Connections in Principal Fibre Bundles and applications to Field Equations

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Outline

1. Fibre Bundles
   - Vector Bundles
   - Principal Fibre Bundles
2. Connections
3. Field Equations

The Tangent Bundle

M smooth m-dimensional manifold
T_xM tangent space of M at a point x of M

The tangent bundle of M, denoted by TM consists of the collection of the tangent spaces of M, i.e.,

$$TM = \bigsqcup_{x \in M} T_xM$$

An element of TM is represented by V_x, where $V \in \mathbb{R}^m$, $x \in M$ and $V_x \in T_xM$.

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There is a natural projection:

$$\begin{aligned} \pi : TM &\longrightarrow M \\ V_x &\longmapsto x \end{aligned}$$

which is a surjective submersion that is locally a cross product, i.e.,

- π and $d\pi$ are surjectives
- around each $x \in M$ there exists an open neighborhood U such that $\pi^{-1}(U) \simeq U \times \mathbb{R}^m$

The Vector Bundle

- M smooth m-dimensional manifold -base space
- V n-dimensional vector space -fibre
- E smooth $n+m$-dimensional manifold -total space
- $\pi : E \to M$ smooth surjective submersion satisfying:
  - around each $x \in M$ there exists an open neighborhood U such that $\pi^{-1}(U)$ and $U \times V$ are diffeomorphic
  - $\pi^{-1}(x)$ is a vector space isomorphic to V
  - coordinates change linearly on the fibres

$\pi : E \to M$ is said to be a vector bundle over M of typical fibre V.

- A vector bundle is said to be trivial if $\pi^{-1}(M) = M \times V$

The Frame Bundle

M smooth m-dimensional manifold

F_xM set of all possible bases of T_xM, where $x \in M$

The frame bundle of M, denoted by FM consists of the collection of the bases of the tangent spaces of M, i.e.,

$$FM = \bigsqcup_{x \in M} F_xM$$

An element of FM is represented by B_x, where $B = \{v_1, \cdots, v_m\}$ is a basis of $\mathbb{R}^m$, $x \in M$ and $v_i|_x \in T_xM$.

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- F_xM is bijective with $Gl(m;\mathbb{R})$, which is a Lie group
- There is a natural projection $B_x \mapsto x$
- Furthermore $Gl(m;\mathbb{R})$ acts freely on FM (i.e. without fixed points):

$$g \cdot B_p = (gB)_p \in FM \text{ for } g \in Gl(m;\mathbb{R})$$

The Principal Fibre Bundle

- M smooth m-dimensional manifold -base space
- P smooth $n+m$-dimensional manifold -total space
- G n-dimensional Lie group right-acting freely on P -typical fibre
- $\pi : P \to M$ smooth surjective submersion satisfying that the orbits of the action coincide with the fibres:

$$\{p \cdot g \mid g \in G\} = \pi^{-1}(\pi(p)) \text{ for all } p \in P$$

$\pi : P \to M$ is said to be a principal fibre bundle over M of typical fibre G or more simply a principal G-bundle.

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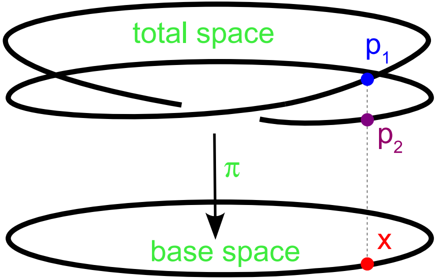
- For $x \in M$, $\pi^{-1}(x)$ has a Lie group structure but not canonical since there is no preferred choice of an identity element
- Around each $x \in M$ there exists an open neighborhood U such that $\pi^{-1}(U)$ and $U \times G$ are diffeomorphic. This is called a trivialization
- There is a special vector bundle over P called the vertical bundle which consists of the tangent vectors of P with null projection to TM. It is denoted by $\mathcal{V}P$

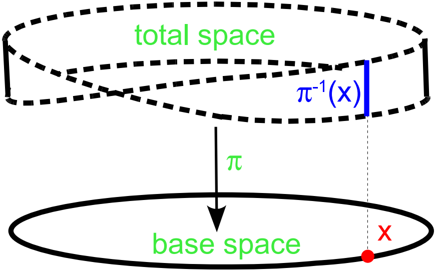
Examples

1. A principal fibre bundle with discrete fiber and whose base space coincides with the total space:



Principal $\mathbb{S}^0$-bundle over $\mathbb{S}^1$

2. A line bundle:



Vector $\mathbb{R}$-bundle over $\mathbb{S}^1$

Connections in Principal Fibre Bundles

Let $\pi : P \to M$ be a principal G-bundle and let $\mathfrak{g}$ be le Lie algebra of G.

Definition I A connection $\mathcal{H}$ in P is a G-invariant complementary distribution to the vertical bundle of P, meaning:

- G-invariant $(R_g)_*\mathcal{H}_p = \mathcal{H}_{p\cdot g}$ for every $p \in P$ and $g \in G$, where $R_g : P \to P$ is the right translation defined by g, i.e., $p \mapsto p \cdot g$
- complementary distribution $\mathcal{H}_p \oplus \mathcal{V}_pP = T_pP$ for every $p \in P$

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Definition II A connection in P is defined as a G-equivariant $\mathfrak{g}$-valued one-form ω on P well-behaved with the vertical bundle, meaning:

- G-equivariant $(R_g)^* \omega = Ad_{g^{-1}} \circ \omega$ for every $g \in G$
- well behaved $\omega(B_p^*) = B$ for all $B \in \mathfrak{g}$ and $p \in P$, where $B_p^* \in \mathcal{V}P$ is defined as $\frac{d}{dt}|_{t=0} p \cdot exp(tB)$

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- Given a connection in a principal G-bundle we define curvature as

$$\Omega = d\omega + [\omega, \omega]$$

Remarks

I⇒II Given $X_p \in TP$, it can be decomposed as $X^{\mathcal{H}} + X^{\mathcal{V}}$. Since

$$X^{\mathcal{V}} \in \mathcal{V}_p P$$

and

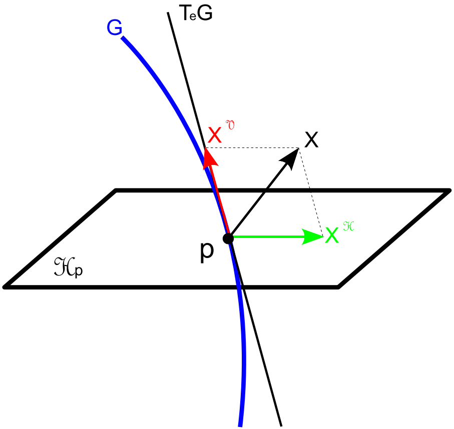
$$\mathcal{V}_p P \simeq T_e G = \mathfrak{g}$$

we can define

$$\omega(X) = B \text{ with } B_p^* = X^{\mathcal{V}}$$

II⇒I The distribution is defined as

$$\mathcal{H} = \ker(\omega)$$



Electromagnetism

- The electric $\vec{E}$ and magnetic $\vec{B}$ fields in a domain of $\mathbb{R}^3$ free of charges satisfy the so-called Maxwell equations

$$\operatorname{div}\vec{E} = 0 \qquad \operatorname{div}\vec{B} = 0$$
$$\operatorname{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \operatorname{rot}\vec{B} = \frac{\partial \vec{E}}{\partial t}$$

- On the other hand, the fields can be given through potentials

$$\vec{E} = -\operatorname{Grad}\phi + \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \operatorname{rot}\vec{A}$$

where ϕ is the electric potential and $\vec{A}$ the magnetic potential

- The potentials are not unique since for any function f

$$\phi' = \phi + \frac{\partial f}{\partial t} \qquad \text{and} \qquad \vec{A}' = \vec{A} + \operatorname{Grad} f$$

give the same fields. The transformation of a potential that gives the same field is called a gauge transformation

Electromagnetism

- We consider the trivial principal bundle $P = M \times \mathbb{S}^1$, with $M = \mathbb{R}^4$ space-time with the Minkowski metric $(-1, +1, +1, +1)$
- We define the connection form

$$A = d\theta + \phi dt + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$$

  Then the curvature is

$$\Omega = dA = -\sum E_i dt \wedge dx^i + \sum B_i dx^j \wedge dx^k$$

  and a gauge transformation is $A' = A + df$
- The Hodge star operator in pseudo-Riemannian manifolds (M, g) is defined as

$$*\colon \Omega^p(M) \to \Omega^{n-p}(M)$$

  characterized by $\quad g(\alpha, \beta)\mathbf{v}_g = (*\alpha) \wedge \beta, \ \forall\, \beta \in \Omega^p(M)$

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- Maxwell equations now read

$$d * \Omega = 0$$

Yang-Mills

- The introduction of bundles for Electromagnetism may seem complicated, but there are other similar field theories for which this geometrical construction is needed:
  - We take non-trivial bundles P and other groups G ($G = SU(2)$ related with weak interaction, $SU(3)$ related with nuclear strong interaction)
  - The equation is

$$d^A * \Omega = 0$$

- The goal now is the study of the Poisson bracket of functions in connections and the first derivatives, invariant with respect to the gauge transformations
- This is useful to give the formulation of the field equation using the gauge symmetry

The Hopf Fibration