Weak Amenability Constant in Banach Algebras

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47$^{\text{th}}$ Summer Symposium in Real Analysis,
Madrid, Spain,
June 16–20, 2025.

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Outline

1. Amenable groups.
2. Amenable algebras.
3. Weakly amenable vector-valued sequence algebras.
4. Weak amenability constant in Banach algebras.

Amenable Groups

Banach-Tarski Paradox (1924)

An orange can be cut into finitely many pieces and those pieces can be reassembled to yield two oranges each of which has the same size as the original one. [V. Runde - "Amenable Banach Algebras. A Panorama"]

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$\mathbb{F}_2 := \{a, b, a^{-1}, b^{-1}, a^2 b^3 a^{-1}, \ldots\}$ – free group in 2 generators,

$$\mathbb{F}_2 = \{1_{\mathbb{F}_2}\} \amalg W(a) \amalg W(a^{-1}) \amalg W(b) \amalg W(b^{-1}) \qquad \&$$

$$W(a) \cup aW(a^{-1}) = \mathbb{F}_2 = W(b) \cup bW(b^{-1}).$$

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Define $\quad A = \begin{pmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}.$

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Proposition
$\mathbb{F}_2 \cong \langle A, B \rangle < SO(3)$.

Banach-Tarski Paradox (1924)
Special orthogonal group $SO(3)$ contains a group isomorphic copy of the free group in two generators $\mathbb{F}_2$ and $\mathbb{F}_2$ admits a paradoxical decomposition.

$\mathbb{F}_2 := \{a, b, a^{-1}, b^{-1}, a^2b^3a^{-1}, \ldots\}$ – free group in 2 generators,

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Proposition
Let G be a discrete group. TFAE:

(i) *there is $\mu \in L^\infty(G)^*$ satisfying:*
- $\langle 1_G, \mu \rangle = \|\mu\| = 1$,
- $\langle L_s f, \mu \rangle = \langle f, \mu \rangle$ *for all* $s \in G, f \in L^\infty(G)$;

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Examples

1. Amenable: compact (Haar measure is finite), abelian (Markov-Kakutani fixed point theorem).
2. Non-amenable: $\mathbb{F}_n, GL(n, \mathbb{K}), SL(n, \mathbb{K})$.

Amenable Banach Algebras

Theorem (B.E. Johnson, 1972)
Let G be a locally compact group. TFAE:

(1) *G is amenable,*

(2) *Every continuous derivation $\delta\colon L^1(G) \to X^*$ from the group algebra $L^1(G)$ into any dual Banach $L^1(G)$-bimodule X^* is inner.*

Remark. A linear map $\delta\colon A \to X$ is a derivation if $\delta(ab) = a \cdot \delta(b) + \delta(a) \cdot b$.
A derivation δ is inner if $\delta(a) = \mathrm{ad}_x(a) := a \cdot x - x \cdot a$ for some $x \in X$.

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We say that a Banach algebra A is:

(i) amenable if every cont. derivation into any dual A-bimodule is inner,

(ii) weakly amenable if every cont. derivation into A^* is inner,

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Let G be a locally compact group. Then:

1. *(Johnson, 1972) $L^1(G)$ is amenable $\Leftrightarrow$ G is amenable.*
2. *(Johnson, 1991) $L^1(G)$ is always weakly amenable.*
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Theorem

1. *(Connes, 1978 + Haagerup, 1983) A C^*-algebra A is amenable $\Leftrightarrow$ it is nuclear, i.e. for every C^*-algebra B there exists a unique C^*-norm on $A \otimes B$.*
2. *(Haagerup, 1983) Every C^*-algebra is weakly amenable.*
3. *(Selivanov, 1976) A C^*-algebra is strongly amenable $\Leftrightarrow$ it is finite dimensional.*

Towards Quantitative Approach

Theorem (B.E. Johnson, 1972)
A Banach algebra A is amenable if and only if there is an approximate diagonal, i.e. a bounded net $(d_\alpha)_\alpha \in A \widehat{\otimes} A$ such that for every $a \in A$ we have

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \qquad \wedge \qquad a \cdot \pi(d_\alpha) \to a.$$

We then say A is C-amenable where $C := \sup_\alpha \|d_\alpha\|$.

Remark. Define $a \cdot (x \otimes y) := ax \otimes y$, $(x \otimes y) \cdot a := x \otimes ya$ and $\pi(a \otimes b) := ab$ and extend linearly.

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If A is a Banach algebra then

$\mathrm{AM}\,(A) := \inf\{C > 0 \colon\ A \text{ is } C\text{-amenable}\}.$

Proposition
If A is either a convolution algebra or a C^*-algebra then

$\mathrm{AM}\,(A) = 1 \qquad \textit{or} \qquad \mathrm{AM}\,(A) = \infty.$

Theorem (B.E. Johnson, 1972)
A Banach algebra A is strongly amenable if and only if there is

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an element $d \in A \widehat{\otimes} A$ such that for every $a \in A$ we have

$$a \cdot d - d \cdot a = 0 \qquad \wedge \qquad \pi(d) = 1.$$

We then say A is C-strongly amenable where $C := \|d\|$.

Remark. Define $a \cdot (x \otimes y) := ax \otimes y$, $(x \otimes y) \cdot a := x \otimes ya$ and $\pi(a \otimes b) := ab$ and extend linearly.

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If A is a Banach algebra then

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Theorem (KK–KP, 2024)

A Banach algebra A is weakly amenable if and only if there is $C > 0$ s.t. it is C-weakly amenable, i.e. for every $\delta \in \mathcal{Z}(A, A^*)$ there is $\varphi \in A^*$ with $\delta = \mathrm{ad}_\varphi \wedge \|\varphi\| \leq C\|\delta\|$.

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Easy Observations
Let A be a weakly amenable Banach algebra. Then:

(i) if A is commutative then $\mathrm{WAM}(A) = 0$,

(ii) if A is non-commutative then $\mathrm{WAM}(A) \geq \frac{1}{2}$,

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Compute $\mathrm{WAM}(A)$ for weakly amenable and non-commutative A.

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Amenability vs Vector-Valued

Theorem
Let A be a Banach algebra.

(1) $c_0(A)$ *is amenable if and only if so is* A.

(2) *(Zhang, 2015)* $C(K, A)$ *is amenable if and only if so is* A. *If* A *is commutative then* $C(K, A)$ *is weakly amenable if and only if so is* A.

(3) *(Lau–Loy–Willis, '96) If* A *is a* C^**-algebra then* $\ell_\infty(A)$ *is amenable if and only if* $A^{**} \in (\mathrm{AP})$. *In particular,* $\mathcal{K}(\ell_2)$ *is amenable whereas* $\ell_\infty(\mathcal{K}(\ell_2))$ *is not.*

Remark. If X is a Banach space and $1 \le p \le \infty$ then

$$\ell_p(X) := \{(x_n)_{n\in\mathbb{N}} \subset X : \quad \|(\|x_n\|_X)_n\|_{\ell_p} < \infty\}.$$

Theorem (KK–KP, 2024)
Let A be a Banach algebra and let $1 \leq p < \infty$.

(1) $\ell_p(A)$ *is never amenable.*

(2) *If A is* <u>*commutative*</u> *and weakly amenable then so is $\ell_p(A)$.*

(3) *If A is non-commutative and $p \neq 1$ then $\ell_p(A)$ is not weakly amenable.*

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Theorem (KK-KP, 2024)
Let A be a Banach algebra. TFAE:

(i) *A is weakly amenable,*

(ii) *$\ell_1(A)$ is weakly amenable,*

(iii) *$c_0(A)$ is weakly amenable.*

WAM **in Matrix Algebras**

Recall that $\mathrm{WAM}(A) = \inf\{C > 0\colon\ A \text{ is } C\text{-weakly amenable}\}$ and $\mathrm{WAM}(A) \geq \frac{1}{2}$ for non-commutative A. Moreover, if $a \in \mathcal{M}_n(\mathbb{C})$ and $1 \leq p \leq \infty$ then

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where λ_k's are eigenvalues of a^*a. Denote $A_p := (\mathcal{M}_2(\mathbb{C}), \|\cdot\|_p)$ and $B_p := (T_2, \|\cdot\|_p)$, where $T_2 := \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} :\ \alpha, \beta, \gamma \in \mathbb{C} \right\}$.

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where λ_k's are eigenvalues of a^*a. Denote $A_p := (\mathcal{M}_2(\mathbb{C}), \|\cdot\|_p)$ and $B_p := (T_2, \|\cdot\|_p)$, where $T_2 := \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} :\ \alpha, \beta, \gamma \in \mathbb{C} \right\}$.

Theorem (KK–KP, 2024)

(i) $\mathrm{WAM}(A_\infty) = \frac{1}{2}, \quad \mathrm{WAM}(A_2) = \frac{\sqrt{2}}{2}, \quad \mathrm{WAM}(A_1) \overset{?}{=} \frac{\sqrt{3}}{2},$

(ii) $\mathrm{WAM}(B_\infty) = \frac{1}{2}, \quad \mathrm{WAM}(B_2) = \frac{\sqrt{2}}{2}, \quad \mathrm{WAM}(B_1) = \frac{\sqrt{3}}{2}.$

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Proof.
$\|a\|_1 = \sqrt{\mathrm{tr}(a^*a) + 2|\det a|}\ +$ for all $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ there are $u, v \in \mathbb{T}$ such that

$$2(|\beta|^2+|\gamma|^2)+|(\alpha-\delta)^2+4\beta\gamma| = |\beta u+\gamma v|^2+|(\alpha-\delta)^2uv+(\beta u+\gamma v)^2|.$$