Contractions of Jordan Algebras in dimension 2

Jonathan Sánchez Hernández

September, 20

Definitions and first properties

Definition

A Jordan Algebra in $\mathbb{R}$, $\mathfrak{J}$, is a symmetric bilinear aplication $\varphi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\varphi(\varphi(X,X),\varphi(X,Y)) = \varphi(X,\varphi(\varphi(X,X),Y)), \quad \forall X,Y \in \mathbb{R}^n$$

Then, we say the dimension of $\mathfrak{J}$ is n.

Definitions and first properties

Definition

A Jordan Algebra in $\mathbb{R}$, $\mathfrak{J}$, *is a symmetric bilinear aplication* $\varphi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *such that*

$$\varphi(\varphi(X,X),\varphi(X,Y)) = \varphi(X,\varphi(\varphi(X,X),Y)), \quad \forall X,Y \in \mathbb{R}^n$$

Then, we say the dimension of $\mathfrak{J}$ *is* n.

This relation becomes in equations system:

$$a_{ii}^h a_{kj}^l a_{lk}^r - a_{ii}^h a_{jk}^l a_{lk}^r - a_{ii}^h a_{jk}^l a_{lh}^r - 2a_{ij}^h a_{ik}^l a_{hl}^r + 2a_{il}^r a_{hj}^l a_{ik}^h = 0,$$
$$1 \le i,j,k,l,r,h \le n,$$

i. e., J^2, the variety of Jordan law 2-dimensional, is a algebraic variety in $\mathbb{R}^6$.

Definitions and first properties

This algebra is similar to other algebras. Thus, we can define notions as ideal or subalgebra.

Definitions and first properties

This algebra is similar to other algebras. Thus, we can define notions as ideal or subalgebra.
Nevertheless, we define isotropy:

Definition

A vectorial subespace V of $\mathbb{R}^n$ is isotropy if there are a vector $v \in \mathbb{R}^n \setminus \{0\}$ such that $\varphi(v, w) = 0$ for all $w \in V$.
We say v is a isotrope vector.

Jordan algebra without isotrope elements has a neutral element.

Definitions and first properties

This algebra is similar to other algebras. Thus, we can define notions as ideal or subalgebra.
Nevertheless, we define isotropy:

Definition

A vectorial subespace V of $\mathbb{R}^n$ is isotropy if there are a vector $v \in \mathbb{R}^n \setminus \{0\}$ such that $\varphi(v, w) = 0$ for all $w \in V$.
We say v is a isotrope vector.

Jordan algebra without isotrope elements has a neutral element.

Definition

If a Jordan algebra has not an ideal, then it is a simple Jordan algebra.

2-dimensional Jordan algebras

If $\{e_1, e_2\}$ is a basis of vector space, then we can express the Jordan law $\circ$ as

$$\begin{aligned} e_1 \circ e_1 &= a_1 e_1 + a_2 e_2 \\ e_2 \circ e_2 &= b_1 e_1 + b_2 e_2 \\ e_1 \circ e_2 &= c_1 e_1 + c_2 e_2 \end{aligned} \longrightarrow \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}$$

We obtain the following equations over a_i, b_i, c_i:

2-dimensional Jordan algebras

$$
\begin{aligned}
a_2\,(c_1\,c_2) &= a_2\,(a_2\,b_1)\\
b_1\,(c_1\,c_2) &= b_1\,(a_2\,b_1)\\
a_2\,(a_1\,b_1 + b_2\,c_1) &= a_2\,\left(b_1\,c_2 + c_1^2\right)\\
b_1\,(a_1\,b_1 + b_2\,c_1) &= b_1\,\left(b_1\,c_2 + c_1^2\right)\\
a_2\,\left(a_2\,c_1 + c_2^2\right) &= a_2\,(a_1\,c_2 + a_2\,b_2)\\
b_1\,\left(a_2\,c_1 + c_2^2\right) &= b_1\,(a_1\,c_2 + a_2\,b_2)\\
a_1\,c_1\,c_2 + 2a_2\,b_1\,c_2 &= 2c_1\,c_2^2 + a_1\,a_2\,b_1\\
a_1\,a_2\,c_1 + 3a_1\,c_2^2 + 2a_2\,b_2\,c_2 &= 2a_2\,c_1\,c_2 + 2c_2^3 + a_1^2\,c_2 + a_1\,a_2\,b_2\\
2c_1^2\,c_2 + 2b_1\,c_2^2 + a_1^2\,b_1 + a_1\,b_2\,c_1 &= 3a_1\,b_1\,c_2 + 2b_2\,c_1\,c_2 + a_1\,c_1^2\\
2c_1\,c_2^2 + 2a_2\,c_1^2 + a_1\,b_2\,c_2 + a_2\,b_2^2 &= b_2\,c_2^2 + 2a_1\,c_1\,c_2 + 3a_2\,b_2\,c_1\\
2a_1\,b_1\,c_1 + 3b_2\,c_1^2 + b_1\,b_2\,c_2 &= a_1\,b_1\,b_2 + b_2^2\,c_1 + 2c_1^3 + 2b_1\,c_1\,c_2\\
2a_2\,b_1\,c_1 + b_2\,c_1\,c_2 &= a_2\,b_1\,b_2 + 2c_1^2\,c_2.
\end{aligned}
$$

Classification of 2-dimensional Jordan algebras

Jordan algebra:

- Without isotropy $\longrightarrow$ with unit element
  - $\psi_0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - $\psi_5 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$

Classification of 2-dimensional Jordan algebras

Jordan algebra:

- Without isotropy $\longrightarrow$ with unit element
  - $\psi_0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - $\psi_5 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$
- With isotropy

Classification of 2-dimensional Jordan algebras

Jordan algebra:

- Without isotropy $\longrightarrow$ with unit element
  - $\psi_0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - $\psi_5 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$
- With isotropy
  - with unit element $\longrightarrow \psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$

Classification of 2-dimensional Jordan algebras

Jordan algebra:

- Without isotropy $\longrightarrow$ with unit element
  - $\psi_0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - $\psi_5 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$
- With isotropy
  - with unit element $\longrightarrow \psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$
  - without unit element
    - $\psi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$
    - $\psi_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
    - $\psi_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

Perturbations on Jordan algebras

We will use the non-standard analysis. Then, we suppose n is standard and then J^n is standard too. Also, if $\varphi \in J^n$ standard and x and y standard, $\varphi(x, y)$ is standard. In the same way, if x and y are limited vector, then $\varphi(x, y)$ is limited.

Perturbations on Jordan algebras

We will use the non-standard analysis. Then, we suppose n is standard and then J^n is standard too. Also, if $\varphi \in J^n$ standard and x and y standard, $\varphi(x, y)$ is standard. In the same way, if x and y are limited vector, then $\varphi(x, y)$ is limited.

We define the perturbation on Jordan algebras:

Definition

Let φ be a standard law of J^n. A perturbation φ_0 of φ is a Jordan law on $\mathbb{R}^n$ that is infinitely near to φ, i. e., for all vectors $x, y \in \mathbb{R}^n$, $\varphi(x, y)$ and $\varphi_0(x, y)$ are infinitely near, and we denote $\varphi \sim \varphi_0$.

Perturbations on Jordan algebras

We will use the non-standard analysis. Then, we suppose n is standard and then J^n is standard too. Also, if $\varphi \in J^n$ standard and x and y standard, $\varphi(x, y)$ is standard. In the same way, if x and y are limited vector, then $\varphi(x, y)$ is limited.

We define the perturbation on Jordan algebras:

Definition

Let φ be a standard law of J^n. A perturbation φ_0 of φ is a Jordan law on $\mathbb{R}^n$ that is infinitely near to φ, i. e., for all vectors $x, y \in \mathbb{R}^n$, $\varphi(x, y)$ and $\varphi_0(x, y)$ are infinitely near, and we denote $\varphi \sim \varphi_0$.

We will show some properties about the perturbations on Jordan algebras.

Jordan algebras in non-standard analysis

We show the good behaviour of Jordan algebras in the theory of non-standard analysis.

Jordan algebras in non-standard analysis

We show the good behaviour of Jordan algebras in the theory of non-standard analysis. The following results show this good behaviour:

Proposition

If $\varphi \in J^n$, then its shadow φ_0 of φ is also a Jordan algebra.

It's a consequence of *standarization axiom*

Jordan algebras in non-standard analysis

We show the good behaviour of Jordan algebras in the theory of non-standard analysis. The following results show this good behaviour:

Proposition

If $\varphi \in J^n$, then its shadow φ_0 of φ is also a Jordan algebra.

It's a consequence of *standarization axiom*

Proposition

Let V be a vector space of $\mathbb{R}^n$ (with n standard). Then, the shadow ${}^0V$ is a subspace of $\mathbb{R}^n$ and it has the same dimension of V.

Jordan algebras in non-standard analysis

It has a direct consequence in Jordan algebras

Corollary

Let φ_0 be a standard Jordan law and let φ be a perturbation of φ_0. Then the shadow of a subalgebra (or a ideal, resp.) of $(\mathbb{R}, \varphi)$ is a subalgebra (a ideal, resp.) of $(\mathbb{R}^n, \varphi)$ with the same dimension.

Jordan algebras in non-standard analysis

It has a direct consequence in Jordan algebras

Corollary

Let φ_0 be a standard Jordan law and let φ be a perturbation of φ_0. Then the shadow of a subalgebra (or a ideal, resp.) of $(\mathbb{R}, \varphi)$ is a subalgebra (a ideal, resp.) of $(\mathbb{R}^n, \varphi)$ with the same dimension.

Proposition

Let φ be a perturbation of a standard Jordan law φ_0 on $\mathbb{R}^n$. If $G_0 = (\mathbb{R}^n, \varphi_0)$ is simple, then $G = (\mathbb{R}^n, \varphi)$ is simple too.

Because the existence of a ideal in G_0 gives a ideal in G.

Jordan algebras in non-standard analysis

It has a direct consequence in Jordan algebras

Corollary

Let φ_0 be a standard Jordan law and let φ be a perturbation of φ_0. Then the shadow of a subalgebra (or a ideal, resp.) of $(\mathbb{R}, \varphi)$ is a subalgebra (a ideal, resp.) of $(\mathbb{R}^n, \varphi)$ with the same dimension.

Proposition

Let φ be a perturbation of a standard Jordan law φ_0 on $\mathbb{R}^n$. If $G_0 = (\mathbb{R}^n, \varphi_0)$ is simple, then $G = (\mathbb{R}^n, \varphi)$ is simple too.

Because the existence of a ideal in G_0 gives a ideal in G.
Finally, the last property:

Proposition

If G_0 has not isotropy, then G has not isotropy.

Rigid Jordan algebras

Definition

A standard Jordan algebra φ_0 *is* *rigid* *if his perturbations are isomorphic to* φ_0.

This is the same notion of rigidity in the classic theory.

Rigid Jordan algebras

Definition

A standard Jordan algebra φ_0 is rigid if his perturbations are isomorphic to φ_0.

This is the same notion of rigidity in the classic theory.
A first example: The rigidity of ψ_0.

$$\psi_0 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Rigid Jordan algebras

Definition

A standard Jordan algebra φ_0 is rigid if his perturbations are isomorphic to φ_0.

This is the same notion of rigidity in the classic theory.
A first example: The rigidity of ψ_0.

$$\psi_0 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let φ a perturbation of ψ_0, that it will be (in the same basis):

$$\varphi \equiv \begin{pmatrix} 1+\epsilon_1 & \epsilon_2 \\ 1+\epsilon_3 & \epsilon_4 \\ \epsilon_5 & 1+\epsilon_6 \end{pmatrix}$$

Rigid Jordan algebras

Definition

A standard Jordan algebra φ_0 is rigid if his perturbations are isomorphic to φ_0.

This is the same notion of rigidity in the classic theory.
A first example: The rigidity of ψ_0.

$$\psi_0 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let φ a perturbation of ψ_0, that it will be (in the same basis):

$$\varphi \equiv \begin{pmatrix} 1+\epsilon_1 & \epsilon_2 \\ 1+\epsilon_3 & \epsilon_4 \\ \epsilon_5 & 1+\epsilon_6 \end{pmatrix} \xrightarrow[e_1' = \frac{1}{1+\epsilon_6}]{e_2' = e_2} \begin{pmatrix} 1+\epsilon_1 & \epsilon_2 \\ 1+\epsilon_3 & \epsilon_4 \\ \epsilon_5 & 1 \end{pmatrix}$$

As φ is a Jordan algebra, then his structure constants must satisfy the equation

$$\epsilon_5 = \epsilon_2(1 + \epsilon_3).$$

As φ is a Jordan algebra, then his structure constants must satisfy the equation

$$\epsilon_5 = \epsilon_2(1 + \epsilon_3).$$

In the other hand, non-isotropy implies that the algebra has a unit element, say $u = \alpha e'_1 + \beta e'_2$.

As φ is a Jordan algebra, then his structure constants must satisfy the equation

$$\epsilon_5 = \epsilon_2(1 + \epsilon_3).$$

In the other hand, non-isotropy implies that the algebra has a unit element, say $u = \alpha e_1' + \beta e_2'$. It holds the following equations:

$$\begin{cases} 1 = \alpha(1 + \epsilon_1) + \beta\epsilon_5, \\ 0 = \alpha\epsilon_2 + \beta, \\ 0 = \alpha\epsilon_5 + \beta(1 + \epsilon_3), \\ 1 = \alpha + \beta\epsilon_4. \end{cases}$$

As φ is a Jordan algebra, then his structure constants must satisfy the equation

$$\epsilon_5 = \epsilon_2(1+\epsilon_3).$$

In the other hand, non-isotropy implies that the algebra has a unit element, say $u = \alpha e_1' + \beta e_2'$. It holds the following equations:

$$\begin{cases} 1 = \alpha(1+\epsilon_1) + \beta\epsilon_5, \\ 0 = \alpha\epsilon_2 + \beta, \\ 0 = \alpha\epsilon_5 + \beta(1+\epsilon_3), \\ 1 = \alpha + \beta\epsilon_4. \end{cases} \rightarrow \begin{cases} 0 = \beta + \alpha\epsilon_2, \\ 1 = \alpha(1-\epsilon_2\epsilon_4). \end{cases}$$

Then, $\alpha \sim 1$ and $\beta \sim 0$.

Then, $\alpha \sim 1$ and $\beta \sim 0$.
In the basis $\{u, e_2\}$ we obtain the following matricial expression of φ:

$$\begin{pmatrix} 1 & 0 \\ 1+\epsilon & \epsilon' \\ 0 & 1 \end{pmatrix},$$

Then, $\alpha \sim 1$ and $\beta \sim 0$.
In the basis $\{u, e_2\}$ we obtain the following matricial expression of φ:

$$\begin{pmatrix} 1 & 0 \\ 1+\epsilon & \epsilon' \\ 0 & 1 \end{pmatrix},$$

As $(\epsilon')^2 + 4(1+\epsilon) > 0$ then, φ is isomorphics to ψ_0.

Dimension of the orbits

$GL(2,\mathbb{R})$ act on the variety J^n as

$$(f,\varphi) := f^{-1} \circ \varphi \circ (f,f)$$

Dimension of the orbits

$GL(2,\mathbb{R})$ act on the variety J^n as

$$(f,\varphi) := f^{-1} \circ \varphi \circ (f,f)$$

Now, we take $Id \in GL(n,\mathbb{R})$ and $f \in \mathfrak{gl}(n,\mathbb{R})$ standard. Let ϵ be infinitely small, and let $Id + \epsilon f$ be the endomorphism in $GL(n,\mathbb{R})$ and

$$(\mathrm{Id}+\varepsilon f)^{-1}\varphi_0\left((\mathrm{Id}+\varepsilon f),(\mathrm{Id}+\varepsilon f)\right) = \varphi_0+\varepsilon\left(\delta_{\varphi_0} f\right)+\varepsilon^2\left(\Delta\left(\varphi_0,f,\varepsilon\right)\right),$$

Dimension of the orbits

$GL(2,\mathbb{R})$ act on the variety J^n as

$$(f,\varphi) := f^{-1} \circ \varphi \circ (f,f)$$

Now, we take $Id \in GL(n,\mathbb{R})$ and $f \in \mathfrak{gl}(n,\mathbb{R})$ standard. Let ϵ be infinitely small, and let $Id + \epsilon f$ be the endomorphism in $GL(n,\mathbb{R})$ and

$$(\mathrm{Id} + \varepsilon f)^{-1} \varphi_0 \left((\mathrm{Id} + \varepsilon f), (\mathrm{Id} + \varepsilon f) \right) = \varphi_0 + \varepsilon \left(\delta_{\varphi_0} f \right) + \varepsilon^2 \left(\Delta \left(\varphi_0, f, \varepsilon \right) \right),$$

where

$$\delta_{\varphi_0} f(x,y) := \varphi_0 \left(f(x), y \right) + \varphi_0 \left(x, f(y) \right) - f\left(\varphi_0 (x,y) \right), \ \forall x,y \in \mathbb{R}^n.$$

Dimension of the orbits

$GL(2,\mathbb{R})$ act on the variety J^n as

$$(f,\varphi) := f^{-1} \circ \varphi \circ (f,f)$$

Now, we take $Id \in GL(n,\mathbb{R})$ and $f \in \mathfrak{gl}(n,\mathbb{R})$ standard. Let ϵ be infinitely small, and let $Id + \epsilon f$ be the endomorphism in $GL(n,\mathbb{R})$ and

$$(\mathrm{Id}+\varepsilon f)^{-1} \varphi_0 ((\mathrm{Id}+\varepsilon f),(\mathrm{Id}+\varepsilon f)) = \varphi_0 + \varepsilon (\delta_{\varphi_0} f) + \varepsilon^2 (\Delta(\varphi_0, f, \varepsilon)),$$

where

$$\delta_{\varphi_0} f(x,y) := \varphi_0(f(x),y) + \varphi_0(x,f(y)) - f(\varphi_0(x,y)), \ \forall x,y \in \mathbb{R}^n.$$

Then, the tangent space of the orbit $\mathcal{O}(\varphi_0)$ is

$$T_{\varphi_0}\mathcal{O}(\varphi_0) = \{\delta_{\varphi_0} f \ : \ f \in \mathfrak{gl}(n,\mathbb{R})\}$$

Example

Let φ_0 be a Jordan algebra without isotropie.

Example

Let φ_0 be a Jordan algebra without isotropie. Somehow, φ_0 can be express like ψ_0 or ψ_5 in a basis $\{e_1, e_2\}$ and his matricial expression is

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example

Let φ_0 be a Jordan algebra without isotropie. Somehow, φ_0 can be express like ψ_0 or ψ_5 in a basis $\{e_1, e_2\}$ and his matricial expression is

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, we take an element of $\mathfrak{gl}(2, \mathbb{R})$,

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If we apply this f on ψ_0 or ψ_5, we obtain the tangent space of $\mathcal{O}(\psi_0)$ and $\mathcal{O}(\psi_5)$:

$$T_{\psi_0} f = \begin{pmatrix} a & b \\ 2d - a & 2c - b \\ b & a \end{pmatrix}, \qquad T_{\psi_5} f = \begin{pmatrix} a & b \\ -2d + a & 2c + b \\ -b & a \end{pmatrix}.$$

If we apply this f on ψ_0 or ψ_5, we obtain the tangent space of $\mathcal{O}(\psi_0)$ and $\mathcal{O}(\psi_5)$:

$$T_{\psi_0} f = \begin{pmatrix} a & b \\ 2d - a & 2c - b \\ b & a \end{pmatrix}, \qquad T_{\psi_5} f = \begin{pmatrix} a & b \\ -2d + a & 2c + b \\ -b & a \end{pmatrix}.$$

Thus, $\dim \mathcal{O}(\psi_0) = \dim \mathcal{O}(\psi_5) = 4$

If we apply this f on ψ_0 or ψ_5, we obtain the tangent space of $\mathcal{O}(\psi_0)$ and $\mathcal{O}(\psi_5)$:

$$T_{\psi_0} f = \begin{pmatrix} a & b \\ 2d - a & 2c - b \\ b & a \end{pmatrix}, \qquad T_{\psi_5} f = \begin{pmatrix} a & b \\ -2d + a & 2c + b \\ -b & a \end{pmatrix}.$$

Thus, $\dim \mathcal{O}(\psi_0) = \dim \mathcal{O}(\psi_5) = 4$
J^2 has two 4-dimensional open components.

Theorem of decomposition of a point

Theorem

Let P_0 a standard point of $\mathbb{R}^n$ with n standard. All point P infinitely close of P_0 has a decomposition like

$$P = P_0 + \epsilon_1 v_1 + \ldots + \epsilon_1 \cdots \epsilon_p v_p$$

that it satisfies the following conditions:

Theorem of decomposition of a point

Theorem

Let P_0 a standard point of $\mathbb{R}^n$ with n standard. All point P infinitely close of P_0 has a decomposition like

$$P = P_0 + \epsilon_1 v_1 + \ldots + \epsilon_1 \cdots \epsilon_p v_p$$

that it satisfies the following conditions:

1. $\epsilon_1, \ldots, \epsilon_p$ *are real numbers such that they are infinitely small.*

Theorem of decomposition of a point

Theorem

Let P_0 a standard point of $\mathbb{R}^n$ with n standard. All point P infinitely close of P_0 has a decomposition like

$$P = P_0 + \epsilon_1 v_1 + \ldots + \epsilon_1 \cdots \epsilon_p v_p$$

that it satisfies the following conditions:

1. $\epsilon_1, \ldots, \epsilon_p$ *are real numbers such that they are infinitely small.*
2. $v_1, \ldots, v_p$ *are standard vectors of $\mathbb{R}^n$ linearly independent.*

Theorem of decomposition of a point

Theorem

Let P_0 a standard point of $\mathbb{R}^n$ with n standard. All point P infinitely close of P_0 has a decomposition like

$$P = P_0 + \epsilon_1 v_1 + \ldots + \epsilon_1 \cdots \epsilon_p v_p$$

that it satisfies the following conditions:

1. *$\epsilon_1, \ldots, \epsilon_p$ are real numbers such that they are infinitely small.*
2. *$v_1, \ldots, v_p$ are standard vectors of $\mathbb{R}^n$ linearly independent.*

If there is a diferent decomposition $P = P_0 + \epsilon'_1 v'_1 + \ldots + \epsilon'_1 \cdots \epsilon'_q v'_q$ with the same conditions

Theorem of decomposition of a point

Theorem

Let P_0 a standard point of $\mathbb{R}^n$ with n standard. All point P infinitely close of P_0 has a decomposition like

$$P = P_0 + \epsilon_1 v_1 + \ldots + \epsilon_1 \cdots \epsilon_p v_p$$

that it satisfies the following conditions:

1. $\epsilon_1, \ldots, \epsilon_p$ *are real numbers such that they are infinitely small.*
2. $v_1, \ldots, v_p$ *are standard vectors of $\mathbb{R}^n$ linearly independent.*

If there is a diferent decomposition $P = P_0 + \epsilon'_1 v'_1 + \ldots + \epsilon'_1 \cdots \epsilon'_q v'_q$ with the same conditions then $p = q$ and

Theorem of decomposition of a point

Theorem

Let P_0 a standard point of $\mathbb{R}^n$ with n standard. All point P infinitely close of P_0 has a decomposition like

$$P = P_0 + \epsilon_1 v_1 + \ldots + \epsilon_1 \cdots \epsilon_p v_p$$

that it satisfies the following conditions:

1. $\epsilon_1, \ldots, \epsilon_p$ *are real numbers such that they are infinitely small.*
2. $v_1, \ldots, v_p$ *are standard vectors of* $\mathbb{R}^n$ *linearly independent.*

If there is a diferent decomposition $P = P_0 + \epsilon'_1 v'_1 + \ldots + \epsilon'_1 \cdots \epsilon'_q v'_q$ *with the same conditions then* $p = q$ *and*

1. $v'_i = \sum_{j=1}^{i} a_i^j v_j$ *which* a_i^j *are standard real numbers for all* i, j*, and* $a_i^i \neq 0$*;*

Theorem of decomposition of a point

Theorem

Let P_0 a standard point of $\mathbb{R}^n$ with n standard. All point P infinitely close of P_0 has a decomposition like

$$P = P_0 + \epsilon_1 v_1 + \ldots + \epsilon_1 \cdots \epsilon_p v_p$$

that it satisfies the following conditions:

1. $\epsilon_1, \ldots, \epsilon_p$ *are real numbers such that they are infinitely small.*
2. $v_1, \ldots, v_p$ *are standard vectors of $\mathbb{R}^n$ linearly independent.*

If there is a diferent decomposition $P = P_0 + \epsilon'_1 v'_1 + \ldots + \epsilon'_1 \cdots \epsilon'_q v'_q$ with the same conditions then $p = q$ and

1. $v'_i = \sum_{j=1}^{i} a_i^j v_j$ *which a_i^j are standard real numbers for all i, j, and $a_i^i \neq 0$;*
2. $\epsilon_i \cdots \epsilon_i = \sum_{j=i}^{p} a_j^i \epsilon'_1 \cdots \epsilon'_j$

Rigidity of Jordan algebras

Let φ be a perturbation of a standard Jordan law of φ_0.

Rigidity of Jordan algebras

Let φ be a perturbation of a standard Jordan law of φ_0. Then, we can split φ in a sum of symmetric bilinear applications:

$$\varphi = \varphi_0 + \epsilon_1 \varphi_1 + \ldots + \epsilon_1 \cdots \epsilon_p \varphi_p$$

and this φ_i $i = 1, \ldots, p$ are independents.

Rigidity of Jordan algebras

Let φ be a perturbation of a standard Jordan law of φ_0. Then, we can split φ in a sum of symmetric bilinear applications:

$$\varphi = \varphi_0 + \epsilon_1 \varphi_1 + \ldots + \epsilon_1 \cdots \epsilon_p \varphi_p$$

and this φ_i $i = 1, \ldots, p$ are independents. As φ is a Jordan law, then its holds the following equations:

Rigidity of Jordan algebras

Let φ be a perturbation of a standard Jordan law of φ_0. Then, we can split φ in a sum of symmetric bilinear applications:

$$\varphi = \varphi_0 + \epsilon_1 \varphi_1 + \ldots + \epsilon_1 \cdots \epsilon_p \varphi_p$$

and this φ_i $i = 1, \ldots, p$ are independents. As φ is a Jordan law, then its holds the following equations:

$$\begin{aligned} \delta_{\varphi_0} \varphi_1 &= 0, \\ \varepsilon_2 \delta_{\varphi_0} \varphi_2 + \varepsilon_1 \delta_{\varphi_1} \varphi_0 + \varepsilon_1^2 \varphi_1^3 + \varepsilon_1 \varepsilon_2 \delta_{\varphi_1} \varphi_3 + \varepsilon_1^2 \varepsilon_2 \delta_{\varphi_1} \varphi_2 + & \\ + \varepsilon_1 \varepsilon_2^2 \delta_{\varphi_2} \varphi_1 + \varepsilon_1 \varepsilon_2 \varepsilon_3 \left(\varphi_0, \varphi_1, \varphi_2 \right) + \ldots + & \\ + \varepsilon_1^2 \varepsilon_2^3 \varepsilon_3^3 \ldots \varepsilon_{p-1}^3 \varepsilon_p^2 \delta_{\varphi_p} \varphi_{p-1} + \varepsilon_1^2 \varepsilon_2^3 \varepsilon_3^3 \ldots \varepsilon_p^3 \varphi_p^3 &= 0, \end{aligned}$$

Rigidity of Jordan algebras

where

$$\begin{aligned}
\varphi_i \circ \varphi_j \circ \varphi_k (X, Y) =& \varphi_i (\varphi_j (X, X), \varphi_k (X, Y)) \\
& - \varphi_i (X, \varphi_j (\varphi_k (X; X), Y)), \\
\varphi_i^3 =& \varphi_i \circ \varphi_i \circ \varphi_i, \\
(\varphi_i, \varphi_j, \varphi_k) =& \varphi_i \circ \varphi_j \circ \varphi_k + \varphi_j \circ \varphi_k \circ \varphi_i + \varphi_k \circ \varphi_i \circ \varphi_j + \\
& + \varphi_i \circ \varphi_k \circ \varphi_j + \varphi_j \circ \varphi_i \circ \varphi_k + \varphi_k \circ \varphi_j \circ \varphi_i \\
\delta_{\varphi_i} \varphi_j =& \varphi_i (\varphi_i (X, X), \varphi_j (X, Y)) + \\
& + \varphi_i (\varphi_j (X, X), \varphi_i (X, Y)) + \\
& + \varphi_j (\varphi_i (X, X), \varphi_i (X, Y)) - \\
& - \varphi_j (X, \varphi_i (\varphi_i (X, X), Y)) - \\
& - \varphi_i (X, \varphi_j (\varphi_i (X, X), Y)) - \\
& - \varphi_i (X, \varphi_i (\varphi_j (X, X), Y)).
\end{aligned}$$

Rigidity of Jordan algebras

Let $G(\varphi_0)$ denote the set of bilinear application φ such that $\delta_{\varphi_0}\varphi = 0$.

Rigidity of Jordan algebras

Let $G(\varphi_0)$ denote the set of bilinear application φ such that $\delta_{\varphi_0}\varphi = 0$.

Lemma

Let $\varphi_1, \varphi_2 \in J^n$ be two Jordan law. The plane span by $\{0, \varphi_1, \varphi_2\}$ is contained in J^n if and only if $\delta_{\varphi_1}\varphi_2 = \delta_{\varphi_2}\varphi_1 = 0$.

Rigidity of Jordan algebras

Let $G(\varphi_0)$ denote the set of bilinear application φ such that $\delta_{\varphi_0}\varphi = 0$.

Lemma

Let $\varphi_1, \varphi_2 \in J^n$ be two Jordan law. The plane span by $\{0, \varphi_1, \varphi_2\}$ is contained in J^n if and only if $\delta_{\varphi_1}\varphi_2 = \delta_{\varphi_2}\varphi_1 = 0$.

I claim that ψ_4 is rigid and $\dim \mathcal{O}(\psi_4) = 2$.

Rigidity of Jordan algebras

Let $G(\varphi_0)$ denote the set of bilinear application φ such that $\delta_{\varphi_0}\varphi = 0$.

Lemma

Let $\varphi_1, \varphi_2 \in J^n$ be two Jordan law. The plane span by $\{0, \varphi_1, \varphi_2\}$ is contained in J^n if and only if $\delta_{\varphi_1}\varphi_2 = \delta_{\varphi_2}\varphi_1 = 0$.

I claim that ψ_4 is rigid and $\dim \mathcal{O}(\psi_4) = 2$. Let φ be a perturbation of ψ_4:

$$\varphi = \psi_4 + \epsilon_1\varphi_1 + \ldots + \epsilon_1 \cdots \epsilon_p\varphi_p, \;\; p \leq 6$$

and let $G(\psi_4) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \right\}$ be.

Rigidity of Jordan algebras

Let $G(\varphi_0)$ denote the set of bilinear application φ such that $\delta_{\varphi_0}\varphi = 0$.

Lemma

Let $\varphi_1, \varphi_2 \in J^n$ be two Jordan law. The plane span by $\{0, \varphi_1, \varphi_2\}$ is contained in J^n if and only if $\delta_{\varphi_1}\varphi_2 = \delta_{\varphi_2}\varphi_1 = 0$.

I claim that ψ_4 is rigid and $\dim \mathcal{O}(\psi_4) = 2$. Let φ be a perturbation of ψ_4:

$$\varphi = \psi_4 + \epsilon_1\varphi_1 + \ldots + \epsilon_1 \cdots \epsilon_p\varphi_p, \;\; p \leq 6$$

and let $G(\psi_4) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \right\}$ be.If $\varphi_1 \in G(\psi_4)$ then $\delta_{\psi_4}\varphi_1 = 0$. By the above result, φ_1 is contain in J^2 and therefore $\varphi_1^3 = 0$. By the equation on φ_i, we obtain $\varphi_2 \in G(\psi_4)$ an so on...

Rigidity of Jordan algebras

Then $\dim \mathcal{O}(\psi_4) = 2$ and a perturbation

$$\begin{pmatrix} 1+\epsilon & 0 \\ 0 & \epsilon' \\ \frac{\epsilon'}{2} & \frac{1+\epsilon}{2} \end{pmatrix}$$

Rigidity of Jordan algebras

Then $\dim \mathcal{O}(\psi_4) = 2$ and a perturbation

$$\begin{pmatrix} 1+\epsilon & 0 \\ 0 & \epsilon' \\ \frac{\epsilon'}{2} & \frac{1+\epsilon}{2} \end{pmatrix} \xrightarrow[e_2'=-\frac{\epsilon'}{1+\epsilon}e_1+(1+\epsilon)e_2]{e_1'=e_1} \psi_4$$

Theorem

The Jordan algebra φ_2 *doesn't perturb on* φ_5*. In particular,* φ_5 *doesn't contract on* φ_2*.*

Theorem

The Jordan algebra φ_2 doesn't perturb on φ_5. In particular, φ_5 doesn't contract on φ_2.

Let $\varphi = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & 1+\epsilon_4 \\ \epsilon_5 & \epsilon_6 \end{pmatrix}$ be a perturbation of ψ_2 with $\epsilon_i \sim 0$.

Theorem

The Jordan algebra φ_2 doesn't perturb on φ_5. In particular, φ_5 doesn't contract on φ_2.

Let $\varphi = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & 1+\epsilon_4 \\ \epsilon_5 & \epsilon_6 \end{pmatrix}$ be a perturbation of ψ_2 with $\epsilon_i \sim 0$.
If $\epsilon_5 \neq 0$ then the equation

$$\epsilon_5(2\epsilon_1\epsilon_3 + 3\epsilon_5 - 1 - 2\epsilon_5^2 - 2\epsilon_3\epsilon_6) = \epsilon_3(\epsilon_1 - \epsilon_6)$$

and then

$$-1 \sim 2\epsilon_1\epsilon_3 + 3\epsilon_5 - 1 - 2\epsilon_5^2 - 2\epsilon_3\epsilon_6 = \frac{\epsilon_3}{\epsilon_5}(\epsilon_1 - \epsilon_6).$$

Theorem

The Jordan algebra φ_2 doesn't perturb on φ_5. In particular, φ_5 doesn't contract on φ_2.

Let $\varphi = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & 1+\epsilon_4 \\ \epsilon_5 & \epsilon_6 \end{pmatrix}$ be a perturbation of ψ_2 with $\epsilon_i \sim 0$.
If $\epsilon_5 \neq 0$ then the equation

$$\epsilon_5(2\epsilon_1\epsilon_3 + 3\epsilon_5 - 1 - 2\epsilon_5^2 - 2\epsilon_3\epsilon_6) = \epsilon_3(\epsilon_1 - \epsilon_6)$$

and then

$$-1 \sim 2\epsilon_1\epsilon_3 + 3\epsilon_5 - 1 - 2\epsilon_5^2 - 2\epsilon_3\epsilon_6 = \frac{\epsilon_3}{\epsilon_5}(\epsilon_1 - \epsilon_6).$$

and this carry out that $\frac{\epsilon_5}{\epsilon_3} \sim 0$

Theorem

The Jordan algebra φ_2 doesn't perturb on φ_5. In particular, φ_5 doesn't contract on φ_2.

Let $\varphi = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & 1+\epsilon_4 \\ \epsilon_5 & \epsilon_6 \end{pmatrix}$ be a perturbation of ψ_2 with $\epsilon_i \sim 0$.
If $\epsilon_5 \neq 0$ then the equation

$$\epsilon_5(2\epsilon_1\epsilon_3 + 3\epsilon_5 - 1 - 2\epsilon_5^2 - 2\epsilon_3\epsilon_6) = \epsilon_3(\epsilon_1 - \epsilon_6)$$

and then

$$-1 \sim 2\epsilon_1\epsilon_3 + 3\epsilon_5 - 1 - 2\epsilon_5^2 - 2\epsilon_3\epsilon_6 = \frac{\epsilon_3}{\epsilon_5}(\epsilon_1 - \epsilon_6).$$

and this carry out that $\frac{\epsilon_5}{\epsilon_3} \sim 0$ then we can take this change of basis

$$\begin{cases} e_1' = e_1 - \frac{\epsilon_5}{\epsilon_3} e_2, \\ e_2' = \lambda e_2, \ \text{with } \lambda \sim 1 \end{cases} \longrightarrow \text{We can suppose that } \epsilon_5 = 0$$

- If $\epsilon_3 \neq 0$,

- If $\epsilon_3 \neq 0$, then $\epsilon_2 = 0$ and $\epsilon_1 = \epsilon_6$. Hence, we have two cases:

- If $\epsilon_3 \neq 0$, then $\epsilon_2 = 0$ and $\epsilon_1 = \epsilon_6$. Hence, we have two cases:
  - If $\epsilon_1 \neq 0$, then, with a change of basis we obtain that φ can be express with the matrix

$$\begin{pmatrix} 1 & 0 \\ \epsilon_1 \epsilon_3 & 1 \\ 0 & 1 \end{pmatrix}$$

and this is isomorphics to ψ_0.

- If $\epsilon_3 \neq 0$, then $\epsilon_2 = 0$ and $\epsilon_1 = \epsilon_6$. Hence, we have two cases:
    - If $\epsilon_1 \neq 0$, then, with a change of basis we obtain that φ can be express with the matrix

$$\begin{pmatrix} 1 & 0 \\ \epsilon_1\epsilon_3 & 1 \\ 0 & 1 \end{pmatrix}$$

      and this is isomorphics to ψ_0.
    - If $\epsilon_1 = 0$ then the perturbation is isomorphics to ψ_2.

- If $\epsilon_3 \neq 0$, then $\epsilon_2 = 0$ and $\epsilon_1 = \epsilon_6$. Hence, we have two cases:
  - If $\epsilon_1 \neq 0$, then, with a change of basis we obtain that φ can be express with the matrix
  $$\begin{pmatrix} 1 & 0 \\ \epsilon_1\epsilon_3 & 1 \\ 0 & 1 \end{pmatrix}$$
  and this is isomorphics to ψ_0.
  - If $\epsilon_1 = 0$ then the perturbation is isomorphics to ψ_2.
- If $\epsilon_3 = 0$ then the perturbation is $\begin{pmatrix} \epsilon_1 & \epsilon_2 \\ 0 & 1 \\ 0 & \epsilon_6 \end{pmatrix}$ and φ is not a simple Jordan algebra ($I = span\{e_2'\}$ is an ideal) and thus, φ is not isomorphics to ψ_5.

Finally, we obtain the rigid algebras: ψ_0, ψ_4 and ψ_5.

Finally, we obtain the rigid algebras: ψ_0, ψ_4 and ψ_5. Let φ_0 be a Jordan algebra different from ψ_0, ψ_4 or ψ_5

Finally, we obtain the rigid algebras: ψ_0, ψ_4 and ψ_5. Let φ_0 be a Jordan algebra different from ψ_0, ψ_4 or ψ_5 and by isotropy we obtain the result:

Finally, we obtain the rigid algebras: ψ_0, ψ_4 and ψ_5. Let φ_0 be a Jordan algebra different from ψ_0, ψ_4 or ψ_5 and by isotropy we obtain the result:

$$\varphi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\varphi_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Finally, we obtain the rigid algebras: ψ_0, ψ_4 and ψ_5. Let φ_0 be a Jordan algebra different from ψ_0, ψ_4 or ψ_5 and by isotropy we obtain the result:

$$\varphi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \xrightarrow[\textit{perturbation}]{} \quad \varphi = \begin{pmatrix} 1 & 0 \\ \epsilon & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\varphi_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Finally, we obtain the rigid algebras: ψ_0, ψ_4 and ψ_5. Let φ_0 be a Jordan algebra different from ψ_0, ψ_4 or ψ_5 and by isotropy we obtain the result:

$$\varphi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow[\textit{perturbation}]{} \varphi = \begin{pmatrix} 1 & 0 \\ \epsilon & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow[\textit{perturbation}]{} \varphi = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\varphi_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Finally, we obtain the rigid algebras: ψ_0, ψ_4 and ψ_5. Let φ_0 be a Jordan algebra different from ψ_0, ψ_4 or ψ_5 and by isotropy we obtain the result:

$$\varphi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow[\textit{perturbation}]{} \varphi = \begin{pmatrix} 1 & 0 \\ \epsilon & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow[\textit{perturbation}]{} \varphi = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\varphi_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow[\textit{perturbation}]{} \varphi = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \\ \epsilon & 0 \end{pmatrix}$$

Finally, we obtain the rigid algebras: ψ_0, ψ_4 and ψ_5. Let φ_0 be a Jordan algebra different from ψ_0, ψ_4 or ψ_5 and by isotropy we obtain the result:

$$\varphi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow[\textit{perturbation}]{} \varphi = \begin{pmatrix} 1 & 0 \\ \epsilon & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow[\textit{perturbation}]{} \varphi = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\varphi_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow[\textit{perturbation}]{} \varphi = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \\ \epsilon & 0 \end{pmatrix}$$

They are not isomorphic to φ_0.

Like Lie algebras, the contractions are define by means of limits on applications.

Definition

Let $\{f_t\}_{t\in\mathbb{R}}$ be a family of non-singular endomorphisms which depends on t. If $\varphi'(x,y) = \lim_{t\to 0} f_t^{-1} \circ \varphi(f_t(x), f_t(y))$ then φ' is a contraction of φ by $\{f_t\}$.

Like Lie algebras, the contractions are define by means of limits on applications.

Definition

Let $\{f_t\}_{t\in\mathbb{R}}$ *be a family of non-singular endomorphisms which depends on* t*. If* $\varphi'(x,y) = \lim_{t\to 0} f_t^{-1} \circ \varphi(f_t(x), f_t(y))$ *then* φ' *is a* *contraction* *of* φ *by* $\{f_t\}$*.*

Property: If $\varphi \to \varphi'$ non-trivial contraction then $\dim \mathcal{O}(\varphi) > \dim \mathcal{O}(\varphi')$.

Like Lie algebras, the contractions are define by means of limits on applications.

Definition

Let $\{f_t\}_{t\in\mathbb{R}}$ be a family of non-singular endomorphisms which depends on t. If $\varphi'(x,y) = \lim_{t\to 0} f_t^{-1} \circ \varphi(f_t(x), f_t(y))$ then φ' is a contraction of φ by $\{f_t\}$.

Property: If $\varphi \to \varphi'$ non-trivial contraction then $\dim \mathcal{O}(\varphi) > \dim \mathcal{O}(\varphi')$.
We use the dimension of the orbits in our classification:

$$\dim \mathcal{O}(\psi_5) = \dim \mathcal{O}(\psi_0) = 4,$$
$$\dim \mathcal{O}(\psi_1) = \dim \mathcal{O}(\psi_2) = 3,$$
$$\dim \mathcal{O}(\psi_3) = \dim \mathcal{O}(\psi_4) = 2.$$

- ψ_1 is a contraction of ψ_0 and ψ_5

$$\psi_0 \text{ or } \psi_5 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}} \psi_1$$

- ψ_1 is a contraction of ψ_0 and ψ_5

$$\psi_0 \text{ or } \psi_5 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}} \psi_1$$

- ψ_2 is a contraction of ψ_0

$$\psi_0 \xrightarrow{\begin{pmatrix} t & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}} \psi_2$$

- ψ_1 is a contraction of ψ_0 and ψ_5

$$\psi_0 \text{ or } \psi_5 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}} \psi_1$$

- ψ_2 is a contraction of ψ_0

$$\psi_0 \xrightarrow{\begin{pmatrix} t & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}} \psi_2$$

- ψ_3 is a contraction of ψ_0, ψ_1, ψ_2 and ψ_5:

$$\psi_0 \xrightarrow{\begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}} \psi_5 \qquad \psi_1 \xrightarrow{\begin{pmatrix} t & t \\ 0 & t^2 \end{pmatrix}} \psi_3$$

$$\psi_2 \xrightarrow{\begin{pmatrix} 1 & t \\ 0 & t^2 \end{pmatrix}} \psi_3 \quad \psi_5 \xrightarrow{\begin{pmatrix} \frac{\sqrt{t}}{2} & \frac{\sqrt{t}}{2} \\ t & 0 \end{pmatrix}} \psi_3$$

Overview of 2-dimensional Jordan algebras