A Quick Overview in Recent Enumerative Geometry

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1 Enumerative Geometry

2 How to solve enumerative problems?

3 Real Enumerative Geometry

Definition

Counting algebraic-geometric objects satisfying certain restrictions (geometric conditions), e.g. number of algebraic curve in X of a fixed degree and genus passing through points and tangents (with a given order) to fixed algebraic curves etc.

Examples

Two problems :

P1 : How many rational plane curves of degree d pass through $3d-1$ generics points in the plane? $\Rightarrow N_d$

P2 : Choose $k+l=\frac{d(d+3)}{2}$, how many smooth curves pass through k generics points and are tangent to l generic lines? $\Rightarrow Z_d(k,l)$

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A. Construct an appropriate moduli space

$\overline{\mathcal{M}}_{g,k}(X,\beta)$: moduli space of stable maps, i.e. curves in X of genus g, provided with k marked points and which realize the homology class $\beta \in H_2(X,\mathbb{Z})$ in a suitable compactification (M. Kontsevich, '94)

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$\Rightarrow$ Intersection theory in this spaces (Gromow-Witten theory) resolve (all most all) complex enumerative problems

B. Tropicalize

Geometry over the tropical semiring $(\mathbb{R}, max, +)$. A tropical polynomial is a piecewise linear function:

$$T(X,Y) = max_{(i,j)}\{a_{i,j} + iX + jY\}$$

Tropical curves (set of nondifferentiable points) can also be seen as deformations of curves with given geometric restrictions
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$\Rightarrow$ Reduce curves in simple objects and the enumerative problem in counting such objects in a combinatoric way

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Same things but over the field of real numbers $\mathbb{R}$

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In a real enumerative problem, we try to find enumerative invariants (and count them) or maximal configurations

P1 : Welschinger's Invariants

Sign of a curve: $m(C) := (-1)^{\#\text{isolated singularities}}$

Theorem (Welschinger, '03)

Let $\mathcal{P} = (p_1, \ldots, p_{3d-1})$ generics points in $\mathbb{R}P^2$, then the algebraic sum :

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Prove the theorem use real counterpart of A: $\mathbb{R}\overline{\mathcal{M}}_{0,3d-1}(\mathbb{C}P^2, d)$
Compute W_d need B (untill now)

P2 : Maximal Configuration

Does there exist a real configuration such that all (complex) curves satisfying the conditions are real?
For **P1**: We don't know (at least when $d \geq 4$)
For **P2**: We know for $l = 1$ or 2, using B (B. Bertrand)

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Other problems: Wich real enumerative problems carrie such integer invariants? (in project: studying characteristic classes of $\mathbb{R}\overline{\mathcal{M}}_{0,k}(X,\beta)$)