete separable metric space and $0 < d < \infty$ then $(X, \mathscr{B}(X), \mathscr{H}^d)$ is semi-finite. $X = \mathbb{R}^m$ due to Davies, general case to Howroyd.
- The measure space $(\mathbb{R}^2, \mathscr{A}_{\mathscr{H}^1}, \mathscr{H}^1)$ is not semi-finite. Here, $\mathscr{A}_{\mathscr{H}^1}$ is the σ-algebra of $\mathscr{H}^1$-measurable sets in the sense of Caratheorody. Due to Fremlin.

Theorem (Grzegorek, 1981). *Define*

$$\mathsf{non}\,(\mathscr{N}_{\mathscr{L}^1}) := \min\{\operatorname{card} S : S \subseteq \mathbb{R} \text{ and } S \notin \mathscr{N}_{\mathscr{L}^1}\}.$$

There exists a universally negligible set $Y \subseteq \mathbb{R}$ *with* $\operatorname{card} Y = \mathsf{non}\,(\mathscr{N}_{\mathscr{L}^1})$.

Hausdorff measures: Surjectivity

- $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathscr{N}_{\mathscr{H}^1,\mathrm{loc}})$ is not localizable. We will explain how this follows from Fubini's Theorem.
- Whether the MSN $(\mathbb{R}^2, \mathscr{A}_{\mathscr{H}^1}, \mathscr{N}_{\mathscr{H}^1,\mathrm{loc}})$ is localizable is undecidable in ZFC. It is localizable under the Continuum Hypothesis.

Define

- $V_s = \{s\} \times \mathbb{R}$, $s \in \mathbb{R}$
- $H_t = \mathbb{R} \times \{t\}$, $t \in \mathbb{R}$

and assume that $A \in \mathscr{B}(\mathbb{R}^2)$ is an $\mathscr{N}_{\mathscr{H}^1,\mathrm{loc}}$-essential supremum of $\{V_s : s \in \mathbb{R}\}$. Then

Fubini

- $\forall s \in \mathbb{R} : \mathscr{H}^1(V_s \setminus A) = 0$
- $\forall t \in \mathbb{R} : \mathscr{H}^1(H_t \cap A) = 0$ (take $B = A \setminus H_t$)

Thus,

$$\mathscr{L}^2(\mathbb{R}^2 \setminus A) = \int_{\mathbb{R}} \mathscr{H}^1(V_s \setminus A) d\mathscr{L}^1(s) = 0$$

and

$$\mathscr{L}^2(A) = \int_{\mathbb{R}} \mathscr{H}^1(H_t \cap A) d\mathscr{L}^1(s) = 0,$$

a contradiction.

Refining needed

This reasoning does not apply if:

- A is not $\mathscr{L}^2$-measurable (because Fubini's Theorem then fails)
- $\mathbb{R}^2$ is replaced by an ambient set X such that $\mathscr{L}^2(X) = 0$ (because no contradiction ensues).

Part of Cichoń diagram

$$\mathsf{non}\left(\mathscr{N}_{\mathscr{L}^1}\right) := \min\{\mathrm{card}\, S : S \subseteq \mathbb{R} \text{ and } S \notin \mathscr{N}_{\mathscr{L}^1}\},$$

$$\mathsf{cov}\left(\mathscr{N}_{\mathscr{L}^1}\right) := \min\{\mathrm{card}\, I : \mathbb{R} \subseteq \cup_{i\in I} N_i \text{ and } N_i \in \mathscr{N}_{\mathscr{L}^1}, i \in I\}$$

Theorem. *It is consistent with ZFC to assume that*

$$\mathsf{non}\left(\mathscr{N}_{\mathscr{L}^1}\right) < \mathsf{cov}\left(\mathscr{N}_{\mathscr{L}^1}\right).$$

Necessity of a larger ambient X

Theorem (DP, 2024). *Assume that*

- $\operatorname{non}(\mathscr{N}_{\mathscr{L}^1}) < \operatorname{cov}(\mathscr{N}_{\mathscr{L}^1})$
- $C \subseteq [0,1]$ *is some Cantor set of dimension 0*
- $X = C \times [0,1]$
- $\mathscr{B}(X) \subseteq \mathscr{A} \subseteq \mathscr{P}(X)$
- $\mathscr{N} = \mathscr{A} \cap \mathscr{N}_{\mathscr{H}^1}$ *or* $\mathscr{N} = \mathscr{A} \cap \mathscr{N}_{p.u.}$

Then $(X, \mathscr{A}, \mathscr{N})$ *is not localizable. (If we replace the first condition by CH instead then* $(X, \mathscr{A}_{\mathscr{H}^1}, \mathscr{N}_{\mathscr{H}^1})$ *is localizable.)*

General Theorem

Stone-Čech compactification

Objets: Topological Spaces
Arrows: Continuous Maps

$$\begin{tikzcd} Y & & \hat{X} \arrow[ll, dotted, "\exists!\zeta"'] \\ & X \arrow[ul, "\forall\xi"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

Alternatively, the forgetful functor

$$\mathsf{Forget} : \mathsf{Comp} \to \mathsf{Top}$$

has a left adjoint.

Universal property: first attempt

Objects: Saturated MSN
Arrows: Equivalence Classes of Measurable Morphisms

$$\begin{tikzcd}
(Y, \mathscr{B}, \mathscr{M}) \arrow[rr, dotted, "\exists!\mathbf{r}"] \arrow[dr, "\forall\mathbf{q}"'] & & (\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}) \arrow[dl, "\mathbf{p}"] \\
& (X, \mathscr{A}, \mathscr{N}) &
\end{tikzcd}$$

This is a first attempt at defining the *localizable version* of an arbitrary MSN.

The categories MSN and LOC

- An MSN $(X, \mathscr{A}, \mathscr{N})$ is *saturated* if for each $N \in \mathscr{N}$ and each $N' \subseteq N$ one has $N' \in \mathscr{A}$
- A *measurable morphism* between two MSNs $(X, \mathscr{A}, \mathscr{N})$ and $(Y, \mathscr{B}, \mathscr{M})$ is a map $f : X \to Y$ such that
  - For all $B \in \mathscr{B}$: $f^{-1}(B) \in \mathscr{A}$
  - For all $M \in \mathscr{M}$: $f^{-1}(M) \in \mathscr{N}$
- Two measurable morphisms f, f' between $(X, \mathscr{A}, \mathscr{N})$ and $(Y, \mathscr{B}, \mathscr{M})$ are *equivalent* if $X \cap \{f \neq f'\} \in \mathscr{N}$.
- This defines the categories MSN and LOC

Localizable version of $(X, \mathscr{A}, \{\emptyset\})$

If $\mathscr{A}$ contains all singletons then the localizable version of $(X, \mathscr{A}, \{\emptyset\})$ should be $(X, \mathscr{P}(X), \{\emptyset\})$.

$$\begin{tikzcd}
(\mathbb{R}, \mathscr{A}_{\mathscr{L}^1}, \mathscr{N}_{\mathscr{L}^1}) \arrow[rr, dotted, "\exists!\mathbf{r}"] \arrow[dr, "\forall\mathbf{q}"'] & & (\mathbb{R}, \mathscr{P}(\mathbb{R}), \{\emptyset\}) \arrow[dl, "\mathbf{p}"] \\
& (\mathbb{R}, \mathscr{A}_{\mathscr{L}^1}, \{\emptyset\}) &
\end{tikzcd}$$

For $\mathbf{q} = \mathrm{id}_{\mathbb{R}}$ and $r \in \mathbf{r}$ one has $\mathbb{R} \cap \{r \neq \mathrm{id}_{\mathbb{R}}\} \in \mathscr{N}_{\mathscr{L}^1}$, contradicting the existence of non $\mathscr{L}^1$-measurable sets. In fact, $\mathbf{q}$ should not be an arrow of the category.

Supremum preserving morphisms

- An equivalence class of measurable morphisms between $(X, \mathscr{A}, \mathscr{N})$ and $(Y, \mathscr{B}, \mathscr{M})$ is *supremum preserving* if the following holds. For every $\mathscr{F} \subseteq \mathscr{B}$, if $\mathscr{F}$ admits an $\mathscr{M}$-essential supremum F then $f^{-1}(F)$ is an $\mathscr{N}$-essential supremum of $f^{-1}(\mathscr{F})$.
- This defines the categories $\mathsf{MSN}_{\mathsf{sp}}$ and $\mathsf{LOC}_{\mathsf{sp}}$.
- If X is uncountable and $\mathscr{C}(X)$ is the countable - cocountable σ-algebra in X then $(X, \mathscr{P}(X), \{\emptyset\})$ is *not* the (new) localizable version of $(X, \mathscr{C}(X), \{\emptyset\})$.

Local determinacy

- In $(X, \mathscr{A}, \mathscr{N})$ we say $\mathscr{E} \subseteq \mathscr{A}$ is $\mathscr{N}$*-generating* if X is an $\mathscr{N}$-essential supremum of $\mathscr{E}$. For example, if $(X, \mathscr{A}, \mu)$ is semi-finite then $\mathscr{A}^f$ is $\mathscr{N}_\mu$-generating.
- $(X, \mathscr{A}, \mathscr{N})$ is *locally determined* if for every $\mathscr{N}$-generating family $\mathscr{E} \subseteq \mathscr{A}$ one has:

$$\forall A \in \mathscr{P}(X) : [\forall E \in \mathscr{E} : A \cap E \in \mathscr{A}] \Rightarrow A \in \mathscr{A}$$

- For instance if ϕ is an outer measure on X and has measurable hulls then $(X, \mathscr{A}_\phi, \mathscr{N}_\phi)$ is locally determined.
- This defines the category $\mathsf{LLD}_{\mathsf{sp}}$.

lld versions

Arrows are in $\mathsf{MSN_{sp}}$ and $(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}})$ is (saturated) localizable and locally determined.

$$
\begin{tikzcd}
(Y, \mathscr{B}, \mathscr{M}) \arrow[rr, dotted, "\exists! \mathbf{r}"] \arrow[dr, "\forall \mathbf{q}"'] & & (\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}) \arrow[dl, "\mathbf{p}"] \\
& (X, \mathscr{A}, \mathscr{N}) &
\end{tikzcd}
$$

It is an open question whether each saturated MSN admits an lld version. It boils down to whether coequalizers exist in $\mathsf{MSN_{sp}}$.

Countable chain condition

- $(X, \mathscr{A}, \mathscr{N})$ satisfies the Countable Chain Condition (ccc) if each almost disjointed family $\mathscr{E} \subseteq \mathscr{A}$ (i.e. $E \cap E' \in \mathscr{N}$ whenever $E, E' \in \mathscr{E}$ are distinct) is at most countable. For example, if $(X, \mathscr{A}, \mu)$ is σ-finite then $(X, \mathscr{A}, \mathscr{N}_\mu)$ satisfies the ccc.
- If $(X, \mathscr{A}, \mathscr{N})$ is ccc then it is lld.
- (Zorn) In an MSN $(X, \mathscr{A}, \mathscr{N})$ if $\mathscr{C} \subseteq \mathscr{A}$ is $\mathscr{N}$-generating then there exists an almost disjointed and $\mathscr{N}$-generating $\mathscr{E} \subseteq \mathscr{A}$ each member of which is contained in some member of $\mathscr{C}$.
- $(X, \mathscr{A}, \mathscr{N})$ is cccc if it is a coproduct (in $\mathsf{MSN}_{\mathsf{sp}}$) of ccc MSNs. A cccc MSN is also lld.

General Theorem

Theorem (Bouafia - DP, 2023)**.** *Let* $(X, \mathscr{A}, \mathscr{N})$ *be a saturated MSN such that the collection* $\mathscr{E}_{ccc} := \{Z : (Z, \mathscr{A}_Z, \mathscr{N}_Z) \text{ is ccc}\}$ *is* $\mathscr{N}$*-generating. Then*

(1) $(X, \mathscr{A}, \mathscr{N})$ *admits a cccc version.*

(2) *If furthermore* $\mathscr{E}_{ccc}$ *contains an* $\mathscr{N}$*-generating subcollection* $\mathscr{E}$ *such that* $\operatorname{card} \mathscr{E} \leqslant \mathfrak{c}$ *and each* $(Z, \mathscr{A}_Z)$ *is countably separated, for* $Z \in \mathscr{E}$*, then* $(X, \mathscr{A}, \mathscr{N})$ *admits an lld version which is also its cccc version.*

The hypothesis is satisfied by $(X, \mathscr{A}, \mathscr{N}_\mu)$ in case $(X, \mathscr{A}, \mu)$ is complete and semi-finite, since $\mathscr{A}^f$ is $\mathscr{N}_\mu$-generating.

Radon-Nikodýmification

Theorem (Bouafia - DP, 2023). *Let* $(X, \mathscr{A}, \mu)$ *be a complete semi-finite measure space and* $[(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}), \mathbf{p}]$ *its corresponding cccc version. Let* $p \in \mathbf{p}$. *There exists a unique (and independent of the choice of* p*) measure* $\hat{\mu}$ *defined on* $\hat{\mathscr{A}}$ *such that* $p_{\#}\hat{\mu} = \mu$ *and* $\mathscr{N}_{\hat{\mu}} = \hat{\mathscr{N}}$.
Furthermore $(\hat{X}, \hat{\mathscr{A}}, \hat{\mu})$ *is a strictly localizable measure space, and the Banach spaces* $\mathbf{L}_1(X, \mathscr{A}, \mu)$ *and* $\mathbf{L}_1(\hat{X}, \hat{\mathscr{A}}, \hat{\mu})$ *are isometrically isomorphic.*

Integral Geometric Measure

Integral geometric measure

We consider the measure space $(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m), \mathscr{I}^k_\infty)$ where $1 \leqslant k \leqslant m-1$ are integers and $\mathscr{I}^k_\infty$ is the integral geometric measure. It is not semi-finite. Thus, we replace it with its complete semi-finite version $(\mathbb{R}^m, \widetilde{\mathscr{B}(\mathbb{R}^m)}, \tilde{\mathscr{I}}^k_\infty)$.

We let $\mathscr{E}$ be the collection of k-dimensional submanifolds $M \subseteq \mathbb{R}^m$ of class C^1 such that $\phi_M = \mathscr{H}^k \llcorner M$ is locally finite. It follows from the Besicovitch - Federer - Mickle Structure Theorem that $\mathscr{E}$ is $\mathscr{N}_{\tilde{\mathscr{I}}^k_\infty}$-generating.

Integral geometric measure

For each $x \in \mathbb{R}^m$ we define $\mathscr{E}_x = \mathscr{E} \cap \{M : x \in M\}$ and we define on $\mathscr{E}_x$ an equivalence relation as follows. We declare that $M \sim_x M'$ if and only if

$$\lim_{r \to 0^+} \frac{\mathscr{H}^k(M \cap M' \cap \mathbf{B}(x,r))}{\boldsymbol{\alpha}(k) r^k} = 1.$$

Letting $[M]_x$ denote the equivalence class of $M \in \mathscr{E}_x$, we prove that the underlying set of the cccc, lld, and strictly localizable version of the MSN $(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m), \mathscr{N}_{\tilde{\mathscr{I}}^k_\infty})$ can be taken to be

$$\hat{X} = \{(x, [M]_x) : x \in \mathbb{R}^m \text{ and } M \in \mathscr{E}_x\}.$$

Thank you!