Triality and fixed points of Spin-bundles

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Representations of $\mathfrak{so}(2n, \mathbb{C})$

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Let (V, Q) be an m-dimensional $\mathbb{C}$-vector space ($m \geq 2$) equipped with a non-degenerate symmetric bilinear form, Q.

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Let (V, Q) be an m-dimensional $\mathbb{C}$-vector space ($m \geq 2$) equipped with a non-degenerate symmetric bilinear form, Q.

The group $G = \mathrm{SO}(Q) = \mathrm{SO}(n, \mathbb{C})$ is the group of automorphisms of V preserving Q and with determinant one. Its Lie algebra is $\mathfrak{g} = \mathfrak{so}(Q) = \mathfrak{so}(n, \mathbb{C})$.

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Let E_{ij} be the $2n \times 2n$ matrix with 1 in the (i,j) position and 0 in the others.

Then, the matrices

$$H_i = E_{ii} - E_{n+i,n+i}$$

generate a Cartan subalgebra of $\mathfrak{g}$. Let $L_1, \ldots, L_n$ be the duals.

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The adjoint representation

$$ad : \mathfrak{so}(2n, \mathbb{C}) \to \mathfrak{gl}(\mathfrak{so}(2n, \mathbb{C}))$$

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The spin representation of $\mathfrak{so}(2n, \mathbb{C})$

$$\Delta : \mathfrak{so}(2n, \mathbb{C}) \to \mathfrak{gl}\left(\bigwedge^{\bullet} W\right),$$

where W is an isotropic n-dimensional subspace of V breaks into the two half-spin representations

Representations of $\mathfrak{so}(2n, \mathbb{C})$

$$\Delta^+ : \mathfrak{so}(2n, \mathbb{C}) \to \mathfrak{gl}\left(\bigwedge^{+} W\right),$$

$$\Delta^- : \mathfrak{so}(2n, \mathbb{C}) \to \mathfrak{gl}\left(\bigwedge^{-} W\right),$$

both irreducible and with maximal weights

$$\frac{1}{2}\left(L_1 + \cdots + L_{n-1} + L_n\right) \text{ and}$$
$$\frac{1}{2}\left(L_1 + \cdots + L_{n-1} - L_n\right)$$

respectively.

The case of $\mathfrak{so}(8, \mathbb{C})$

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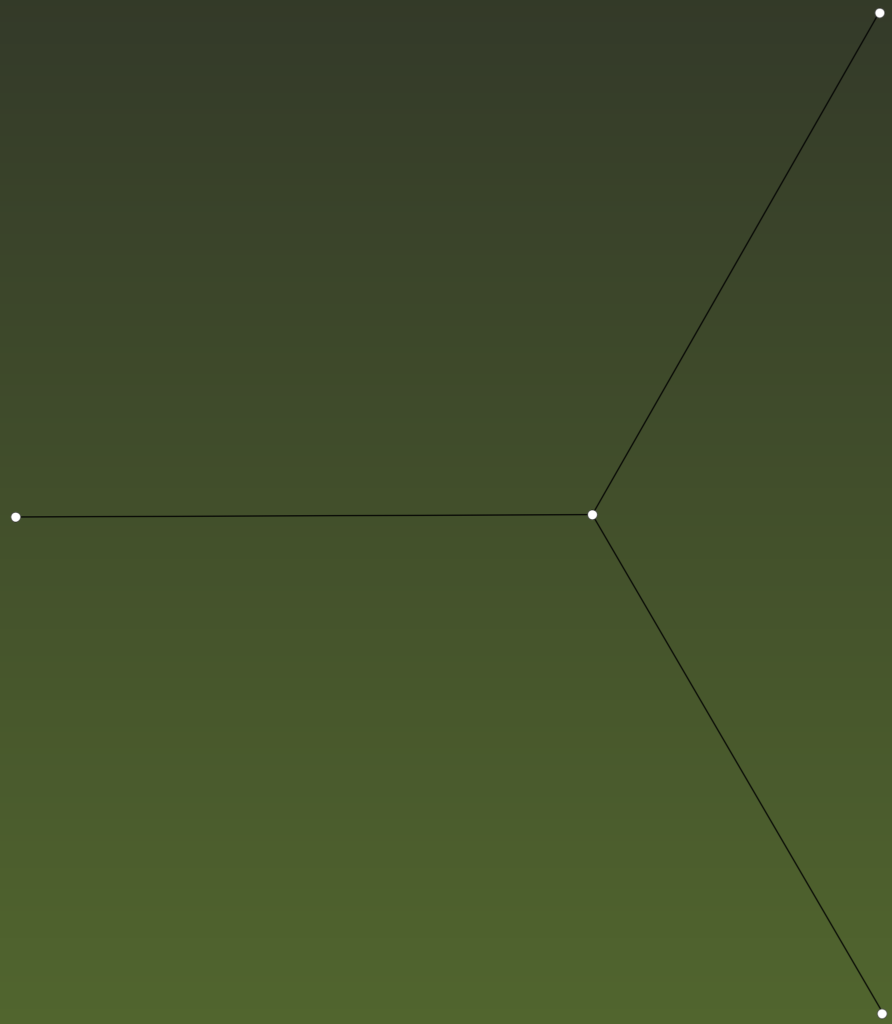
The Dynkin diagram of the algebra $\mathfrak{so}(8, \mathbb{C})$ is D_4,



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Theorem. Let $\mathfrak{g}$ be a simple algebra. The group $\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g}) / \mathrm{Int}(\mathfrak{g})$ is isomorphic to the group of symmetries of the Dynkin diagram of $\mathfrak{g}$.

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From the theorem it follows that $\mathrm{Out}(\mathfrak{so}(8, \mathbb{C}))$ is isomorphic to S_3. In particular, $\mathfrak{so}(8, \mathbb{C})$ admits two different automorphisms (modulo inner automorphisms) of order 3 (those corresponding to the symmetries of D_4). These automorphisms are mutually inverse.

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This table sums up the basic information about representations:

Representation	Complex dimension	Dominant weight
ad	28	$L_1 + L_2$
st	8	L_1
Δ^+	8	$\frac{1}{2}(L_1 + L_2 + L + L_4)$
Δ^-	8	$\frac{1}{2}(L_1 + L_2 + L - L_4)$

The case of $\mathfrak{so}(8, \mathbb{C})$

The family of weights $\{L_1, L_1 + L_2, \frac{1}{2}(L_1 + L_2 + L_3 + L_4), \frac{1}{2}(L_1 + L_2 + L_3 - L_4)\}$ is a fundamental system of weights of $\mathfrak{so}(8, \mathbb{C})$.

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In fact, the automorphisms of order three of $\mathfrak{so}(8, \mathbb{C})$ leaves the standard representation invariant and moves the other three (ad to Δ^-, Δ^- to Δ^+ and Δ^+ to ad), so these automorphisms correspond to the order three symmetries of the Dynkin diagram

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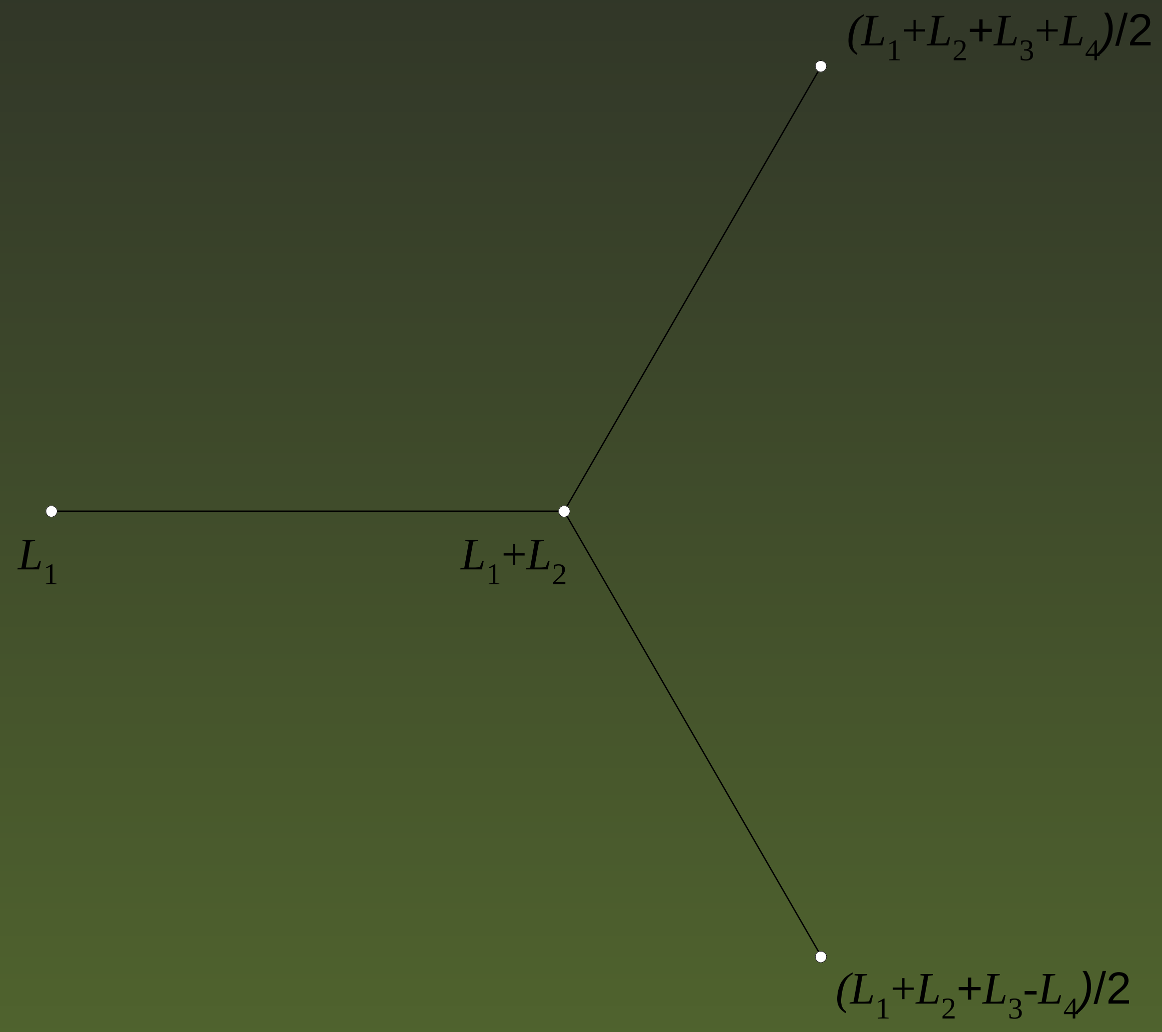

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Let τ and τ^{-1} be the automorphisms of order three. In $\mathrm{Aut}(\mathfrak{so}(8, \mathbb{C}))$ we consider other equivalence relation, $\sim_i$, given by conjugation by inner automorphisms, that is, if $\alpha, \beta \in \mathrm{Aut}(\mathfrak{so}(8, \mathbb{C}))$,

$$\alpha \sim_i \beta \Leftrightarrow \exists \theta \in \mathrm{Int}(\mathfrak{so}(8, \mathbb{C})) \ : \ \alpha = \theta \beta \theta^{-1}.$$

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Proposition. $\mathrm{Aut}_3(\mathfrak{so}(8, \mathbb{C}))/\sim_i$ has four elements, the classes of τ and τ^{-1} and the classes of other two automorphisms of order three, τ' and τ'^{-1}.

$\mathrm{Out}_3(\mathfrak{so}(8, \mathbb{C}))$ is given by the classes of τ y τ^{-1}.

The case of $\mathfrak{so}(8,\mathbb{C})$

Proposition. Via the natural map

$$\mathrm{Aut}_3(\mathfrak{so}(8,\mathbb{C}))/\sim_i \longrightarrow \mathrm{Out}_3(\mathfrak{so}(8,\mathbb{C})) \cup \{1\},$$

the classes of τ and τ' modulo inner conjugation are sent to the class of τ modulo inner automorphisms and the classes of τ^{-1} and τ'^{-1} modulo inner conjugation are sent to the class of τ^{-1} modulo inner automorphisms.

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Proposition. The algebra of fixed points of τ is isomorphic to the algebra $\mathfrak{g}_2$ and the algebra of fixed points of τ' is isomorphic to the algebra $\mathfrak{a}_2$.

Triality automorphism

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Let $\mathbb{K}$ be a field and (V, Q) a $\mathbb{K}$-vector space equipped with a symmetric bilinear form. In the tensor algebra of V, $\bigotimes V$, we consider the ideal

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Let $Cl(Q)_0$ be the subalgebra of the Clifford algebra generated by the products of an even number of elements of V and the automorphism $* : Cl(Q) \longrightarrow Cl(Q)$ defined by

$$(v_1 \cdot \cdots \cdot v_r)^* = (-1)^r v_r \cdot \cdots \cdot v_1 .$$

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- $\mathrm{Spin}(8, \mathbb{C})$ is simply connected. So it is the simply connected group of algebra $\mathfrak{so}(8, \mathbb{C})$.
- The map $\rho : \mathrm{Spin}(8, \mathbb{C}) \to \mathrm{SO}(8, \mathbb{C})$, $x \mapsto \rho(x)(v) = xvx^*$ is a $2 : 1$ covering map. So $\mathrm{Spin}(8, \mathbb{C})$ is the universal cover of $\mathrm{SO}(8, \mathbb{C})$.

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Proposition. Let $\mathfrak{g}$ be a complex Lie algebra and G the unique connected and simply connected group with Lie algebra $\mathfrak{g}$. Then, there is a natural automorphism of short exact sequences of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Int}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & \mathrm{Out}(G) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Int}(\mathfrak{g}) & \longrightarrow & \mathrm{Aut}(\mathfrak{g}) & \longrightarrow & \mathrm{Out}(\mathfrak{g}) & \longrightarrow & 1. \end{array}$$

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We obtain that the automorphism τ lifts uniquely to an automorphism of $\mathrm{Spin}(8, \mathbb{C})$. this automorphism is called *triality automorphism*.

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From all this we deduce that the group of fixed points of j is isomorphic to G_2 and the group of fixed points of j' is isomorphic to $\mathrm{SL}(3, \mathbb{C})$.

Moduli of principal bundles

Moduli of principal bundles

Definition. Let G be a reductive group. A holomorphic principal G-bundle E is said to be stable (resp. semistable) if for each reduction of the structure group of E to a parabolic subgroup P of G (that is, for each global section $\sigma : X \to E/P$), we have that $\deg \sigma^*(T_{G/P}) > 0$ (resp. ≥ 0), where $T_{G/P}$ is the sub-bundle of TE/P, tangent along the fibres of E/P.

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The moduli of principal G-bundles, $\mathcal{M}(G)$, is, then, an algebraic variety that parametrizes classes of S-equivalence of semistable bundles.

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From this, $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ is a variety of dimension $28(g-1)$ and the varieties $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$, $\mathcal{M}(G_2)$ and $\mathcal{M}(\mathrm{SL}(3,\mathbb{C}))$ are irreducible.

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Recall the notion of simplicity.

Definition. A G-bundle is said to be simple if its unique automorphisms are multiplication by elements of the center of G.

Moduli of principal bundles

We will study stable and simple fixed points for certain automorphisms of the moduli of principal $\mathrm{Spin}(8, \mathbb{C})$-bundles.

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Proposition. If G is semisimple, $\mathcal{M}(G)$ is smooth in the open set of stable simple points.

From this proposition we know that we will study smooth fixed points of the moduli.

Action of $\mathrm{Out}(G)$ in $\mathcal{M}(G)$

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Let G a complex semisimple Lie group. The group $\mathrm{Aut}(G)$ acts on $\mathcal{M}(G)$ in this way: if E is a G-bundle and $A \in \mathrm{Aut}(G)$, $A(E)$ will be equal to E as a variety but equipped with the following action of G,

$$e \Diamond g = eA^{-1}(g)$$

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If $\{\psi_{ij}\}$ are transition functions of E, then $\{A \circ \psi_{ij}\}$ are transition functions of $A(E)$.

With the point of view of transition functions, it is easy to show that if A is an inner automorphism of G, then E is isomorphic to $A(E)$, that is, the action of $\mathrm{Aut}(G)$ is trivial on $\mathrm{Int}(G)$.

Action of $\mathrm{Out}(G)$ in $\mathcal{M}(G)$

We can assure that the action of $\mathrm{Aut}(G)$ defines an action of $\mathrm{Out}(G)$, that is, this action of $\mathrm{Out}(G)$ in $\mathcal{M}(G)$ is well defined: if $\sigma \in \mathrm{Out}(G)$, $A \in \mathrm{Aut}(G)$ is a representant of σ and $E \in \mathcal{M}(G)$, we define

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Observe that, in order to prove that the preceding action is well defined, it is necessary to see that, if $A \in \mathrm{Aut}(G)$, then $A(E)$ is semistable if E is semistable (which is immediate form the definition of semistability) and the following result.

Proposition. If E_1 and E_2 are semistable S-equivalent bundles, then $A(E_1)$ and $A(E_2)$ are S-equivalent.

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The proof of the preceding proposition follows from the definition.

Fixed points of $\mathrm{Spin}(8, \mathbb{C})$-bundles

Fixed points of $\mathrm{Spin}(8,\mathbb{C})$-bundles

The main result of this work is the following.

Theorem. Let X be a compact Riemann surface. Let $\sigma \in \mathrm{Out}(G)$ with $G = \mathrm{Spin}(8,\mathbb{C})$ an element of order three. Let $\mathcal{M}^\sigma(G)$ be the subset of fixed points of $\mathcal{M}(G)$ for the action of σ and $\mathcal{M}^\sigma_{stable,simple}(G)$ be the subset of stable and simple fixed points. Then,

$$\widetilde{\mathcal{M}(G_2)} \cup \mathcal{M}(\widetilde{\mathrm{SL}(3,\mathbb{C})}) \subseteq \mathcal{M}^\sigma(\mathrm{Spin}(8,\mathbb{C}))$$

and

$$\mathcal{M}^\sigma(\mathrm{Spin}(8,\mathbb{C}))_{stable,simple} \subseteq \widetilde{\mathcal{M}(G_2)} \cup \mathcal{M}(\widetilde{\mathrm{SL}(3,\mathbb{C})}),$$

where, if H is a subgroup of G, $\widetilde{\mathcal{M}(H)}$ is the image of the map

$$\mathcal{M}(H) \to \mathcal{M}(G), \quad E \mapsto E \times_H G$$

induced by the inclusion of groups $H \hookrightarrow G$.

Fixed points of $\mathrm{Spin}(8,\mathbb{C})$-bundles

Suppose that A is a lifting of σ and E is a simple fixed point. Then E and $A(E)$ are isomorphic. There exists an isomorphism

$$f : E \longrightarrow A(E).$$

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A acts on the bundles and on the morphisms. Taking into account that $A^3 = 1$, we have a chain

$$E \xrightarrow{f} A(E) \xrightarrow{A(f)} A^2(E) \xrightarrow{A^2(f)} E,$$

and, so, an automorphism of E

$$A^2(f) \circ A(f) \circ f : E \longrightarrow E.$$

Fixed points of $\mathrm{Spin}(8, \mathbb{C})$-bundles

From the simplicity of E we deduce the existence of $\lambda \in Z(\mathrm{Spin}(8, \mathbb{C}))$ such that

$$A^2(f) \circ A(f) \circ f = \lambda.$$

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It will be $\lambda \in \mathrm{Fix}(A)$. We know that $\mathrm{Fix}(A) \cong G_2$ or $\mathrm{Fix}(A) \cong \mathrm{SL}(3,\mathbb{C})$. From this and the fact that λ is in the center, we can deduce that $\lambda = 1$. So we have the equation

$$A^2(g)A(g)g = 1.$$

Fixed points of $\mathrm{Spin}(8,\mathbb{C})$-bundles

This equation determines a variety H with tangent space at 1

$$\ker\left(dA^2\big|_1 + dA\big|_1 + id\right).$$

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We consider the map $X \longrightarrow E(G/H)$ defined in the following way. For $x \in X$ and (U,ϕ) a local trivialization of E, the image of x is $\left[\phi^{-1}(x,1),H\right]$. It is well defined and defines a global section of $E(G/H)$.

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Global section of $E(G/H)$ induce global sections of

$$E\left(\mathrm{Gr}_k\left(\ker\left(dA^2\big|_1 + dA|_1 + id\right)\right)\right) \to X$$

(details are due to A. Ramanathan), where k is the dimension of the subalgebra $\ker\left(dA^2\big|_1 + dA|_1 + id\right)$.

Fixed points of $\mathrm{Spin}(8,\mathbb{C})$-bundles

It can be seen that the subalgebras $\ker\left(dA^2\big|_1 + dA|_1 + id\right)$ and $\ker\left(dA|_1 - id\right)$ are semisimple and mutually orthogonal with respect to the Killing form. From this, it is equivalent a global section of

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The converse (a bundle that reduces to $\mathrm{Fix}(A)$ is a fixed point) is easy.

Fixed points of $\mathrm{Spin}(8, \mathbb{C})$-bundles

Then, we have proved the desired theorem

Theorem. Let X be a compact Riemann surface. Let $\sigma \in \mathrm{Out}(G)$ with $G = \mathrm{Spin}(8, \mathbb{C})$ an element of order three. Let $\mathcal{M}^{\sigma}(G)$ be the subset of fixed points of $\mathcal{M}(G)$ for the action of σ and $\mathcal{M}^{\sigma}_{stable,simple}(G)$ be the subset of stable and simple fixed points. Then,

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Appendix

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THE END

THANK YOU VERY MUCH

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