When is the typical operator norm attaining?

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Máster en Matemáticas Avanzadas

Fac. CC. Matemáticas - UCM

Madrid, May 24-25, 2023

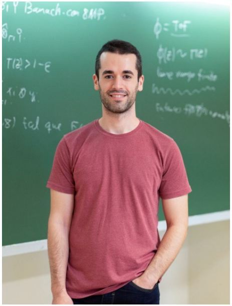
Preliminaries

Section 1

The minicourse is mainly based on the paper

M. Jung, M. Martín, and A. Rueda Zoca.
Residuality in the set of norm attaining operators between Banach spaces.
J. Funct. Anal. 284 (2023), 109746, 46pp.

Mingu Jung (KIAS, Korea)



Abraham Rueda Zoca (Granada, Spain)

Preliminaries

Section 1

Notation

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X, Y real or complex Banach spaces

- $\mathbb{K}$ base field $\mathbb{R}$ or $\mathbb{C}$,
- $\mathbb{T}$ modulus one scalars,
- $B_X = \{x \in X \colon \|x\| \leqslant 1\}$ closed unit ball of X,
- $S_X = \{x \in X \colon \|x\| = 1\}$ unit sphere of X,
- $\overline{\operatorname{conv}}(C)$ closed convex hull of C,
- $\mathcal{L}(X, Y)$ bounded linear operators from X to Y,
  - $\|T\| = \sup\{\|T(x)\| \colon x \in S_X\}$,
- $\mathcal{K}(X, Y)$ compact linear operators from X to Y,
- $\mathcal{F}(X, Y)$ bounded linear operators from X to Y with finite rank,
- $X^* = \mathcal{L}(X, \mathbb{K})$ topological dual of X.

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Examples and comments

- $\dim(X) < \infty \implies \mathrm{NA}(X, \mathbb{K}) = \mathcal{L}(X, \mathbb{K})$ (Heine-Borel).
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- $\mathrm{NA}(c_0,\mathbb{K}) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1,\mathbb{K}) = \Big\{x \in \ell_\infty : \|x\|_\infty = \max_n\{|x(n)|\}\Big\} \subseteq \ell_\infty$, residual, contains c_0,
- $\mathrm{NA}(X,\mathbb{K})$ may be “wild”, for instance:
  - it may contain NO two-dimensional subspaces (Read, 2017; Rmoutil, 2017),
  - it can be NOT norm Borel (Kaufman, 1991).

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  - it may contain NO two-dimensional subspaces (Read, 2017; Rmoutil, 2017),
  - it can be NOT norm Borel (Kaufman, 1991).
- (Petunin–Plichko 1974; Godefroy 1987): X separable, $Z \leqslant X^*$ closed, separating for X, $Z \subseteq \mathrm{NA}(X,\mathbb{K}) \implies Z$ is an isometric predual of X.

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  $\mathrm{NA}(X,\ell_\infty) = \left\{(x_n^*) \in \ell_\infty(X^*) \colon \exists k \in \mathbb{N},\ \|x_k^*\| = \|(x_n^*)\|_\infty,\ x_k^* \in \mathrm{NA}(X,\mathbb{K})\right\}$.
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- $\mathrm{NA}(L_1[0,1], L_\infty[0,1])$???

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Theorem (E. Bishop & R. Phelps, 1961)

The set of norm attaining functionals is dense in X^* (for the norm topology).

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Modified problem

When is $\mathrm{NA}(X,Y)$ dense in $\mathcal{L}(X,Y)$**?**

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Modified problem

When is $\mathrm{NA}(X, Y)$ dense in $\mathcal{L}(X, Y)$**?**

The study of this problem was initiated by J. Lindenstrauss in 1963,
who provided the first negative and positive examples.

An overview on “classical” results on norm attaining operators

Section 2

Bibliography for this overview

M. D. Acosta
Denseness of norm attaining mappings
RACSAM (2006)

A. Capel
Norm-attaining operators
Master thesis. Universidad Autónoma de Madrid, Spain. 2015
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M. Martín
The version for compact operators of Lindenstrauss properties A and B
RACSAM (2016)

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Y LUR, $T\colon X \longrightarrow Y$ bounded from below (monomorphism).
If T attains its norm, then it does at a strongly exposed point.

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X separable without strongly exposed points (e.g. c_0, $C[0,1]$, $L_1[0,1]$), Y LUR renorming of X. Then, $\mathrm{NA}(X,Y)$ is not dense in $\mathcal{L}(X,Y)$.

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Lindenstrauss properties A and B

Observation

- The question then is for which X and Y the density holds.
- As this problem is too general, Lindenstrauss introduced two properties.

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X, Y Banach spaces,

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- c_0, $C[0,1]$, $L_1[0,1]$ fail property A,
- if Y is strictly convex, $Y \supset c_0$, then Y fails property B.

Positive results I

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Theorem (Lindenstrauss, 1963)

X, Y Banach spaces. Then

$$\big\{T \in \mathcal{L}(X,Y) \colon \; T^{**} \colon X^{**} \longrightarrow Y^{**} \text{ attains its norm}\big\}$$

is dense in $\mathcal{L}(X,Y)$.

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Given $T \in \mathcal{L}(X,Y)$, there is $S \in \mathcal{K}(X,Y)$ such that $[T+S]^{**} \in \mathrm{NA}(X^{**},Y^{**})$.

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If X is reflexive, then X has property A.

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An improvement (Zizler, 1973)

X, Y Banach spaces. Then

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Let Z be a Banach space. Consider for two sets $\{z_i : i \in I\} \subset S_Z$, $\{z_i^* : i \in I\} \subset S_{X^*}$ and a constant $0 \leqslant \rho < 1$, the following four conditions:

1. $z_i^*(z_i) = 1$, $\forall i \in I$;
2. $|z_i^*(z_j)| \leqslant \rho < 1$ if $i, j \in I, i \neq j$;
3. B_Z is the absolutely closed convex hull of $\{z_i : i \in I\}$ (i.e. $\|z^*\| = \sup\{|z^*(z_i)| : i \in I\}$ for ever $z^* \in Z^*$);
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- Property α implies property A.

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3. B_Z is the absolutely closed convex hull of $\{z_i : i \in I\}$ (i.e. $\|z^*\| = \sup\{|z^*(z_i)| : i \in I\}$ for ever $z^* \in Z^*$);
4. B_{Z^*} is the absolutely weakly*-closed convex hull of $\{z_i^* : i \in I\}$ (i.e. $\|z\| = \sup\{|z_i^*(z)| : i \in I\}$ for every $z \in Z$).

- Z has property α if 1, 2, and 3 are satisfied (e.g. ℓ_1).
- Z has property β if 1, 2, and 4 are satisfied (e.g. c_0, ℓ_∞).

Theorem (Lindenstrauss, 1963)

- Property α implies property A.
- Property β implies property B.

Positive results III

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- The following spaces have property α:
  - ℓ_1,
  - finite-dimensional spaces whose unit ball has finitely many extreme points (up to rotation).

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- The following spaces have property A: ℓ_1 and **all** finite-dimensional spaces.
- The following spaces have property B: every Y such that $c_0 \subset Y \subset \ell_\infty$, finite-dimensional spaces such that the dual unit ball has finitely many extreme points (up to rotation).
- Every finite-dimensional space has property A, but the only known (in the 1960's) finite-dimensional real spaces with property B were the polyhedral ones. Only a little bit more is known nowadays...

An overview on "classical" results on norm attaining operators

Section 2

The Radon-Nikodým property

The Radon-Nikodým property

Definitions

X Banach space.

- X has the Radon-Nikodým property (RNP) if the Radon-Nikodým theorem is valid for X-valued vector measures (with respect to every finite positive measure).
- $C \subset X$ is dentable if it contains slices of arbitrarily small diameter.
- $C \subset X$ is subset-dentable if every subset of C is dentable.

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Theorem (Rieffel, Maynard, Huff, David, Phelps, 1970's)

X RNP $\iff$ every bounded $C \subset X$ is dentable $\iff$ B_X subset-dentable.

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Theorem (Rieffel, Maynard, Huff, David, Phelps, 1970's)

$$X \text{ RNP} \iff \text{every bounded } C \subset X \text{ is dentable} \iff B_X \text{ subset-dentable.}$$

Remark

In the book

J. Diestel and J. J. Uhl
Vector Measures
Math. Surveys **15**, AMS, Providence 1977.

there are more than 30 different reformulations of the RNP.

The RNP and property A: positive results

The RNP and property A: positive results

Theorem (Bourgain, 1977)

X Banach space, $C \subset X$ absolutely convex closed bounded subset-dentable, Y Banach space. Then

$$\{T \in \mathcal{L}(X,Y) \colon \text{the norm of } T \text{ attains its supremum on } C\}$$

is dense in $\mathcal{L}(X,Y)$.

★ In particular, RNP $\implies$ property A.

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It is actually shown that for every bounded linear operator there are arbitrary closed **compact** perturbations of it attaining the norm.

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Non-linear Bourgain-Stegall variational principle (Stegall, 1978)

X, Y Banach spaces, $C \subset X$ bounded subset-dentable, $\varphi : C \longrightarrow Y$ uniformly bounded such that $x \longmapsto \|\varphi(x)\|$ is upper semicontinuous.
Then for every $\delta > 0$, there exists $x_0^* \in X^*$ with $\|x_0^*\| < \delta$ and $y_0 \in S_Y$ such that the function $x \longmapsto \|\varphi(x) + x^*(x)y_0\|$ attains its supremum on C.

The RNP and property A: negative results

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Lindenstrauss actually showed that if X is separable and has property A
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A refinement (Huff, 1980)

X Banach space failing the RNP.

Then there exist X_1 and X_2 equivalent renorming of X such that

$$\mathrm{NA}(X_1, X_2) \text{ is NOT dense in } \mathcal{L}(X,Y).$$

The RNP and property A: isomorphic characterization

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Main consequence

$$\text{Every renorming of } X \text{ has property A} \quad \Longleftrightarrow \quad X \text{ has the RNP.}$$

The RNP and property A: isomorphic characterization

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Every renorming of X has property A $\iff$ X has the RNP.

Example

ℓ_1 has property A in every equivalent norm.

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Another consequence

Every renorming of X has property B $\implies$ X has the RNP.

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1. The converse of the implication above is NOT TRUE (Gowers, 1990)

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Another consequence

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Observations

1. The converse of the implication above is NOT TRUE (Gowers, 1990)
2. To get an equivalence, a weaker property is needed, quasi norm attainment:

G. Choi, Y.-S. Choi, M. Jung, M. M.
On quasi norm attaining operators between Banach spaces
RACSAM (2022)

An overview on "classical" results on norm attaining operators

Section 2

Counterexamples for property B

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It was an open question in the 1980's whether RNP $\implies$ property B

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- (Gowers, 1990): ℓ_p does not have property B for any $1 < p < \infty$.
- (Acosta, 1999): No infinite-dimensional strictly convex space has property B.
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Consequence

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The main open problem

★ Do all finite-dimensional spaces have property B**?**
Equivalently, does $\mathcal{F}(X,Y) \subset \overline{\mathrm{NA}(X,Y)}$ for all Banach spaces X and Y**?**

An overview on "classical" results on norm attaining operators

Section 2

Some classical spaces: positive results

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Example (Johnson-Wolfe, 1979)

In the real case, $\mathrm{NA}(C(K_1), C(K_2))$ is dense in $\mathcal{L}(C(K_1), C(K_2))$.

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Every weakly compact operator from $C(K)$ can be approximated by (weakly compact) norm attaining operators.

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Example (Finet-Payá, 1998)

$\mathrm{NA}(L_1[0,1], L_\infty[0,1])$ is dense in $\mathcal{L}(L_1[0,1], L_\infty[0,1])$.

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Consequence

$C[0,1]$ does not have property B and it was the first "classical" example.

An overview on “classical” results on norm attaining operators

Section 2

The question of norm attainment for compact operators

Question (open from 1970's till 2014)

Can every compact operator be approximated by norm-attaining operators?

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- Actually, the idea of the proofs is to use that the operator which is not going to be approximated is not compact or, even, it is an isomorphism.

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- Actually, the idea of the proofs is to use that the operator which is not going to be approximated is not compact or, even, it is an isomorphism.
- In most examples, it was even known that compact operators attaining the norm are dense.

Positive results on norm attaining compact operators

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- Some classical spaces (Johnson-Wolfe, 1979):
  - $X = C_0(L)$ or $X = L_1(\mu)$, Y arbitrary;
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  - X arbitrary, $Y \leqslant c_0$ with AP.
- More recent results:
  - (Cascales-Guirao-Kadets, 2013) X arbitrary, Y uniform algebra;
  - (M. 2014) $X^* = \ell_1$, Y arbitrary.

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Some examples

- When $X \leqslant c_0$ with X^* failing AP, exists Y such that...
- Exists $X \leqslant c_0$ with AP and Y such that...
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$\mathcal{K}(X,Y)$ is not contained in $\overline{\mathrm{NA}(X,Y)}$.

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Can every **finite-rank** operator be approximated by norm-attaining operators?

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Note

We do not even know whether it may exists a Banach space X such that the elements in $\mathrm{NA}(X,\ell_2)$ are of rank one.

What is residuality and why can it be interesting here?

Section 3

What is residuality or typicality?

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Equivalent reformulation

M complete metric space, $C \subset M$. TFAE:

- C is residual
- $C = \bigcap_{n=1}^{\infty} C_n$ and $\mathrm{int}(C_n)$ dense for all n
- $C \supseteq \bigcap_{n=1}^{\infty} O_n$ and O_n open dense for all n (i.e. C contains a subset which is G_δ in M and dense in M)

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Main property (Baire's Category Theorem)

The countable intersection of residual sets is residual, hence dense.

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Consequences of the residuality of norm attaining operators

X, Y Banach spaces, suppose $\mathrm{NA}(X,Y)$ is residual in $\mathcal{L}(X,Y)$. Then:

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- Given $\{S_n\} \subset \mathcal{L}(X,Y)$ (maybe unbounded), the set

$$\big\{T \in \mathcal{L}(X,Y) \colon S_n + T \in \mathrm{NA}(X,Y)\big\}$$

  is residual (in particular, dense) in $\mathcal{L}(X,Y)$.
  ★ In particular, given $\varepsilon > 0$ there is $T \in \mathcal{L}(X,Y)$ with $\|T\| < \varepsilon$ such that $S_n + T \in \mathrm{NA}(X,Y)$ for all $n \in \mathbb{N}$.

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Example

$\mathrm{NA}(c_0,\mathbb{K}) = \ell_1 \cap c_{00} \subsetneq \ell_1$, so it is not residual. Besides:

- $\mathrm{NA}(c_0,\mathbb{K}) - \mathrm{NA}(c_0,\mathbb{K}) = \ell_1 \cap c_{00} \neq \ell_1$,
- Given $x_1^* = 0$ and $x_2^* \in \ell_1 \setminus c_{00}$, there is NO $x^* \in \mathcal{L}(c_0,\mathbb{K})$ such that $x_1^* + x^* \in \mathrm{NA}(c_0,\mathbb{K})$ and $x_2^* + x^* \in \mathrm{NA}(c_0,\mathbb{K})$.

Necessary conditions on property A

Section 4

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Definitions

Definitions

Definition 1: strong exposition

$C \subset X$ bounded. $x_0 \in C$ is strongly exposed if there is $x^* \in X^*$ such that whenever $\{x_n\} \subset C$ satisfies $\operatorname{Re} x^*(x_n) \longrightarrow \sup \operatorname{Re} x^*(C)$, then $\{x_n\} \longrightarrow x_0$.
Equivalently, the slices of C defined by x^* contain x_0 and are arbitrarily small.

- In this case, we say that x^* strongly exposes C (at x_0).
- $\text{str-exp}(C)$ set of strongly exposed points of C.
- $\text{SE}(C)$ functionals which strongly expose C at some (strongly exposed) point.
- $\text{SE}(C)$ is a G_δ subset of X^*.

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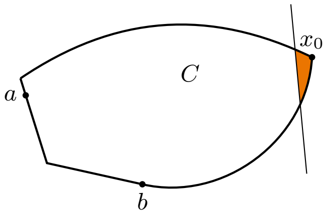

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Definition 1: strong exposition

$C \subset X$ bounded. $x_0 \in C$ is strongly exposed if there is $x^* \in X^*$ such that whenever $\{x_n\} \subset C$ satisfies $\operatorname{Re} x^*(x_n) \longrightarrow \sup \operatorname{Re} x^*(C)$, then $\{x_n\} \longrightarrow x_0$.
Equivalently, the slices of C defined by x^* contain x_0 and are arbitrarily small.

- In this case, we say that x^* strongly exposes C (at x_0).
- $\text{str-exp}(C)$ set of strongly exposed points of C.
- $\mathrm{SE}(C)$ functionals which strongly expose C at some (strongly exposed) point.
- $\mathrm{SE}(C)$ is a G_δ subset of X^*.

For the case $C = B_X \dots$

- If $\mathrm{SE}(B_X)$ is dense, $\mathrm{NA}(X, \mathbb{K})$ is residual.
- Šmulyan's test:

$$x^* \in \mathrm{SE}(B_X) \iff \text{the norm of } X^* \text{ is Fréchet-differentiable at } x^*$$

Lindenstrauss's and Bourgain's necessary conditions on property A

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Lindenstrauss, 1963

If X admits a LUR renorming and has property A
$\implies$ B_X is the closed convex hull of $\mathrm{str\text{-}exp}(B_X)$.

Lindenstrauss's and Bourgain's necessary conditions on property A

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If X admits a LUR renorming and has property A
$\implies$ B_X is the closed convex hull of $\text{str-exp}(B_X)$.

Bourgain, 1977

$C \subseteq X$ *separable bounded closed convex* such that for every Y the set

$$\left\{T \in \mathcal{L}(X,Y) \colon \exists \max_{x \in C} \|Tx\|\right\}$$

is dense in $\mathcal{L}(X,Y)$ (C has the Bishop-Phelps property in Bourgain's terminology)
$\implies$ C is dentable (i.e. C contains slices of arbitrarily small diameter).

Necessary conditions on property A

Section 4

The new result vs Lindenstrauss's and Bourgain's ones

The new result vs Lindenstrauss's and Bourgain's ones

Our result

X admitting a LUR renorming, $C \subseteq X$ bounded with the Bishop-Phelps property
$\implies$ $\mathrm{SE}(C)$ dense in X^* (hence $C = \overline{\mathrm{conv}}\big(\text{str-exp}(B_X)\big)$).
★ In particular, X admitting a LUR renorming, X with property A
$\implies$ $\mathrm{SE}(B_X)$ is dense in X^*, hence $\mathrm{NA}(X, \mathbb{K})$ is residual.

Compare with...

Lindenstrauss, 1963

If X admits a LUR renorming and has property A
$\implies$ B_X is the closed convex hull of $\text{str-exp}(B_X)$.

Bourgain, 1977

$C \subseteq X$ *separable bounded closed convex* such that for every Y the set

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is dense in $\mathcal{L}(X, Y)$ $\implies$ C contains slices of arbitrarily small diameter.

Sketch of the proof of a particular case

Our result (particular case)

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Then, x_0 is strongly exposed by S^*y^* for every $y^* \in S_{Y^*}$ with $\operatorname{Re} y^*(Sx_0) = \|S\|$.

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- Consider a LUR norm $|\!|\!| \cdot |\!|\!|$ on X and let $Y = (X, |\!|\!| \cdot |\!|\!|) \oplus_2 \mathbb{K}$ which is LUR.

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- Consider a LUR norm $\|\cdot\|$ on X and let $Y = (X, \|\cdot\|) \oplus_2 \mathbb{K}$ which is LUR.
- For $x^* \in S_{X^*}$, define $T_n \in \mathcal{L}(X, Y)$ by $T_n(x) = (n^{-1}x, x^*(x))$, which are monomorphisms, and $S \in \mathcal{L}(X, Y)$ by $S(x) = (0, x^*(x))$. Observe $\{T_n\} \longrightarrow S$.

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- By the lemma, there are $y_n^* = (x_n^*, \lambda_n) \in Y^* = X^* \oplus_2 \mathbb{K}$ such that $S_n^* y_n^* \in \mathrm{SE}(X)$ and $\|S_n^* y_n^*\| = \|S_n\| = 1$.

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$$\|\lambda_0 x^* - S_n^* y_n^*\| = \|\lambda_0 x^* - (\lambda_n x^* - S^* y_n^*) - S_n^* y_n^*\| \leqslant |\lambda_0 - \lambda_n| + \|S^* - S_n^*\| \longrightarrow 0.$$

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- As $\lambda_0 \neq 0$, $x^* = \lambda_0^{-1}(\lambda_0 x^*) \in \overline{\lambda_0^{-1}\,\mathrm{SE}(X)} = \overline{\mathrm{SE}(X)}$.

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- $\mathrm{SE}\big(\mathcal{F}(\mathbb{T})\big)$ is not dense in $\mathcal{F}(\mathbb{T})^* \equiv \mathrm{Lip}_0(\mathbb{T}, \mathbb{R})$ (C-GL-M-RZ, 2021),
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Compare with...

- (Lindenstrauss): If $B_X = \overline{\mathrm{conv}}(C)$ and the elements of C are *uniformly* strongly exposed, then X has property A.

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Compare with...

- (Lindenstrauss): If $B_X = \overline{\mathrm{conv}}(C)$ and the elements of C are *uniformly* strongly exposed, then X has property A.
- (Bourgain): X has RNP

$$
\begin{aligned}
&\iff \text{str-exp}(B_Z) \neq \emptyset \ \forall Z \simeq X \\
&\iff B_Z = \overline{\mathrm{conv}}(\text{str-exp}(B_Z)) \ \forall Z \simeq X \\
&\iff \mathrm{SE}(Z) \text{ dense } \forall Z \simeq X.
\end{aligned}
$$

Getting residuality of norm-attaining operators

Section 5

Getting residuality of norm-attaining operators

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Lindenstrauss: uniform strong exposition

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Set of uniformly strongly exposed points

X Banach space, $A \subset S_X$ is a set of uniformly strongly exposed points if for every $a \in A$ there is $a^* \in S_{X^*}$ with $\operatorname{Re} a^*(a) = 1$ satisfying that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$x \in B_X, \quad \operatorname{Re} a^*(x) > 1 - \delta \quad \Longrightarrow \quad \|x - a\| < \varepsilon.$$

(That is, the elements in B are strongly exposed in a uniform way).

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Examples

- Obviously, if X is uniformly convex.
- Property α.

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This not cover...

Reflexivity (also Lindenstrauss), Property quasi-α (Choi-Song), RNP (Bourgain).

The RNP and absolutely strongly exposing operators

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Definition (Bourgain, 1977)

$T \in \mathcal{L}(X,Y)$ is absolutely strongly exposing ($T \in \mathrm{ASE}(X,Y)$) iff there exists $x_0 \in S_X$ such that whenever $\{x_n\} \subset B_X$ satisfies $\|T(x_n)\| \longrightarrow \|T\|$ then $\exists \{\theta_n\} \subset \mathbb{T}$ for which $\{\theta_n x_n\} \longrightarrow x_0$.

★ $\mathrm{ASE}(X,Y)$ is a G_δ-set. Therefore, if $\mathrm{ASE}(X,Y)$ is dense, $\mathrm{NA}(X,Y)$ is residual.

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Theorem (Bourgain, 1977)

X RNP, Y arbitrary $\implies \mathrm{ASE}(X,Y)$ is dense.

★ It is only used in the proof that B_X is subset-dentable.

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Much more: the non-linear Bourgain-Stegall variational principle (Stegall, 1978)

$C \subset X$ bounded RNP set, $\varphi\colon C \longrightarrow \mathbb{R}$ bounded upper semicontinuous.

Then, the set

$$\left\{x^* \in X^* \colon \varphi + \operatorname{Re} x^* \text{ strongly exposed } C\right\}$$

is residual in X^*.

Getting residuality of norm-attaining operators

Section 5

Some comments and questions (I)

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Observation (Chiclana–GarcíaLirola–M.–RuedaZoca, 2021)

ALL known sufficient conditions for property A actually imply that absolutely strongly exposing operators are dense:

- RNP,
- properties α and quasi-α,
- $B_X = \overline{\text{conv}}(A)$, A uniformly strongly exposed.

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This is because of the following lemma...

If $T \in \mathcal{L}(X, Y)$ attains its norm at an element of $\mathrm{str\text{-}exp}(B_X)$, then $T \in \overline{\mathrm{ASE}(X, Y)}$.

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This is because of the following lemma...

If $T \in \mathcal{L}(X, Y)$ attains its norm at an element of $\text{str-exp}(B_X)$, then $T \in \overline{\text{ASE}(X, Y)}$.

Open problem 1 (still open)

Does the property A of X imply that $\text{ASE}(X, Y)$ is dense for every Y**?**

Some comments and questions (II)

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If $\mathrm{ASE}(X, Y)$ is dense for some $Y \implies \mathrm{SE}(X)$ is dense.

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If $\mathrm{ASE}(X, Y)$ is dense for some $Y \implies \mathrm{SE}(X)$ is dense.

Open problem 2 (still open)

Does the denseness of $\mathrm{SE}(X)$ imply that $\mathrm{ASE}(X, Y)$ is dense for every Y**?**

Some comments and questions (III)

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Less ambitious question

If $\mathrm{SE}(X)$ is dense, for which Ys is $\mathrm{ASE}(X,Y)$ dense**?**

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If $\mathrm{SE}(X)$ is dense, for which Ys is $\mathrm{ASE}(X, Y)$ dense**?**

Examples of when $\mathrm{SE}(X)$ is dense

- If X has RNP,
- If X has property A and admits a LUR renorming,
- If X is LUR (a property which is not known to imply property A),
- If $\text{str-exp}(B_X) = S_X$ (a property which is not known to imply property A),
- $X = JT^*$ (the dual of the James-tree space, not known if it has property A).

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- $X = JT^*$ (the dual of the James-tree space, not known if it has property A).

The objective

To find spaces Y which are not known to have property B such that $\mathrm{ASE}(X, Y)$ is dense whenever $\mathrm{SE}(X)$ is dense.

Getting residuality of norm-attaining operators

Section 5

A first family of new examples. The general result

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Theorem

X, Y Banach spaces, $\mathcal{I}(X,Y) \leqslant \mathcal{L}(X,Y)$ containing rank-one operators. Suppose:

- $\mathrm{SE}(X)$ is dense,
- there is $\{y_n^*\} \subset S_{Y^*}$ such that the set $\mathcal{A} = \{T \in \mathcal{I}(X,Y) \colon \|T\| = \|T^* y_n^*\| \text{ for some } n \in \mathbb{N}\}$ is residual in $\mathcal{I}(X,Y)$.

Then, $\mathrm{ASE}(X,Y) \cap \mathcal{I}(X,Y)$ is dense in $\mathcal{I}(X,Y)$.

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- The set $\mathcal{B} = \{T \in \mathcal{I}(X,Y) \colon T^* y_n^* \in \mathrm{SE}(X)\ \forall n \in \mathbb{N}\}$ is residual.

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Lemma

$T \in \mathcal{L}(X,Y)$, $y^* \in S_{Y^*}$ with $T^* y^* \in \mathrm{SE}(X)$, $\|T^* y^*\| = \|T\|$, then there is $x_0 \in \text{str-exp}(B_X)$ such that $|[T^* y^*](x_0)| = \|T x_0\| = \|T\|$, so $T \in \overline{\mathrm{ASE}(X,Y)}$.

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- $\mathrm{SE}(X)$ is dense,
- there is $\{y_n^*\} \subset S_{Y^*}$ such that the set
  $\mathcal{A} = \{T \in \mathcal{I}(X,Y) \colon \|T\| = \|T^* y_n^*\| \text{ for some } n \in \mathbb{N}\}$ is residual in $\mathcal{I}(X,Y)$.

Then, $\mathrm{ASE}(X,Y) \cap \mathcal{I}(X,Y)$ is dense in $\mathcal{I}(X,Y)$.

Idea of the proof:

- The set $\mathcal{B} = \{T \in \mathcal{I}(X,Y) \colon T^* y_n^* \in \mathrm{SE}(X)\ \forall n \in \mathbb{N}\}$ is residual.

Lemma

$T \in \mathcal{L}(X,Y)$, $y^* \in S_{Y^*}$ with $T^* y^* \in \mathrm{SE}(X)$, $\|T^* y^*\| = \|T\|$, then there is $x_0 \in \text{str-exp}(B_X)$ such that $|[T^* y^*](x_0)| = \|T x_0\| = \|T\|$, so $T \in \overline{\mathrm{ASE}(X,Y)}$.

- $\mathcal{A} \cap \mathcal{B}$ is residual and contained in $\overline{\mathrm{ASE}(X,Y) \cap \mathcal{I}(X,Y)}$.

A first family of new examples. Consequences I

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Consequence 1

$\mathrm{SE}(X)$ dense, Y^* RNP with $\text{str-exp}(B_{Y^*})$ countable up to rotations. Then:

$$\mathrm{ASE}(X,Y) \text{ dense in } \mathcal{L}(X,Y), \quad \mathrm{ASE}(X,Y)\cap\mathcal{K}(X,Y) \text{ dense in } \mathcal{K}(X,Y).$$

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This result applies to...

- Y being a predual of ℓ_1,
- Y being finite-dimensional such that $\mathrm{ext}\,(B_{Y^*})$ is countable (up to rotation),
- $Y = \mathrm{lip}_0(M)$ when M is a countable compact metric space.

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This result applies to...

- $Y = \mathcal{F}(M)$ (so $Y^* = \mathrm{Lip}_0(M)$) when M is a countable proper metric space.

A first family of new examples. Consequences II

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Consequence 3

$\mathrm{SE}(X)$ dense, Y such that every separable subspace of Y admits a countable James boundary. Then:

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This result applies to...

- Y polyhedral real Banach space,
- Y closed subspace of (the real or complex space) $C(K)$ where K is a Hausdorff scattered compact space.

(no separability condition is needed!)

A second family of new examples. The general result

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Theorem

X, Y Banach spaces, $\mathcal{I}(X,Y) \leqslant \mathcal{L}(X,Y)$ containing rank-one operators. Suppose:

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  (i.e. for every sequence $\{y_n\}$ of elements of $\text{str-exp}(B_Y)$ converging to an element $y_0 \in \text{str-exp}(B_Y)$, there is a sequence $\{\theta_n\} \subset \mathbb{T}$ such that $y_n = \theta_n y_0$ for large n).

Then, $\mathrm{ASE}(X,Y^*) \cap \mathcal{I}(X,Y^*)$ is dense in $\mathcal{I}(X,Y^*)$.

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- We use Stegall variational principle in $\mathcal{L}(Y,X^*) \equiv \mathcal{L}(X,Y^*)$.

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- We use Bourgain's ideas, the discreteness hypothesis, and the residuality of $\mathrm{SE}(X)$, to get operators $T\colon Y \longrightarrow X^*$ and norm-one elements y such that $\|Ty\| = \|T\|$ and $Ty \in \mathrm{SE}(X)$.

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- The (pre)adjoints of these operators attains their norms at strongly exposed points of B_X. Hence, they belong to $\overline{\mathrm{ASE}(X,Y^*)}$.

A second family of new examples. Consequence

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Consequence 4

$\mathrm{SE}(X)$ dense, Y RNP with $\text{str-exp}(B_Y)$ discrete up to rotations. Then:

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This result applies to...

- $Y = \mathcal{F}(M)$ (hence $Y^* = \mathrm{Lip}_0(M)$) when M is a discrete metric space.

When is the typical operator norm attaining?

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Madrid, May 24-25, 2023