Proyecto Fin de Máster en Investigación Matemática
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid

Aproximación numérica para la ecuación del calor con flujo no lineal en la frontera

Gustavo Ito

Trabajo dirigido por: Raúl Ferreira.

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Abstract

In this work we study numerical approximation for positive solutions of a heat equation with a nonlinear flux condition which produces blow up of the solution. By a semidiscretization in space we obtain a system of ordinary differential equations which is expected to be an approximation of the original problem.

We describe in terms of the nonlinearity when solutions of this system exists globally in time and when they blow up in infinite time. We also find the blow-up rates and the blow-up set.

Moreover, under certain condition in the initial data, we also prove that the numerical blow-up time converges to the real blow-up time when the mesh-size goes to zero.

Keywords: Reaction-diffusion equations, numerical blow-up, nonlinear boundary condition.

AMS Subject Classification: 35B40, 35B35, 35K50.

Resumen

En este trabajo estudiamos una aproximación numérica para soluciones positivas de la ecuación del calor con condición de flujo no lineal en la frontera. Mediante una semidiscretización en espacio obtenemos un sistema de ecuaciones diferenciales ordinarias, que esperamos sea una buena aproximación del problema original.

Describimos, en términos de la no linearidad, cuando las soluciones del sistema de ecuaciones están definidas globalmente y cuando explotan en tiempo finito. También calculamos la tasa de explosión y el conjunto de explosión.

Además, bajo hipótesis adicionales en el dato inicial, también demostramos la convergencia de los tiempos de explosión al tiempo de explosión del problema original, cuando el tamaño de la malla tiende a cero.

Keywords: Ecuaciones de reacción-difusión, explosión numérica, condiciones de frontera no lineales.

AMS Subject Classification: 35B40, 35B35, 35K50.

Contents

1 Introduction

In this work we study the behavior of a numerical approximation of the following problem,

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) & \text{in } (0,1)\times[0,T) \\ u_x(0,t) = -u^p(0,t) & \text{in } [0,T) \\ u_x(1,t) = 0 & \text{in } [0,T) \\ u(x,0) = \psi(x) & \text{in } [0,1] \end{cases} \tag{1}$$

Where $p>0$ and the initial data $\psi(x)$ is a smooth positive function which satisfies the boundary condition.

Such problems can be interpreted of a model for heat propagation. In this case u stands for the temperature, and $-u_x$ represents the heat flux. Hence the boundary condition represent a nonlinear radiation law at the boundary. This kind of boundary condition appears also in combustion problems when the reaction happens only at the boundary of the container, for example because of the presence of a solid catalyzer.

Local in time existence and uniqueness can be obtained by using a contraction mapping principle. The time T is the maximal existence time for the solution, which may be finite or infinite. If $T<\infty$, then

$$\limsup_{t\to T} \|u(\cdot,t)\|_{L^\infty([0,1])} = \infty,$$

and we say that it blows up. If $T=\infty$ we say that the solution is global. For references on blow-up problem, see [5] and the references there in.

The major questions that have been studied since then are:

1. For which values of p does blow-up occur?
2. For which initial functions does blow-up occur?
3. With which rate (in t) does the solution approach the blow-up time?
4. Where are the blow-up points located?

In the survey, [6], this questions have been analyzed. Let us summarize the known results for the solutions of (1).

1. For $p \leq 1$ the solution is global in time.
2. For $p > 1$ all solution blows up in finite time.
3. Let u be a blow-up solution, then
$$\|u(\cdot, t)\|_{L^\infty([0,1])} \sim (T - t)^{-\frac{1}{2(p-1)}}$$
4. The only blow-up point is $x = 0$.

This work is devoted to studying a numerical approximation of (1). The main point we are interested in is to know how the answers of the above questions (respect to the continuous problem) are reproduced by numerical methods. In other words, we ask ourselves all these questions related to numerical approximations for blow-up problem (1); and we compare the answers with the ones concerning the continuous problem.

Inspired in [7], where the author study the heat equation with reaction in the interior of the domain,

$$\begin{cases} u_t = u_{xx} + u^p & \text{in } (-1,1) \times (0,T) \\ u(-1,t) = u(1,t) = 0 & \text{in } [0,T) \\ u(x,0) = u_0(x) & \text{in } [0,1] \end{cases}$$

We analyze a semidiscretizations in space, ie, discretizing the spatial variable x keeping t continuous. Therefore, we replace the original problem by a system of ordinary differential equations. For references of this type of numerical approximation see the survey [8].

More precisely, we consider a uniform mesh for the space variable, and we approximate the diffusion term u_{xx} using a standard central finite difference second order scheme. We denote by $U(t) = (u_1(t), \cdots, u_N(t))$ the values of the numerical approximation at the nodes $x_i = (i-1)\,h$ with $h = 1/(N-1)$, at time t. Then $U(t)$ is a solution of the following system

$$\begin{cases} u_1'(t) = \dfrac{2}{h^2}(u_2(t) - u_1(t)) + \dfrac{2}{h} u_1^p(t)\,, \\ u_j'(t) = \dfrac{1}{h^2}(u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)), \qquad j = 2, \cdots, N-1\,, \\ u_N'(t) = \dfrac{2}{h^2}(u_{N-1}(t) - u_N(t))\,. \end{cases} \tag{2}$$

To begin our analysis we prove that numerical approximations given by (2) converge uniformly if we consider a regular bounded solution of the continuous problem. Hence our scheme is uniformly convergent in sets of the form $[0, L] \times [0, T - \tau]$.

Our main results concern the behavior of the numerical approximations given by (2). Significant differences appear between the continuous and the discrete problem.

First we prove that positive solutions of the numerical problem blow up if and only if $p > 1$. Hence, the blow-up condition is the same that for the continuous problem.

Next, we turn our attention to the blow-up rate. For decreasing initial data we find that the blow-up rate for the numerical scheme is given by

$$\|U(t)\|_\infty \sim (T_h - t)^{-\frac{1}{p-1}} \qquad \text{if } p > 1.$$

Therefore, the blow-up rate does not coincide with the continuous one. In order to reproduce the correct blow-up rate we need to use a adaptive mesh refinement near of the boundary $x = 0$. Some references that use adaptive numerical methods are [1], [2] and [4].

Concerning to the blow-up time, we show the convergence of the discrete blow-up time T_h to the continuous one T, i.e.

$$T_h \to T \qquad \text{as } h \to 0\,.$$

Finally, for the blow-up set of the numerical approximations we prove that

$$B(U) = \begin{cases} \{x_1\} & p > 2, \\ \{x_1, \cdots, x_K\} & p \le 2, \end{cases}$$

where the constant $K \ge 2$ depends only on p. In fact is the integer that verifies

$$\frac{K+1}{K} < p \le \frac{K}{K-1}\,.$$

In particular, for $p > 2$ the only blow-up point is $x = 0$ and the blow-up set coincides with the continuous one. However, for $p \le 2$, the blow-up set is larger than a single point, $x = 0$. But, as $x_K \to 0$ as $h \to 0$, our result show that

$$B(U) \to B(u), \qquad \text{as } h \to 0.$$

Moreover, the asymptotic behavior of the blow-up nodes is given by

$$u_j \sim (T_h - t)^{j-1-\frac{1}{p-1}}, \qquad \text{if} \quad p \ne \frac{K}{K-1} \quad \text{or} \quad j \ne K\,,$$

and by

$$u_j \sim \ln(T_h - t), \qquad \text{if} \quad p = \frac{K}{K-1}\,.$$

This work is organized as follows: In Section 2 we describe our numerical approximation. In Section 3 we describe some properties of the numerical scheme and prove the convergence of the method. In Section 4 we prove the numerical blow-up results and find the numerical blow-up rates. In Section 5 we shows the convergence of the blow-up times to the continuous one. Section 6 is devoted to the numerical blow-up sets. In Section 7 we present some numerical experiments.

2 The method of lines

The basic idea of the method of lines is to replace the spatial (boundary-value) derivatives in the PDE with algebraic approximations. Once this is done, the spatial derivatives are no longer stated explicitly in terms of the spatial independent variables. Thus, only the time variable remains and we have a system of ODEs that approximate the original PDE.

In our case, we approximate the spacial derivatives by finite difference. To do that, we take a uniform partition $\{x_j\}$ of the interval $[0,1]$ of size h,

$$x_j = (j-1)\,h \qquad j = 1, \cdots, N \qquad \text{and} \qquad h = \frac{1}{N-1}\,,$$

and we use a truncated Taylor expansion for small h,

$$u\left(x_{j+1}, t\right) = u\left(x_j, t\right) + h u_x\left(x_j, t\right) + \frac{h^2}{2!} u_{xx}\left(x_j, t\right) + \frac{h^3}{3!} u_{xxx}\left(x_j, t\right) + \frac{h^4}{4!} u_{xxxx}\left(\eta_j, t\right)\,,$$

$$u\left(x_{j-1}, t\right) = u\left(x_j, t\right) - h u_x\left(x_j, t\right) + \frac{h^2}{2!} u_{xx}\left(x_j, t\right) - \frac{h^3}{3!} u_{xxx}\left(x_j, t\right) + \frac{h^4}{4!} u_{xxxx}\left(\eta_{j-1}, t\right)\,,$$

where $\eta_j \in (x_j, x_{j+1})$. Summing this two expressions we get the following approximation of $u_{xx}(x_j, t)$

$$u_{xx}\left(x_j, t\right) = \frac{1}{h^2}\Big(u(x_{j+1}, t) - 2u(x_j, t) + u(x_{j-1}, t)\Big) - \frac{h^2}{24}\left(u_{xxxx}\left(\eta_j, t\right) - u_{xxxx}\left(\eta_{j-1}, t\right)\right).$$

Observe that in the approximation of $u_{xx}(x_j, t)$ appears the nodes x_{j-1}, x_j, x_{j+1}, so this is not a valid approximation for the first and the last nodes.

Since u is a solution of the heat equation, we have

$$u_t\left(x_j, t\right) = \frac{1}{h^2}\Big(u(x_{j+1}, t) - 2u(x_j, t) + u(x_{j-1}, t)\Big) - \frac{h^2}{24}\left(u_{xxxx}\left(\eta_j, t\right) - u_{xxxx}\left(\eta_{j-1}, t\right)\right).$$

In order to obtain and approximating equation for the first node, we use the boundary condition.

$$\begin{aligned} u(x_2,t) = &\ u(x_1,t) + h u_x(x_1,t) + \frac{h^2}{2!} u_{xx}(x_1,t) + \frac{h^3}{3!} u_{xxx}(\eta_1,t) \\ = &\ u(x_1,t) - h u^p(x_1,t) + \frac{h^2}{2!} u_t(x_1,t) + \frac{h^3}{3!} u_{xxx}(\eta_1,t). \end{aligned}$$

Then,

$$u_t(x_1,t) = \frac{2}{h^2}(u(x_2,t) - u(x_1,t)) + \frac{2}{h} u^p(x_1,t) - \frac{h}{3!} u_{xxx}(\eta_1,t).$$

In a similar way we obtain the approximating equation for the last node,

$$u_t(x_N,t) = \frac{2}{h^2}(u(x_{N-1},t) - u(x_N,t)) + -\frac{h}{3!} u_{xxx}\left(\eta_{N-1},t\right).$$

Summing up, we get that a smooth solution of (1) satisfies

$$\begin{cases} u_t(x_1,t) = \dfrac{2}{h^2}(u(x_2,t) - u(x_1,t)) + \dfrac{2}{h} u^p(x_1,t) + O(h), \\ u_t(x_j,t) = \dfrac{1}{h^2}\Big(u(x_{j+1},t) - 2u(x_j,t) + u(x_{j-1},t)\Big) + O(h^2), \qquad j = 2, \cdots, N-1, \\ u_t(x_N,t) = \dfrac{2}{h^2}(u(x_{N-1},t) - u(x_N,t)) + O(h). \end{cases}$$

Finally, removing the truncation error terms, we obtain the following ode's system

$$\begin{cases} u_1'(t) = \dfrac{2}{h^2}(u_2(t) - u_1(t)) + \dfrac{2}{h} u_1^p(t), \\ u_j'(t) = \delta^2 u_j(t), \qquad j = 2, \cdots, N-1, \\ u_N'(t) = \dfrac{2}{h^2}(u_{N-1}(t) - u_N(t)). \end{cases} \tag{3}$$

where the operator δ^2 is defined by

$$\delta^2 u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}.$$

We remark that for a fixed h the right hand side of this system is a Lipstchiz continuous function. Then, for a bounded initial data, we can apply the Picar's Theorem to obtain local existence and uniqueness.

3 Convergence

In this section we collect some preliminary results for our numerical method. In particular we prove convergence for regular solutions.

First, we prove a comparison principle, for a more general problem

$$\begin{cases} v_1'(t) = \dfrac{2}{h^2}(v_2(t) - v_1(t)) + \dfrac{2}{h} m\,(t)\, f\,(v_1(t)) + K_1\, h & \\ v_j'(t) = \delta^2\,(v_j\,(t)) + K_j\, h^2 & j = 2, \ldots, N-1 \\ v_N'(t) = \dfrac{2}{h^2}(v_{N-1}\,(t) - v_N\,(t)) + K_N\, h & \\ v_j(0) = g(x_j) & j = 1, \ldots, N \end{cases} \tag{4}$$

where m is a positive function and the function f is increasing.

Definition 1 *We call W a supersolution (resp. a subsolution) if it satisfies (4) with upper (resp. lower) inequalities instead of equalities.*

Theorem 2 *Let V and W be a subsolution and a supersolution respectively such that $w_j(0) > v_j(0)$. Then, $W\,(t) > V\,(t)$, for every $t > 0$.*

Proof. Let β such that

$$0 < \beta < \min_{j=1,\cdots,N} (w_j\,(0) - v_j(0))$$

Let us define the function $E := W - V$. Which satisfies

$$\begin{cases} e_1' \geq \frac{2}{h^2}(e_2 - e_1) + \frac{2}{h}m\,(t)\,(f\,(w_1) - f\,(v_1)) & \\ e_j' \geq \delta^2\,(e_j) & j = 2, \dots, N-1 \\ e_N' \geq \frac{2}{h^2}(e_{N-1} - e_N) & \\ e_j(0) > \beta & j = 1, \dots, N \end{cases}$$

We argue by contradiction, assume that there exists a first time $t_0 > 0$ where the vector E reaches the level β. Let us define

$$j_0 = \min\{j = 1, \dots, N \,:\, u_j(t_0) = \beta\}.$$

Then, at time t_0, we have

$$e_j(t_0) > \beta \quad j = 1, \dots, j_0$$

$$e_j(t_0) \geq \beta \quad j = j_0, \dots, N$$

$$e_{j_0}'\,(t_0) \leq 0$$

$$m(t_0) > 0$$

$$f(w_1(t_0)) - f(v_1(t_0)) > 0$$

From this inequalities and the equation satisfies by e_{j_0} we arrive a contradiction. Indeed,

i) If $j_0 = 1$,

$$0 \geq e_1'\,(t_0) \geq \frac{2}{h^2}(e_2\,(t_0) - \beta) + \frac{2}{h}m\,(t)\,(f\,(w_1) - f\,(v_1)) > 0,$$

ii) If $2 \leq j_0 \leq N-1$,

$$0 \geq e'_{j_0}(t_0) \geq \delta^2 e_{j_0}(t_0) = \frac{e_{j_0+1}(t_0) - 2\beta + e_{j_0-1}(t_0)}{h^2} > 0.$$

iii) If $j = N$,

$$0 \geq e'_N(t_0) > \frac{2}{h^2}(e_{N-1}(t_0) - \beta) > 0.$$

Thus, we have that $E > \beta$ and the result follows. ■

Now, we are able to show our convergence result for regular solutions.

Theorem 3 *Let $u \in C^4([0,1] \times [0, T-\tau))$ be a positive solution of (1) and u_h the numerical approximation given by (3). Then, there exists a constant D such that for every h small enough the following estimate holds*

$$\max_{j=1,\dots,N} |u(x_j, t) - u_j(t)| \leqslant D \left(\|\psi(x_j) - u_j(0)\|_{L^\infty} + h^2 \right)$$

for all time $t \in [0, T-\tau]$.

Proof. We define the error function as

$$e_j(t) = u(x_j, t) - u_j(t).$$

Let c be a constant such that $u \leq c$ for every $t \in [0, T-\tau]$ and

$$\widetilde{t} = \max\{t \in [0, T-\tau] \text{ such that } |e_j| \leq c/2\} \tag{5}$$

so as to ensure that, up to time $\widetilde{t}$, none of the solutions, neither the approximation nor the continuous solution, blow up. The following estimates will be performed restricting ourselves to $t \in [0, \widetilde{t}]$. Afterwards, we will show that $\widetilde{t} = T - \tau$.

Since $u \in C^4([0,1] \times [0, T-\tau))$ there exists positive constants

$$\frac{\|u_{xxxx}\|_{L^\infty}}{12} + 1 \leqslant K_1, \qquad \frac{\|u_{xxx}\|_{L^\infty}}{6} + 1 \leqslant K_2, \qquad p\left(\|u\|_{L^\infty} + 1\right)^{p-1} \leqslant K_3,$$

where $\|\cdot\|_{L^\infty}$ denote the norm in $L^\infty\left([0,1] \times [0, T-\tau]\right)$. With this notation the error function satisfies

$$\begin{cases} e_1'(t) & < & \frac{2}{h^2}\left[e_2(t) - e_1(t)\right] + \frac{2}{h}\left(u_1^p(t) - u^p(x_1, t)\right) + K_2 h \\ e_j'(t) & < & \frac{1}{h^2}\left(e_{j+1}(t) - 2e_j(t) + e_{j-1}(t)\right) + K_1 h^2 \qquad j = 2, \cdots, N-1 \\ e_N'(t) & < & \frac{2}{h^2}\left(e_{N-1}(t) - e_N(t)\right) + K_2 h \\ e_j(0) & = & \psi(x_j) - u_j(0) \qquad j = 1, \cdots, N \end{cases}$$

Applying the Mean Value Theorem to the function $g(s) = s^p$ in the first equation we obtain

$$e_1'(t) - \frac{2}{h^2}\left[e_2(t) - e_1(t)\right] < \frac{2K_3}{h}\, e_1(t) + K_2 h.$$

Then, the error function is a subsolution of the following system

$$\begin{cases} w_1'(t) & = & \frac{2}{h^2}(w_2 - w_1) + \frac{2K_3}{h} w_1 + K_2 h \\ w_j'(t) & = & \delta^2 w_j(t) + K_1 h^2 \qquad j = 2, \cdots, N-1 \\ w_N'(t) & = & \frac{2}{h^2}(w_{N-1} - w_N) + K_2 h \\ w_j(0) & = & \psi(x_j) - u_j(0) + \frac{h^2}{2} \qquad j = 1, \cdots, N \end{cases} \tag{6}$$

Let us look for a supersolution of the form

$$V_j(t) := A(h)\varphi(t)\gamma(x_j) \qquad j = 1, \cdots, N,$$

where

$$A(h) = (\|\psi(x_j) - u_j(0)\|_{L^\infty([0,1])} + h^2), \quad \varphi(t) = e^{2K_1 t}, \quad \gamma(x) = e^{-C_1(x-1)} + C_2 x.$$

- For the first equation, we get

$$V_1'(t) = 2K_1 e^{K_1 t} A(h)\gamma(0) > 0$$

and

$$\frac{2}{h^2}(V_2 - V_1) + \frac{2K_3}{h}V_1 + K_2 h = \frac{2}{h}A(h)\varphi(t)\left(\frac{\gamma(h)-\gamma(0)}{h} + K_3\gamma(0)\right) + K_2 h.$$

Using the Taylor expansion for the function γ, we obtain that

$$\frac{2}{h^2}(V_2 - V_1) + \frac{2K_3}{h}V_1 + K_2 h = \left(2\frac{A(h)}{h^2}\left(\gamma'(0) + \gamma''(\xi)\frac{h}{2} + K_3\gamma(0)\right) + K_2\right)h,$$

where $\xi \in (0, h)$. Now, we impose that

$$2\gamma'(0) + \gamma''(\xi)h + 2K_3\gamma(0) + K_2 < 0, \tag{7}$$

then using the fact $A(h)/h^2 > 1$, we obtain that

$$\frac{2}{h^2}(V_2 - V_1) + \frac{2K_3}{h}V_1 + K_2 h < 0 < V_1'(t).$$

- For the interior nodes. Using, again, the taylor expansion we get that

$$\delta^2 V_j + K_1 h^2 < \left(\frac{1}{12}A(h)\varphi(t)\|\gamma_{xxxx}\|_{L^\infty([0,1])} + K_1\right)h^2.$$

Since $A(h) \to 0$ as $h \to 0$, we get that for h small enough

$$\delta^2 V_j + K_1 h^2 < 2K_1 h^2.$$

On the other hand, $A(h) > h^2$, $\varphi \geq 1$ and $\gamma(x_j) \geq 1$, then

$$V_j'(t) = 2K_1 A(h)\varphi(t)\gamma(x_j) > 2K_1 h^2.$$

So,

$$V_j'(t) > 2K_1h^2 > \delta^2 V_j + K_1h^2.$$

- For the last node.

$$V_N'(t) = 2K_1A(h)\varphi(t)\gamma(1) > 0$$

and

$$\begin{aligned}\tfrac{2}{h^2}(V_{N-1} - V_N) + K_2h = & \left(2\varphi(t)\,\tfrac{A(h)}{h^2}\,\tfrac{\gamma(1-h)-\gamma(1)}{h} + K_2\right)h \\ = & \left(2\varphi(t)\,\tfrac{A(h)}{h^2}\,(-\gamma'(1) + \gamma''(\xi)\tfrac{h}{2}) + K_2\right)h\end{aligned}$$

where $\xi \in (1-h, h)$. Now, we impose that

$$-\gamma'(1) + \gamma''(\xi)\frac{h}{2} + K_2 < 0, \tag{8}$$

then using the fact $A(h)/h^2 > 1$, we obtain that

$$\frac{2}{h^2}(V_2 - V_1) + \frac{2K_3}{h}V_1 + K_2h < 0 < V_1'(t).$$

- For the initial data.

$$V_j(0) = \gamma(x_j)A(h) > A(h) > \psi(x_j) - u_j(0) + h^2 > w_j(0).$$

- Finally, we verify the conditions (7) and (8). We start by (8)

$$-\gamma'(1) + \gamma''(\xi)\frac{h}{2} + K_2 = C_1 - C_2 + \|\gamma''\|_{L^\infty([0,1])}\frac{h}{2} + K_2.$$

Then taking $C_2 = 2(C_1 + K_2)$ condition (8) holds for h small enough. For (7) we have that

$$\begin{aligned}2\gamma'(0) + \gamma''(\xi)h + K_3\gamma(0) + 2K_2 \leq & -2C_1e^{C_1} + 2C_2 + \|\gamma''\|_{L^\infty([0,1])}h + 2K_3e^{C_1} + K_2 \\ = & \ 2(K_3 - C_1)e^{C_1} + 2C_1 + 5K_2 + \|\gamma''\|_{L^\infty([0,1])}h.\end{aligned}$$

Notice that the function $G(s) = 2(K_3 - s)e^s + 2s + 5K_2$ goes to $-\infty$ as $s \to \infty$. Then, taking C_1 large we obtain that $G(C_1) < 0$. Therefore, for h small condition (7) also holds.

Therefore, V_j is a supersolution of system (6). Hence, applying the comparison principle (see 2), we get

$$e_j(t) \le w_j(t) \le V_j(t) \le e^{C_1} e^{K_1 T} A(h).$$

Arguing in the same way with $-e_j$ we obtain

$$|e_j(t)| \le w_j(t) \le V_j(t) \le e^{C_1} e^{2K_1 T} A(h).$$

Since $A(h) \to 0$ as $h \to 0$ we can take h small enough such that,

$$e^{C_1} e^{2K_1 T} A(h) \le c/2, \qquad t \in [0, T - \tau].$$

Hence $\widetilde{t} = T - \tau$ and the theorem is proved. ■

Remark 4 *In order to observe only the error of the method, we consider initial data for problem (3) which satisfies*

$$\max_{1 \le j \le N} |\psi(x_j) - u_j(0)| = 0(h^\gamma) \qquad \gamma > 2\,.$$

4 Blow-up versus global solution. Blow-up rate

This chapter shows the existence of solutions that exploit, the rate with which they do (which coincides with the ongoing problem) and the convergence of blow-up time of the discrete problem to the continuous when $h \to 0$.

Lemma 5 *Let a, b, p be three positive constants. If $x : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\begin{cases} ax^p(t) \leq x'(t) \leq bx^p(t) & t > t_0 \\ x(t_0) = x_0 > 0 \end{cases}$$

Then, for $p \leq 1$ the function x is global in time, while for $p > 1$ the function x blows up at a finite time T. Moreover, in the blow-up case there exists two positive constants such that

$$C_1(T-t)^{\frac{-1}{p-1}} \leq x(t) \leq C_2(T-t)^{\frac{-1}{p-1}}.$$

Proof. Take the case $p > 1$. Integrating the inequality, as $x'(t) \geqslant ax^p(t)$, we have

$$\int_{t_0}^{t} \frac{x'(t)}{x^p(t)} ds \geqslant a \int_{t_0}^{t} ds$$

changing variables

$$\int_{x_0}^{x(t)} \frac{1}{s^p} ds \geqslant a(t - t_0)$$

$$\implies \quad \tfrac{(x(t))^{1-p}}{1-p} \geqslant a(t-t_0) + \tfrac{x_0^{1-p}}{1-p}$$

$$\implies \quad (x(t))^{1-p} \leqslant x_0^{1-p} - a(p-1)(t-t_0)$$

If $x(t)$ be global, then it is always positive, but as $a > 0$, then for t sufficiently large the right member is negative, a contradiction. Therefore, $x(t)$ can not exist beyond time $t_0 + \frac{x_0^{1-p}}{(p-1)a}$, which explodes in finite time, T. To estimate the rate, we integrate the inequality $x'(t) \geqslant ax^p(t)$ between t and T the time of explosion:

$$\int_t^T \frac{x'(t)}{x^p(t)} ds \geqslant a \int_t^T ds$$

changing variables

$$\int_{x(t)}^{+\infty} \frac{1}{s^p} ds \geqslant a(T-t)$$

$$\implies \frac{(x(t))^{1-p}}{p-1} \geqslant a(T-t)$$

$$\implies x(t) \leqslant C(T-t)^{-\frac{1}{p-1}}$$

thus $x(t)$ blows up at most with the rate indicated. Now, consider the case $p < 1$. Proceeding as before, we have

$$\frac{(x(t))^{1-p}}{1-p} \leqslant a(t-t_0) + \frac{x_0^{1-p}}{1-p}$$

$$\implies (x(t))^{1-p} \leqslant x_0^{1-p} + a(1-p)(t-t_0)$$

$$\implies x(t) \leqslant \left[x_0^{1-p} + a(1-p)(t-t_0)\right]^{\frac{1}{1-p}}$$

and $x(t)$ is bounded by $\left[x_0^{1-p} + a(1-p)(t-t_0)\right]^{\frac{1}{1-p}}$, that does not blows up. If $p = 1$, then

$$\log(x(t)) \leqslant a(t-t_0) + \log x_0$$

$$\Longrightarrow \quad \log(x(t)) \leqslant \log e^{a(t-t_0)} + \log x_0$$

$$\Longrightarrow \quad \log(x(t)) \leqslant \log\left(x_0 * e^{a(t-t_0)}\right)$$

$$\Longrightarrow \quad x(t) \leqslant x_0 * e^{a(t-t_0)}$$

and as before, $x_0 * e^{a(t-t_0)}$ does not blows up. Thus we see that $x(t)$ is dominated by global functions. ■

Lemma 6 *Let U be a solution of (3) with $u_j(0) > u_{j+1}(0)$, $j = 1, \cdots, N$. Then*

$$u_j(t) > u_{j+1}(t), \quad j = 1, \cdots, N$$

Proof. We argue by contradiction, assume that there exists a first time $t_0 > 0$ and a first node j_0 where $u_{j_0}(t_0) = u_{j_0+1}(t_0)$.

Let us define the function

$$e_j(t) := u_j(t) - u_{j+1}(t) \qquad j = 1, \cdots, N-1.$$

Which satisfies,

$$\begin{cases} e_1' = \dfrac{1}{h^2} e_2 + \dfrac{2}{h} u_1^p \\ e_j' = \delta^2 e_j & j = 2, \cdots, N-2 \\ e_{N-1}' = \dfrac{1}{h^2}(e_{N-2} - 3e_{N-1}) \end{cases}$$

Notice that $e_j(0) > 0$ for all $j = 1, \ldots, N-1$, then at time t_0 we have that

$$e_j(t_0) \geq 0 \qquad j = 1, \cdots, N-1.$$

and at node j_0,

$$e_{j_0}(t_0) = 0 \qquad e'_{j_0}(t_0) \leq 0.$$

From this inequality and the equation satisfies by e_{j_0} we arrive a contradiction. Indeed,

i) If $j_0 = 1$,

$$0 \geq e'_1(t_0) = \frac{1}{h^2}\, e_2(t_0) + \frac{2}{h} u_1^p(t_0) \geq \frac{2}{h} u_1^p(t_0) > 0.$$

ii) If $2 \leq j_0 \leq N-2$,

$$0 \geq e'_{j_0}(t_0) = \frac{1}{h^2}(e_{j_0+1}(t_0) + e_{j_0-1}(t_0)) \geq 0.$$

Then, $e_{j_0-1}(t_0) = 0$, and

$$0 \geq e'_{j_0-1}(t_0) = \frac{1}{h^2}(e_{j_0}(t_0) + e_{j_0-2}(t_0)) \geq 0.$$

So, $e_{j_0-2}(t_0) = 0$. Iterating this procedure, we get $e_j(t_0) = 0$ for all $j \leq j_0$. In particular, $e_1(t_0) = 0$ which is a contradiction by i).

iii) Finally, if $j_0 = N-1$, we get

$$0 \geq e'_{N-1} = \frac{1}{h^2} e_{N-2} \geq 0.$$

Which implies that $e_{N-2} = 0$ and by ii) we get a contradiction. ■

Theorem 7 *A solution of (3) blows up if and only if $p > 1$. Moreover, if $p > 1$ and $U(0)$ is decreasing, then there exists two constants, $C_1 = C_1(h)$ and $C_2 = C_2(h)$, such that*

$$C_1(T-t)^{\frac{-1}{p-1}} \leq \|U(t)\|_\infty \leq C_2(T-t)^{\frac{-1}{p-1}}.$$

Proof. First we consider a decreasing initial data $U(0)$. By Lemma 6, $U(t)$ is decreasing and then its maximum will be $u_i(t)$.

We define the function

$$w(t) = \frac{1}{2}u_1(t) + \sum_{j=2}^{N-1} u_j(t) + \frac{1}{2}u_N(t)$$

which satisfies

$$\begin{aligned}
w'(t) &= \tfrac{1}{2}u_1'(t) + \textstyle\sum_{j=2}^{N-1} u_j'(t) + \tfrac{1}{2}u_N'(t) \\
&= \tfrac{1}{2}\left[\tfrac{2}{h^2}(u_2 - u_1) + \tfrac{2}{h}u_1^p(t)\right] + \textstyle\sum_{j=2}^{N-1} \delta^2 u_j(t) + \tfrac{1}{2}\tfrac{2}{h^2}(u_{N-1} - u_N) \\
&= \tfrac{1}{h^2}(u_2 - u_1) + \tfrac{1}{h}u_1^p(t) + \tfrac{1}{h^2}\textstyle\sum_{j=2}^{N-1}(u_{j-1} - 2u_j + u_{j+1}) + \tfrac{1}{h^2}(u_{N-1} - u_N) \\
&= \tfrac{1}{h^2}(u_2 - u_1) + \tfrac{1}{h}u_1^p(t) + \tfrac{1}{h^2}(u_1 - 2u_2 + u_3) + (u_2 - 2u_3 + u_4) + \cdots \\
&\quad + (u_{N-2} - 2u_{N-1} + u_N) + \tfrac{1}{h^2}(u_{N-1} - u_N) \\
&= \tfrac{1}{h}u_1^p(t)\ .
\end{aligned}$$

On the other hand, since $U(t)$ is decreasing vector, we have

$$\frac{1}{2}u_1(t) \leqslant w(t) \leqslant \frac{1}{2}u_1(t) + \sum_{j=2}^{N-1} u_1(t) + \frac{1}{2}u_1(t) = (N-1)\,u_1(t) = \frac{1}{h}u_1(t)\ .$$

Therefore,

$$\frac{1}{2^p}u_1^p(t) \leqslant w^p(t) \leqslant \frac{1}{h^p}u_1^p(t)$$

and the function w satisfies

$$h^{p-1}w^p(t) \leqslant w'(t) \leqslant \frac{2^p}{h}w^p(t) \tag{9}$$

Then Lemma 5 gives us the desired result for decreasing solution.

Finally, we observe that this blow-up result is valid for every initial data as we can use a comparison argument with an increasing supersolution or subsolution. ■

Remark 8 *Notice that integrating the lower estimate in (9), for $p \leq 1$ we get that*

$$w(t) \to \infty \qquad as \quad t \to \infty\,.$$

Then the numerical solution grows up for $p \leq 1$, i.e.,

$$\|U\|_\infty \to \infty \qquad as \quad t \to \infty\,.$$

5 Blow-up time

In this section we study the convergence of the numerical blow-up time to the continuous one. Inspired in [3] we take convex initial data, that is, we impose the hypothesis

$$\psi''(x) \geq \mu > 0\,. \tag{10}$$

Using Taylor expansion in the nodes of the mesh, it is easy to see that

$$\begin{cases} \frac{2}{h^2}(\psi(x_2)-\psi(x_1))+\frac{2}{h}\psi^p(x_1)=\psi''(x_1)+O(h) \\ \\ \delta^2\psi(x_j)(0)=\psi''(x_j)+O(h^2) \qquad j=2,\dots,N-1 \\ \\ \frac{2}{h^2}(\psi(x_{N-1})-\psi(x_N)=\psi''(x_N)+O(h) \end{cases}$$

On the other hand, if we consider a nice approximation of the initial data (nice in the senses of Remark 4) we get that for all $\beta < \mu$ and h small enough

$$\begin{cases} \frac{2}{h^2}(u_2(0)-u_1(0))+\frac{2}{h}(u_1(0))^p \geqslant \beta > 0 \\ \\ \delta^2\left(u_j(0)\right) \geqslant \beta > 0 \qquad j=2,\dots,N-1 \\ \\ \frac{2}{h^2}(u_{N-1}(0)-u_N(0)) \geqslant \beta > 0 \end{cases} \tag{11}$$

This estimates allow us to obtain and upper blow-rate estimate independent of h.

Lemma 9 *Let U be a solution of (3). If the initial data satisfies (11) with β independent of h, then exists $C>0$, independent of h, such that*

$$u_j' \leq C u_j^p\,. \tag{12}$$

Proof. Notice that as $p>1$ the function $f(x)=x^p$ is convex. So, it satisfies

$$x^p - y^p \geq p\, y^{p-1}\,(x-y)\,. \tag{13}$$

Now, we look for the equations satisfies for vector

$$e_j := u_j' - C u_j^p\,.$$

For the first node,

$$\begin{aligned}
e_1' - \tfrac{2}{h^2}\,(e_2-e_1) &= \left(u_1'' - Cpu_1^{p-1}u_1'\right) - \tfrac{2}{h^2}\left[(u_2'-Cu_2^p)-(u_1'-Cu_1^p)\right] \\
&= \left(u_1'' - Cpu_1^{p-1}u_1'\right) - \tfrac{2}{h^2}\left[u_2'-u_1'-Cu_2^p+Cu_1^p\right] \\
&= u_1'' - Cpu_1^{p-1}u_1' - \tfrac{2}{h^2}\left[u_2'-u_1'\right] - \tfrac{2C}{h^2}\left[u_1^p-u_2^p\right] \\
&= u_1'' - Cpu_1^{p-1}\left[\tfrac{2}{h^2}\,(u_2-u_1)+\tfrac{2}{h}u_1^p\right] - \tfrac{2}{h^2}\left[u_2'-u_1'\right] \\
&\quad -\tfrac{2C}{h^2}\left[u_1^p-u_2^p\right] \\
&= \left[u_1'' - \tfrac{2}{h^2}\,(u_2'-u_1')\right] + \tfrac{2C}{h^2}\left[(u_2^p-u_1^p) - pu_1^{p-1}\,(u_2-u_1)\right] \\
&\quad -\tfrac{2}{h}Cpu_1^{p-1}\,(u_1^p) \\
&= \tfrac{2}{h}pu_1^{p-1}\left(u_1'\right) - \tfrac{2}{h}pu_1^{p-1}\,(Cu_1^p) \\
&\quad +\tfrac{2C}{h^2}\left[(u_2^p-u_1^p) - pu_1^{p-1}\,(u_2-u_1)\right] \\
&= \tfrac{2}{h}pu_1^{p-1}e_1 + \tfrac{2C}{h^2}\left[(u_2^p-u_1^p) - pu_1^{p-1}\,(u_2-u_1)\right] \\
&\geqslant \tfrac{2}{h}pu_1^{p-1}e_1\,.
\end{aligned}$$

In middle nodes:

$$
\begin{aligned}
e_j' - \delta^2 e_j &= \left(u_j'' - Cpu_j^{p-1}u_j'\right) - \tfrac{1}{h^2}\left(u_{j+1}' - 2u_j' + u_{j-1}'\right) \\
&\quad + \tfrac{C}{h^2}\left(u_{j+1}^p - 2u_j^p + u_{j-1}^p\right) \\
&= -\tfrac{C}{h^2}\left[pu_j^{p-1}\left(u_{j+1} - 2u_j + u_{j-1}\right) - \left(u_{j+1}^p - 2u_j^p + u_{j-1}^p\right)\right] \\
&= -\tfrac{C}{h^2}\left\{\left[pu_j^{p-1}\left(u_{j+1} - u_j\right) - \left(u_{j+1}^p - u_j^p\right)\right]\right. \\
&\quad \left. + \left[pu_j^{p-1}\left(u_{j-1} - u_j\right) - \left(u_{j-1}^p - u_j^p\right)\right]\right\} \\
&\geqslant 0\,.
\end{aligned}
$$

Finally, in last node:

$$
\begin{aligned}
e'_N - \tfrac{2}{h^2}\left(e_{N-1} - e_N\right) &= \left(u''_N - Cpu_N^{p-1}u'_N\right) - \tfrac{2}{h^2}\left(u'_{N-1} - Cu^p_{N-1}\right) \\
&\quad + \tfrac{2}{h^2}\left(u'_N - Cu^p_N\right) \\
&= \left(u''_N - Cpu_N^{p-1}u'_N\right) \\
&\quad - \tfrac{2}{h^2}\left[u'_{N-1} - u'_N - Cu^p_{N-1} + Cu^p_N\right] \\
&= u''_N - Cpu_N^{p-1}u'_N - \tfrac{2}{h^2}\left[u'_{N-1} - u'_N\right] \\
&\quad - \tfrac{2C}{h^2}\left[u^p_N - u^p_{N-1}\right] \\
&= -\tfrac{2C}{h^2}\left[pu_N^{p-1}\left(u_{N-1} - u_N\right) + \left(u^p_N - u^p_{N-1}\right)\right] \\
&= -\tfrac{2C}{h^2}\left[pu_N^{p-1}\left(u_{N-1} - u_N\right) - \left(u^p_{N-1} - u^p_N\right)\right] \\
&\geqslant 0\,.
\end{aligned}
$$

Moreover, by the hypothesis of the initial data we have that for C small enough

$$e_i(0) \geq \beta - Cu_i^p \geq \frac{\beta}{2}\,.$$

Then, E is a supersolution of the problem

$$\begin{cases} w_1' = \frac{2}{h^2}(w_2 - w_1) + \frac{2}{h} p u_1^{p-1}(t)\, w_1 & \\ w_j' = \delta^2 w_j & j = 2, \cdots, N-1 \\ w_N' = \frac{2}{h^2}(w_{N-1} - w_N) & \\ w_j(0) = 0 & j = 1, \cdots, N \end{cases}$$

Which the unique solution is given by $W = 0$. So, applying the comparison principle we get

$$e_j(t) = u_j'(t) - C u_j^p > 0 \qquad j = 1, \ldots, N\,.$$

■

Notice that, from (12) and Lemma 5 we obtain, in another way, that the solution of the problem (3) blows-up in finite time for $p > 1$. Moreover, integrating (12), we get

$$(T_h - t) \leq \frac{1}{C} \int_{\|U(t)\|_\infty}^{\infty} \frac{1}{s^p}\, ds = \frac{\|U(t)\|^{1-p}}{K}\,. \tag{14}$$

The independence of h in the upper estimate gives us the key to proof the convergence of the blow-up times.

Theorem 10 *Let u be a blowing up regular solution of problem (1), with an strictly convex initial data ($\psi'' \geq \mu > 0$) and let U be the approximation given by (3) with initial data $U(0)$, which satisfies*

$$\max_{1 \leq j \leq N} |\psi(x_j) - u_j(0)| = 0(h^\gamma) \qquad \gamma > 2\,.$$

If T and T_h be the blow up times of u and U respectively, then

$$T_h \to T \qquad as \qquad h \to 0\,.$$

Proof. Given $\varepsilon > 0$ we choose M large enough such that

$$\frac{M^{1-p}}{K} \le \frac{\varepsilon}{2},$$

where the constant K is given in (14).

Since u blows up a time T, we can choose $\tau < \varepsilon/2$ such that

$$\|u(\cdot, T-\tau)\|_{L^\infty([0,1])} \ge 2M .$$

On the other hand, by the convergence result (Theorem 3), we have that for h small enough,

$$\max_{j=1,\dots,N} |u(x_j, T-\tau) - u_j(T-\tau)| \leqslant D\left(\|\psi(x_j) - u_j(0)\|_{L^\infty} + h^2\right) < M .$$

Hence,

$$\|U(T-\tau)\|_\infty \ge M .$$

Now, thanks to the upper estimate (14), we get

$$|T_h - (T-\tau)| \le \frac{1}{K}\|U(T-\tau)\|^{1-p} \le \frac{1}{K}M^{1-p} \le \frac{\varepsilon}{2} .$$

Finally,

$$|T_h - T| \le |T_h - (T-\tau)| + |\tau| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

■

6 Blow-up set

In this section we study the blow-up set, which is define as

$$B(U) = \{x_i \,:\, u_i \to \infty \text{ as } t \to T_h\}.$$

Since we consider positive and decreasing solutions, we know that the first node is in the blow-up set. Moreover the behavior near the blow-up time of this node is given in Section 4,

$$u_1 \backsim (T - t)^{\frac{-1}{p-1}}$$

The question is: Also explodes the next node, the second? Let's see.

For the equation of the second node, we have

$$u_2' \left(t\right) = \frac{1}{h^2} \left(u_3 \left(t\right) - 2u_2 \left(t\right) + u_1 \left(t\right)\right) \leqslant \frac{2}{h^2} u_1 \left(t\right) \,.$$

Then, by integration between $\tau < t < T_h$ we get

$$u_2(t) - u_2(\tau) \leq \frac{2}{h^2} \int_\tau^t u_1(s)\, ds\,.$$

Now, using the behavior of u_1, we obtain

$$u_2(t) - u_2(\tau) \leq \frac{2}{h^2} \int_\tau^t (T_h - s)^{-1/(p-1)}\, ds\,.$$

In the case $p > 2$ we can choose $t = T_h$ to obtain

$$u_2 \left(T_h\right) - u_2 \left(\tau\right) \leqslant \frac{2}{h^2} \frac{p-1}{p-2} \left(T_h - \tau\right)^{\frac{p-2}{p-1}} < C\,.$$

Thus, u_2 remains bounded and $x_2 \notin B(U)$.

For $p = 2$:

$$u_2(t) - u_2(\tau) \leqslant \frac{2}{h^2}\left[\ln(T_h - t) - \ln(T_h - \tau)\right]$$

and for $p < 2$:

$$u_2(t) - u_2(\tau) \leqslant \frac{2}{h^2}\frac{p-1}{2-p}\left[(T_h - t)^{\frac{p-2}{p-1}} - (T_h - \tau)^{\frac{p-2}{p-1}}\right]$$

In order to obtain a lower bound, we observe from the equation of the second node,

$$u_2'(s) = \frac{1}{h^2}(u_1(t) - 2u_2(t) + u_3(t)) \geqslant \frac{1}{h^2}(u_1(t) - 2u_2(t)),$$

because the solutions are positive. And, multiplying both sides by the integrating factor $e^{\frac{2}{h^2}t}$, we have

$$\left(e^{\frac{2}{h^2}t}u_2(t)\right)' = e^{\frac{2}{h^2}t}\left[u_2'(s) + \frac{2}{h^2}u_2(t)\right] \geqslant \frac{e^{\frac{2}{h^2}t}}{h^2}(u_1(t))$$

Integrating this inequality between $\tau < t < T_h$

$$u_2(t) \geq \frac{1}{h^2}\int_\tau^t e^{\frac{2}{h^2}(s-t)}u_1(s)\,ds + e^{\frac{2}{h^2}(\tau-t)}u_2(\tau).$$

Since the time lives in a bounded interval, the exponential in the above inequality are uniformly bounded from bellow. So, there exist positive constant such that

$$u_2(t) \geq C_1\int_\tau^t u_1(s)\,ds + C_2u_2(\tau).$$

Using now the behavior of u_1, we get that for $p = 2$

$$u_2(t) \geq C_1\left[\ln(T_h - t) - \ln(T_h - \tau)\right] + C_2u_2(\tau)$$

and for $p < 2$:

$$u_2(t) \geq C_1\frac{p-1}{2-p}\left[(T_h - t)^{\frac{p-2}{p-1}} - (T_t - \tau)^{\frac{p-2}{p-1}}\right]$$

Then,

$$x_2 \in B(U), \qquad \text{for } p \leq 2\,.$$

Summing up, we get that the behavior of u_2 near the blow-up time is given by

$$u_2\,(t) \sim \begin{cases} 1 & \text{if } p > 2 \\ \\ \ln\,(T_h - t) & \text{if } p = 2 \\ \\ (T_h - t)^{1-\frac{1}{p-1}} & \text{if } p < 2 \end{cases}$$

Now, let's look at the third node. Following the same way with the equation of the third node we get that there exists positive constant such that

$$C_1 + C_2 \int u_2(t)\,dt \leq u_3(t) \leq C_3 + C_4 \int u_2(t)\,dt$$

Therefore, using the behavior of u_2 near the blow-up time we get

$$u_3\,(t) \backsim \begin{cases} 1 & \text{if } p > \frac{3}{2} \text{ (remains bounded)} \\ \\ \ln\,(T - \tau) & \text{if } p = \frac{3}{2} \text{ (blows up)} \\ \\ (T_h - \tau)^{2-\frac{1}{p-1}} & \text{if } p < \frac{3}{2} \text{ (blows up)} \end{cases}$$

Then, we know the asymptotic behavior of u_3, in particular

$$x_3 \in B(U), \qquad \text{for } p \geq \frac{3}{2}\,.$$

Notice that all the interior nodes have the same equation. So, applying the same argument given above, if we assume that the first $j-1$ nodes blow up, then for the node j we have that there exists positives constant such that

$$C_1 + C_2 \int u_{j-1}(t)\,dt \leq u_j(t) \leq C_3 + C_4 \int u_{j-1}(t)\,dt$$

But,

$$u_{j-1} \sim (T - h - t)^{j-2-\frac{1}{p-1}},$$

then

$$u_j\left(t\right) \backsim \begin{cases} 1 & \text{if } p > \frac{j}{j-1} \text{ (remains bounded)} \\ \\ \ln\left(T_h - \tau\right) & \text{if } p = \frac{j}{j-1} \text{ (blows up)} \\ \\ \left(T_h - \tau\right)^{j-1-\frac{1}{p-1}} & \text{if } p < \frac{j}{j-1} \text{ (blows up)} \end{cases}$$

Thus, finally found a general rule for the propagation of the blow-up. For $j \in \{2, 3, ..., N\}$, $u_j\left(t\right)$ blows up if and only if:

$$1 < p \leq \frac{j}{j-1}.$$

7 Numerical experiments

In this Section we present some numerical experiments. Our goal is to show that the results presented in the previous sections can be observed when one perform numerical computations. For the numerical experiments we use an adaptive ODE solver provided by MATLAB (ode23s, which solve stiff differential equations).

First we consider a monotone decreasing initial data, which satisfies the boundary condition,

$$\phi(x) = \frac{1}{2}(x-1)^2 + 1\,.$$

We start with the case $p \le 1$, in Figure 1 and Figure 2 we represent the evolution of the solution. Global existence can be appreciate.

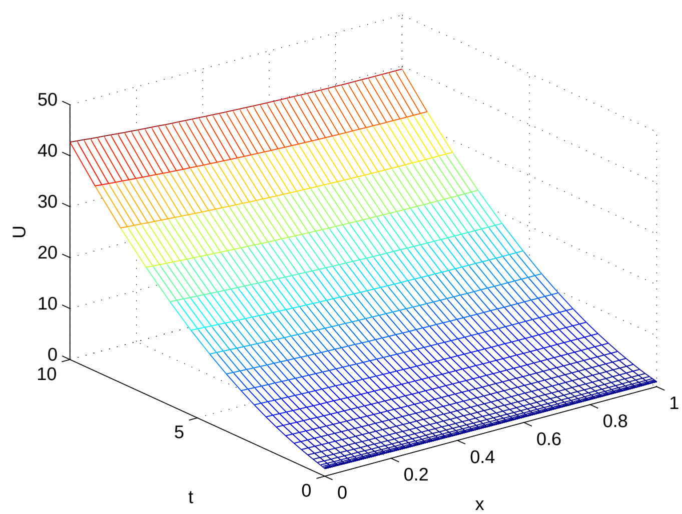

Figure 1. Evolution of the solution with $p = \frac{1}{2}$.

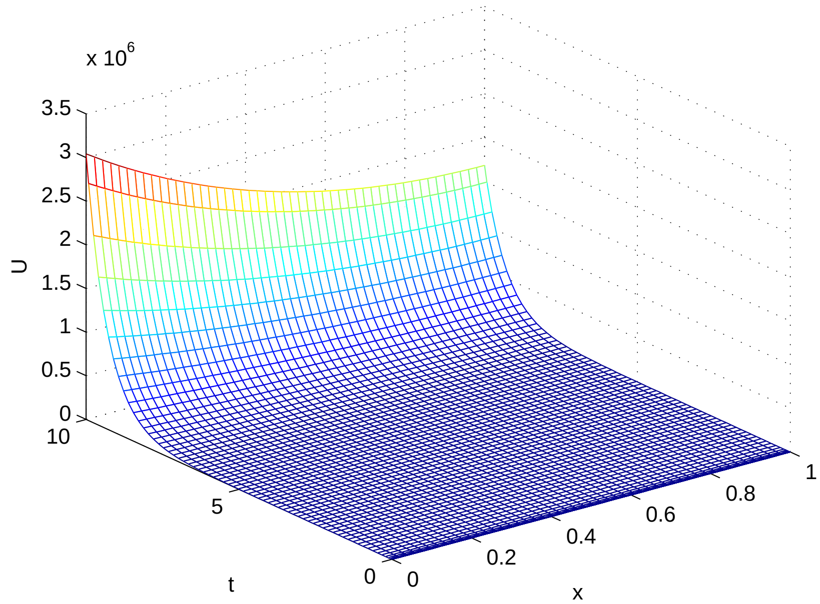

Figure 2. Evolution of the solution with $p = 1$.

In both cases, we can appreciate that the solution grows up, see Remark 8.

We can observe in Figure 3 that for $p = 3$ the solution blows-up near the boundary $x = 0$.

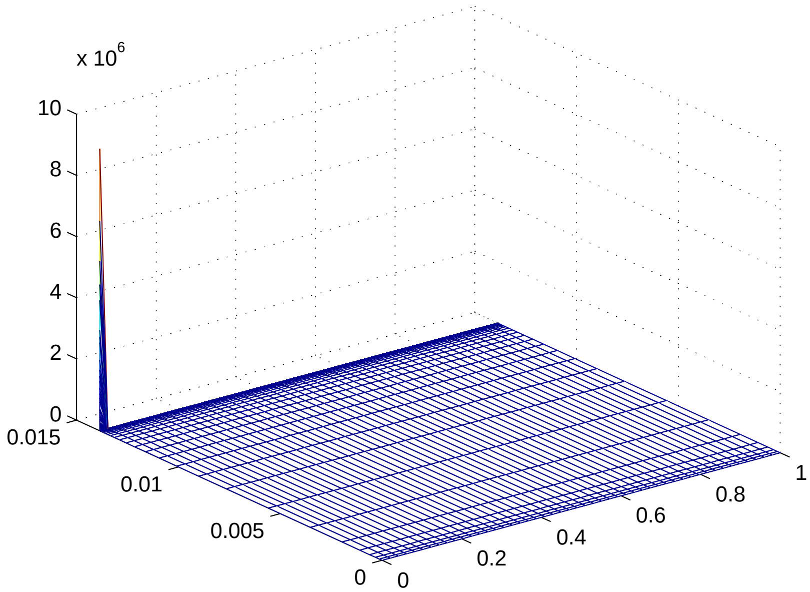

Figure 3. Evolution of the solution with $p = 3$.

The surface looks flat because the solution remains bounded away from zero, then compared to the blow-up points is negligible.

When you run the program, Matlab tells you that can not continue with the numerical integration (which confirms the existence of singularity, according to the theoretical interpretation) at some time. Then we take this time as computational blow-up time.

In the following picture, we show that the computational blow-up time converges as N goes to infinity.

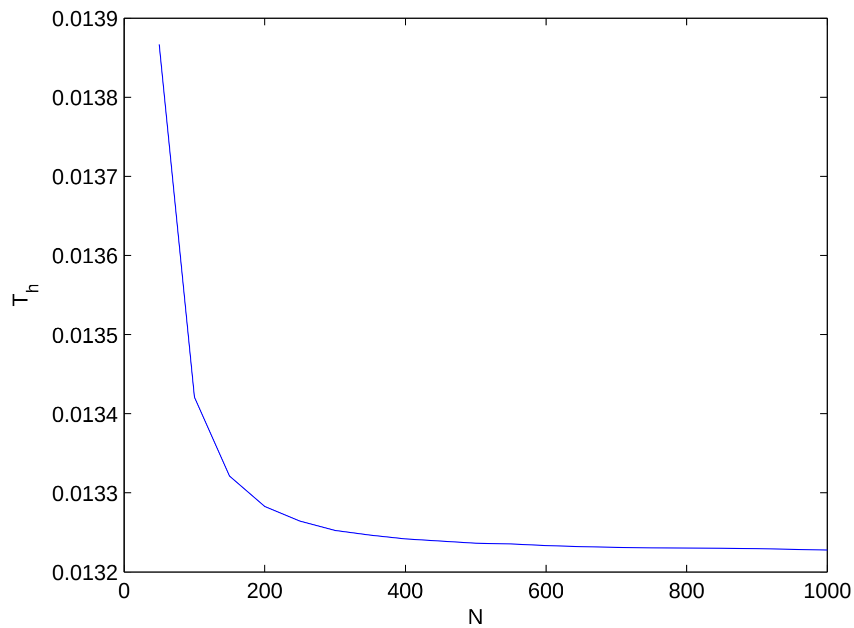

Figure 4 Blow-up time for $p = 3$.

In this particular case

$$T_h \to 0.013227.$$

In order to see the computational blow-up rates, in the next figure we display $\ln(u_i)$ versus $\ln(T_h - t)$ for $i = 1, 2$.

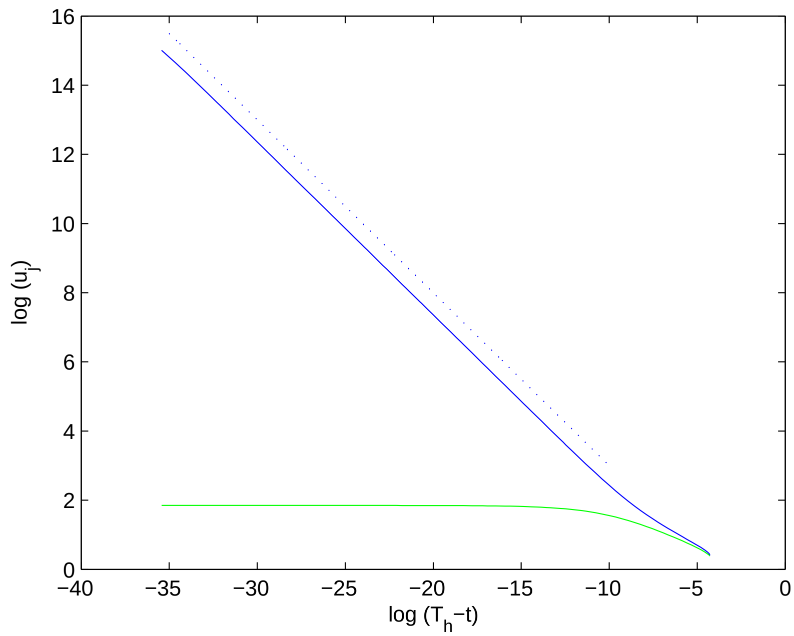

Figure 5 Blow-up rates for $p = 3$.

We can appreciate that the curve for $u_1(t) = \|U(t)\|_\infty$ (blue curve) becomes parallel to the dotted blue line which has slope $-1/2$. The green curve (corresponding to u_2) is flat and then u_2 remains bounded. These behaviors correspond to the expected blow-up set and blow-up rate,

$$B(U) = \{x_0\}\,, \qquad u_1(t) \sim (T_h - t)^{-\frac{1}{2}} \qquad \frac{1}{2} = \frac{1}{p-1}\,.$$

Finally to see the propagation of the blow-up to the interior node, we consider $p = 1.6$ and $p = 1.4$. In this cases, Theorem .... gives

$$B(U) = \begin{cases} \{x_0, x_1\} & p = 1.6 \\ \{x_0, x_1, x_2\} & p = 1.4 \end{cases}$$

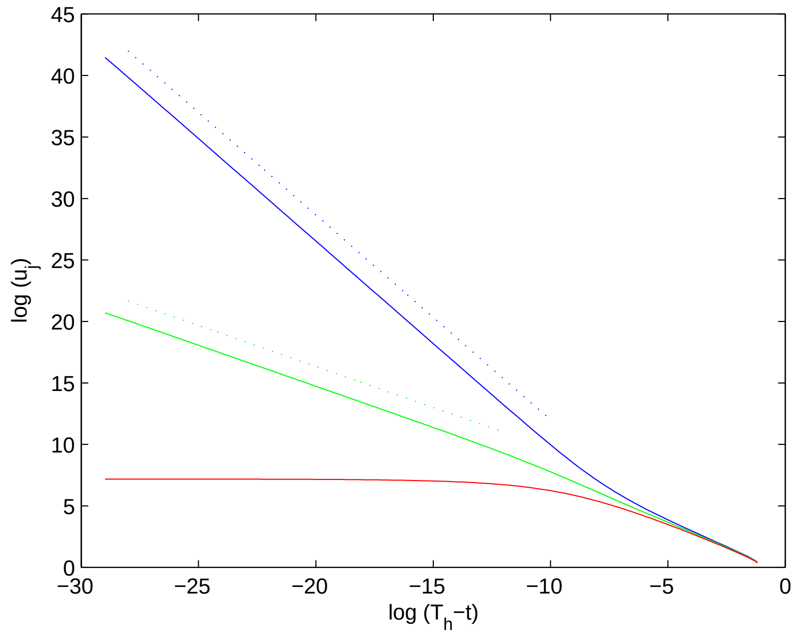

Figure 6 Blow-up rates for $p = 1.6$.

The blue color corresponds to u_1, green for u_2 and red for u_3. Notice that red line is flat, so u_3 remains bounded and we obtain the expected blow-up set.

On the other hand, the doted lines have slopes $-10/6$ (blue) and $-4/6$ (green). Which gives us the expected blow up rates for u_1 and u_2

$$-\frac{1}{p-1} = -\frac{10}{6}, \qquad 1 - \frac{1}{p-1} = -\frac{4}{6}\,.$$

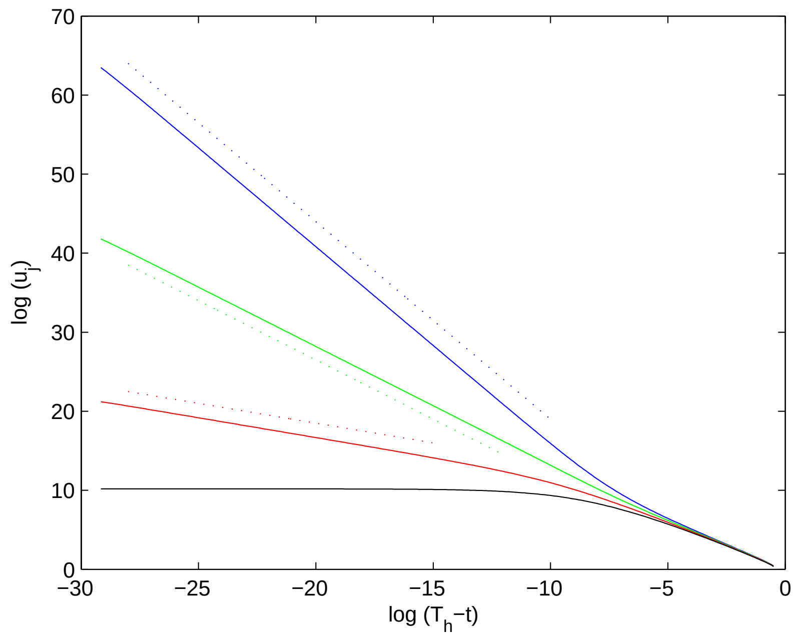

Figure 7 Blow-up rates for $p = 1.4$.

The blue color corresponds to u_1, green for u_2, red for u_3 and black for u_4. Since the flat line is the black line we get the expected blow-up set.

Doted lines have slopes $-10/4$ (blue), $-3/2$ (green) and $-1/2$ (red). Which also gives us the expected blow up rates for u_1, u_2 and u_3

$$-\frac{1}{p-1} = -\frac{10}{4}, \qquad 1 - \frac{1}{p-1} = -\frac{3}{2}, \qquad 2 - \frac{1}{p-1} = -\frac{1}{2}\,.$$

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