Normal tropical $(0, -1)$-matrices and their orthogonal sets*

Bakhad Bakhadly[1], Alexander Guterman[1,2], María Jesús de la Puente[3]

To the memory of Professor V.N. Latyshev

[1]Lomonosov Moscow State University, Moscow, 119991, Russia
[2]Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991, Russia
[3]Departamento de Álgebra, Geometría y Topología, Facultad de Matemáticas, Plaza de Ciencias, 3, Universidad Complutense UCM, 28040–Madrid, Spain

Abstract

Square matrices A, B are orthogonal if $A \odot B = Z = B \odot A$, where Z is the matrix with all entries equal to 0, and $\odot$ is the tropical matrix multiplication. We study orthogonality for normal matrices over the set $\{0, -1\}$, endowed with tropical addition and multiplication. To do this we investigate the orthogonal set of a matrix A, i.e., the set of all matrices orthogonal to A. In particular, we study the family of minimal elements inside the orthogonal set, called a basis. Orthogonal sets and bases are computed for various matrices and matrix sets. Matrices whose bases are singletons are characterized. Orthogonality and minimal orthogonality are described in the language of graphs. The geometric interpretation of the results obtained is discussed.

Keywords: orthogonal, orthogonality, tropical semiring, tropical normal matrix.

1 Introduction

Tropical sum $\oplus$ of real numbers is their maximum (in some papers the authors use minimum instead of maximum), and tropical multiplication $\odot$ is the addition of real numbers. Tropical arithmetic can be lifted to matrices if we define the sum and the product via the standard rules, and use tropical operations to deal with the matrix entries. Tropical linear algebra is a modern actively developing science, which has a lot of deep and interesting mathematical problems and a number of applications, see, for example, [5].

In this paper we continue the study of orthogonality on tropical matrices. Square matrices A, B are orthogonal if $A \odot B = Z = B \odot A$, where Z is the matrix with all entries equal to 0, and $\odot$ is the tropical matrix multiplication. Our attention is focused on orthogonal pairs of tropical normal matrices. Normal matrices over semirings were introduced by Yoeli in 1961 and have proved to be useful in tropical linear algebra and applications, see [5, 8, 15, 16]. Pairwise orthogonality relation on the set of normal matrices has a lot of interesting and non-trivial properties, see [2] for example.

The main goal of this work is to introduce and investigate the orthogonal set to a subset $\mathcal{S}$ of normal matrices, i.e., the set $\operatorname{or}(\mathcal{S})$ of all matrices orthogonal to the matrices from $\mathcal{S}$, and its basis, i.e., the set $\operatorname{b}(\mathcal{S})$ of minimal elements in $\operatorname{or}(\mathcal{S})$. In particular, we obtain a characterization of the matrices A such that $|\operatorname{b}(A)| = 1$, we compute orthogonal sets and bases for various matrices and matrix sets. Orthogonality and minimal orthogonality are described in the language of graphs. We also discuss the geometric interpretation of the obtained results.

For simplicity, we choose to work over the binary tropical semiring $\mathbb{K} := \{0, -1\}$. This set is also a Boolean algebra. Moreover, the set of all order n normal matrices over $\mathbb{K}$, denoted by M_n^N, has a lattice structure, where negation is a natural involution (see Definition 2.2 and Notation 2.4). For the detailed and self-contained information on lattices and their applications we refer the reader to [3, 1].

Our paper is organized as follows. In Section 2, we introduce the main notions to be handled, such as normal matrix, negated normal matrix $\neg A$, lattice structure, top $\top$ and bottom $\bot$ operators, atoms, elementary matrices

*The third author is partially supported by Ministerio de Economía y Competitividad, Proyecto I+D MTM2016-76808-P, Ministerio de Ciencia e Innovación, Proyecto PID–2019–10770 GB-I00 and by UCM research group 910444.

E_{ij}, U_{ij}, U_{ij}^{kl}, R_i, C_j, orthogonality relation (borrowed from [2]), orthogonal set and basis. In Section 3 we introduce the crucial notion of controlled matrix $F(A)$ of a given matrix A, prove the inequality $F(A) \leq \urcorner A$. In section 4 the sets $\mathrm{or}(S)$ and $\mathrm{b}(S)$ are computed for elementary matrices S. Complement sets of orthogonal sets are computed in some cases. In Section 5, we study graphs associated to orthogonality and self–orthogonality and translate orthogonality and minimal orthogonality to the language of graphs with M_n^N as vertex set. They turn out to be strongly connected and with diameter equal to 1 or 2 (see Corollary 5.7). In Section 6 we study the relation between bases and orthogonal sets. We show that the equality $|\mathrm{b}(A)| = 1$ is equivalent to the condition $\mathrm{b}(A) = \{F(A)\}$ (see Corollary 6.10, a direct consequence of Theorem 6.9). In Section 7 we work on the power set $\mathcal{P}(M_n^N)$, finding the orthogonal sets for subsets of matrices and investigating their bases. In Section 8 we compare two different notions of minimality that the orthogonality relation suggests: (a) (A, B) being a minimal orthogonal pair (definition borrowed from [2]) and (b) satisfying two symmetric conditions: $A \in \mathrm{b}(B)$ and $B \in \mathrm{b}(A)$, and we show that the former is stronger than the latter (see Lemma 8.3). In Section 9 the geometric interpretation of the obtained results is considered.

2 Preliminaries

For $n \in \mathbb{N}$, the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$, and $[n] \setminus \{1\}$ is denoted by $[n]'$. The diagonal set in $[n] \times [n]$ is denoted by Δ. Capital letters $A, B, C, E, L, \ldots, R, U, X, Z \ldots$ denote matrices, small letters $a, b, \ldots, u, v, \ldots$ denote vectors. All vectors are assumed to be column vectors. For each i, $1 \leq i \leq n$, let e_i be a column vector with 1 in i-th position and 0 elsewhere. We denote $e := -e_1 - \cdots - e_n = [-1, \ldots, -1]^T$. Sets are denoted with capital calligraphic letters: $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{S}, \ldots$ The *complement* of the set $\mathcal{S}$ is denoted by $\mathcal{S}^{\mathsf{C}}$.

Definition 2.1 (Tropical operations)**.** Take $\mathbb{K}$ equal to $\{0, -1\}$. Consider the *max–plus semiring* or *tropical semiring* $(\mathbb{K}, \oplus, \odot)$, where $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$. The operations $\oplus$, $\odot$ are called *tropical sum*, *tropical multiplication*, respectively. We define $(-1) \odot a = -1 = a \odot (-1)$, for $a \in \{0, -1\}$. In particular, $(-1) \odot (-1) = (-1) \oplus (-1) = -1$. The neutral element for tropical multiplication is 0. The neutral element for tropical addition is -1.

The set of square $n \times n$ matrices with entries in $\mathbb{K}$ is denoted by M_n. The operations $\oplus$ and $\odot$ are extended to vectors and matrices in the usual way. We write $A \leq B$ if and only if $A \oplus B = B$, and $A < B$ whenever $A \leq B$ and $A \neq B$. We use the following agreement: AB stands for $A \odot B$. Often, we use classical addition of matrices $A + B$.

From now on let us assume that $n \geq 2$.

Definition 2.2 (Normal matrix, Yoeli, 1961, see [15])**.** A tropical matrix $A = [a_{ij}] \in M_n$ is called *normal* if $a_{ij} \leq 0$ for all i, j and all its diagonal entries equal 0. The set of order n normal matrices is denoted by M_n^N. A normal matrix $A = [a_{ij}]$ is *strictly normal* if $a_{ij} < 0$ for all $i \neq j$.

Notation 2.3 (Negated vector)**.** For $a = [a_1, a_2, \ldots, a_n]^T$ we denote $\neg a = [b_1, b_2, \ldots, b_n]^T$ such that $b_i = \begin{cases} -1, \text{ if } a_i = 0, \\ 0, \text{ otherwise.} \end{cases}$

The following is the adequate negation in M_n^N. It must not be confused with plain negation of matrix A, denoted $\neg A$, so a different notation is used.

Notation 2.4 (Negated normal matrix)**.** For $A = [a_{ij}] \in M_n^N$ we denote $\urcorner A = [b_{ij}] \in M_n^N$ such that $b_{ij} = \begin{cases} 0, \text{ if } i = j \text{ or } a_{ij} = -1, \\ -1, \text{ otherwise.} \end{cases}$

Notation 2.5 (Zero matrix and identity matrix)**.** For $n \in \mathbb{N}$, consider the following $n \times n$ matrices:

1. Z_n is the matrix with all entries equal to 0,
2. U_n is the matrix having 0 on the diagonal and -1 elsewhere outside the diagonal.

We write Z and U if n is clear from the context. Note that both matrices are normal.

Note that Z is *absorbing* in M_n^N, i.e., $AZ = Z = ZA$, $A \in M_n^N$. Besides, U is the identity for tropical sum and multiplication in M_n^N, i.e., $A \oplus U = A = U \oplus A$ and $AU = A = UA$, $A \in M_n^N$. The same element being the identity for two operations is most unusual in mathematics!

The set M_n^N is a finite bounded *upper semilattice* (because $\oplus$ is associative, commutative and idempotent) with *top element* Z and *bottom element* U. Since $\mathbb{K} = \{0, -1\}$, then $\min\{A, B\} = A + B$, and the set M_n^N is a *Boolean algebra* with operations $(\max, \min, \neg) = (\oplus, +, \neg)$, so M_n^N is a *partially ordered set*, and, hence, a *lattice*. See [7] for more details.

Notation 2.6. The usual *top* and *bottom operators* are $\top(A) := \{B \in M_n^N : B \geq A\}$ and $\bot(A) := \{B \in M_n^N : B \leq A\}$. These sets are sublattices.

Notation 2.7. Let $i, j \in [n]$, $i \neq j$.

1. E_{ij} is the matrix with -1 on the position (i, j), and 0 otherwise.
2. $U_{ij} := \neg E_{ij}$.
3. $U_{ij}^{kl} = U_{kl}^{ij} := \neg (E_{ij} + E_{kl})$.

Remark 2.8 (Atoms)**.** *The matrices U_{ij} are the minimal elements (atoms) in M_n^N. The matrices E_{ij} are the maximal elements in M_n^N. Every non-zero matrix is a classical sum of matrices E_{ij}. Clearly,*

$$M_n^N = \bigcup_{i \neq j} \bot(E_{ij}).$$

Notation 2.9. The i–th *row* of matrix A is denoted by $\operatorname{row}(i, A)$.

Notation 2.10. For $v \in \mathbb{K}^n$, $j \in [n]$, let $L_j(v)$ be the matrix with v in the j–th column and 0 elsewhere. Let $L_j(v^T)$ be the matrix with v in the j–th row and 0 elsewhere.

Notation 2.11.

1. For $i \in [n]$, consider $R_i := \sum_{i \neq j} E_{ij}$. With Notation 2.10, we have $R_i = L_i(e^T + e_i^T)$.
2. For $j \in [n]$, consider $C_j := \sum_{i \neq j} E_{ij}$. We have $C_j = L_j(e + e_j)$.

Lemma 2.12.

1. *$R_i E_{ij} = E_{ij} = E_{ij} C_j$,*
2. *$E_{ij} E_{kl} = Z$, for $n \geq 3$ and all $i, j, k, l \in [n]$, $i \neq j, k \neq l$.*

Proof. Direct computations. □

Recall the following definitions, lemma and its corollary from [2].

Definition 2.13 (Orthogonality relation)**.** Let $A, B \in M_n^N$. If $AB = Z = BA$, then we say that A and B are *mutually orthogonal*. If $A^2 = Z$, then we say that A is *self–orthogonal.*

Definition 2.14 ([2, Definition 3.3] Indicator matrix of a pair)**.** For matrices $A = [a_{ij}], B = [b_{ij}] \in M_n^N$, consider $AB = [l_{ij}]$ and $BA = [r_{ij}] \in M_n^N$. The matrix $C = [c_{ij}] \in M_n^N$ defined by $c_{ij} = 0$ if $l_{ij} = r_{ij} = 0$, and $c_{ij} = -1$ otherwise, is called the *indicator matrix* of the pair (A, B).

Obviously, the matrices A, B are mutually orthogonal if and only if the indicator matrix C is zero.

Lemma 2.15 ([2, Lemma 3.4] Propagation of zeros)**.** *Let $A, B \in M_n^N$, AB, BA as above and $p, q \in [n]$. If $a_{pq} = 0$, then $l_{pq} = r_{pq} = c_{pq} = 0$.*

Corollary 2.16 ([2, Corollary 3.5] Orthogonality by propagation)**.** *Let $A, B \in M_n^N$ and C be the indicator matrix of (A, B). If $A \oplus B = Z$, then $C = Z$.*

Remark 2.17. *Let $A, B \in M_n^N$. Then Lemma 2.15 implies $A \leq AB$ and $B \leq AB$, whence $A \oplus B \leq AB$. Similarly we have $A \oplus B \leq BA$ and we can conclude $A \oplus B \leq \min\{AB, BA\}$. Furthermore, $C = \min\{AB, BA\}$ is the* indicator matrix *of the pair (A, B).*

Notation 2.18 (Orthogonal set and its iterations)**.** For a subset $\mathfrak{S} \subseteq M_n^N$ and a matrix $A \in M_n^N$ we define the *orthogonal set* of matrix A in $\mathfrak{S}$ by $\mathrm{or}(A)_{\mathfrak{S}} = \{B \in \mathfrak{S} : AB = Z = BA\}$. Similarly, the *orthogonal set* of a subset $\mathcal{S} \subseteq M_n^N$ in $\mathfrak{S}$ is $\mathrm{or}(\mathcal{S})_{\mathfrak{S}} = \bigcap_{A \in \mathcal{S}} \mathrm{or}(A)_{\mathfrak{S}}$. We omit the subscript $\mathfrak{S}$ if $\mathfrak{S} = M_n^N$.

Define $\mathrm{or}^j(A) = \mathrm{or}(\mathrm{or}^{j-1}(A))$, for $j > 1$.

Corollary 2.19. $1 \leq |\,\mathrm{or}(A)| \leq |M_n^N| = 2^{n^2-n}$ *for* $A \in M_n^N$.

Lemma 2.20 (or is monotonic)**.** *If* $A, A' \in M_n^N$ *with* $A \leq A'$*, then* $\mathrm{or}(A) \subseteq \mathrm{or}(A')$.

Proof. Monotonicity of $\oplus$ and $\odot$ yield $A'B \leq AB \leq ZB = Z$ and $BA' \leq BA \leq BZ = Z$. If $A'B = Z = BA'$, then $AB = Z = BA$. The result follows. ☐

Corollary 2.21. *Let* $A \in M_n^N$*. Then* $\mathrm{or}(\top(A)) = \mathrm{or}(A)$.

Proof. By Notation 2.18 and Lemma 2.20, $\mathrm{or}(\top(A)) = \bigcap_{A' \in \top(A)} \mathrm{or}(A') = \mathrm{or}(A)$. ☐

Corollary 2.22. *Let* $A, B \in M_n^N$*. If* $A \in \mathrm{or}(B)$*, then* $\top(A) \subseteq \mathrm{or}(B)$.

Proof. Let $A' \in \top(A)$, i.e. $A \leq A'$. By Lemma 2.20, $B \in \mathrm{or}(A')$, hence $A' \in \mathrm{or}(B)$. ☐

Definition 2.23 (Basis)**.** Given $\mathcal{S} \subseteq M_n^N$ the set of matrices that are minimal in the set $\mathrm{or}(\mathcal{S})$ is denoted $\mathrm{b}(\mathcal{S})$. We say that each matrix in $\mathrm{b}(\mathcal{S})$ is *minimally orthogonal* to each matrix in $\mathcal{S}$. We will write $\mathrm{b}(A)$ if $\mathcal{S} = \{A\}$.

Example 2.24. *Notice that* b *is not monotonic. Indeed, Corollary 4.4 yields that* $U_{1n} < U_{1n}^{2,n-1}$ *but* $\mathrm{b}(U_{1n}) \not\subset \mathrm{b}(U_{1n}^{2,n-1})$.

Corollary 2.25. *Let* $\mathcal{S} \subseteq M_n^N$*. Then* $\mathrm{or}(\mathcal{S}) = \bigcup_{B \in \mathrm{b}(\mathcal{S})} \top(B)$.

Proof. Follows from Definition 2.23 and Corollary 2.22. ☐

Lemma 2.26. *Let* $\mathcal{S} \subseteq M_n^N$*. If* $\mathcal{S}' = \bigcup_{S \in \mathcal{S}} \top(S)$*, then* $\mathrm{or}(\mathcal{S}) = \mathrm{or}(\mathcal{S}')$ *and* $\mathrm{b}(\mathcal{S}) = \mathrm{b}(\mathcal{S}')$.

Proof. By Notation 2.18 and Corollary 2.21, we have $\mathrm{or}(\mathcal{S}') = \bigcap_{S \in \mathcal{S}} \mathrm{or}(\top(S)) = \bigcap_{S \in \mathcal{S}} \mathrm{or}(S) = \mathrm{or}(\mathcal{S})$ and clearly $\mathrm{b}(\mathcal{S}) = \mathrm{b}(\mathcal{S}')$. ☐

Lemma 2.27. *If* $A, X \in M_n^N$ *with* $X \in \mathrm{or}(A)$*, then any minimal element in* $\perp(X) \cap \mathrm{or}(A)$ *belongs to the basis* $\mathrm{b}(A)$.

Proof. If $Y \in \perp(X) \cap \mathrm{or}(A)$ is minimal and $Y \notin \mathrm{b}(A)$ then there exists Y' with $Y' < Y \leq X$ such that $Y' \in \mathrm{or}(A)$, a contradiction. ☐

Notation 2.28 (Lattice of subsets)**.** Let $\mathcal{P}(M_n^N)$ denote the family of subsets of M_n^N. Then $(\mathcal{P}(M_n^N), \cup, \cap)$ is a lattice with one binary operation $\oplus$ and three maps from $\mathcal{P}(M_n^N)$ to itself: $\mathrm{or}, \mathrm{Min}$ and the composition $\mathrm{b} := \mathrm{Min} \circ \mathrm{or}$. Here Min stands for the set of minimal elements, it is not the operation of minimization.

3 Almost nowhere zero rows and columns. Controlled matrix.

Definition 3.1 (Almost nowhere zero rows and columns)**.** A row or a column of a matrix is called *almost nowhere zero*, if all its off–diagonal elements are non-zero.

Example 3.2. *All rows (resp. columns) of the matrices* R_i *(resp.* C_j*) in Notation 2.11 are either zero or almost nowhere zero.*

Notation 3.3 (Counting zeros in rows and columns)**.** Let $i \in [n]$, $\mathcal{J} \subseteq [n]$, $A = [a_{ij}] \in M_n^N$. Denote $\mathrm{nr}(A, i, \mathcal{J}) := |\{j : a_{ij} = 0, i \neq j \text{ and } j \notin \mathcal{J}\}|$. Similarly for columns: $\mathrm{nc}(A, i, \mathcal{J}) := |\{j : a_{ji} = 0, i \neq j \text{ and } j \notin \mathcal{J}\}|$.

Notice that $\mathrm{nr}(A, i, \emptyset) \geq 1$ if and only if $\mathrm{row}(i, A)$ is not almost nowhere zero. Similar for columns.

In the following lemma we see how rows and columns in U and Z interplay in order to have orthogonality and minimal orthogonality.

Lemma 3.4 (Controlled initialization)**.** *Let $A \in M_n^N$, $i \in [n]$.*

1. *If $\mathrm{row}(i, A)$ is almost nowhere zero and $B \in \mathrm{or}(A)$, then $\mathrm{row}(i, B)$ is zero.*
2. *If $\mathrm{row}(i, A)$ is zero, then there exists $B \in \mathrm{b}(A)$ such that $\mathrm{row}(i, B)$ is almost nowhere zero.*

The same is true for columns.

Proof. 1. For each $B \in \mathrm{or}(A)$, we have $0 = (AB)_{ij} = (UB)_{ij} = B_{ij}$, all $j \in [n]$.

2. Recall that the i–th row of R_i is almost nowhere zero. By the hypothesis and Corollary 2.16 we get $R_i \in \mathrm{or}(A)$. Now choose B to be a minimal element in $\mathrm{or}(A) \cap \perp(R_i)$.

Similarly for columns. □

Definition 3.5 (Controlled matrix)**.** Let $A \in M_n^N$ and $\mathcal{I}, \mathcal{J} \subseteq [n]$ correspond to almost nowhere zero rows and columns of A, respectively. Then $F(A) := [f_{ij}] \in M_n^N$, where $f_{ij} = 0$ if $i \in \mathcal{I}$ or $j \in \mathcal{J}$ or $i = j$, and $f_{ij} = -1$ otherwise, is called the *controlled matrix* of A.

It is possible that $F(A) \notin \mathrm{or}(A)$, for instance, take $A = U_{1n}^{2,n-1}$. However, $F(A) \in \mathrm{or}(A)$ characterizes $\mathrm{b}(A) = \{F(A)\}$, whence $|\mathrm{b}(A)| = 1$, which is interesting, because then $\mathrm{or}(A) = \top(F(A))$ is a sublattice. The next lemma provides a proof for this fact. In Corollary 6.10 we have the converse.

Lemma 3.6. *Let $A \in M_n^N$ and $F(A) \in M_n^N$ be the controlled matrix of A. Then*

1. $\mathrm{or}(A) \subseteq \top(F(A))$*. In particular,* $\mathrm{b}(A) \subseteq \top(F(A))$.
2. $F(A) \in \mathrm{or}(A)$ *if and only if* $\mathrm{b}(A) = \{F(A)\}$.

Proof.

1. By Item 1 of Lemma 3.4, if $\mathrm{row}(i, A)$ is almost nowhere zero, then $\mathrm{row}(i, B)$ is zero for all $B \in \mathrm{or}(A)$, similarly for the columns. Hence $B \in \top(F(A))$ for all $B \in \mathrm{or}(A)$, i.e. $\mathrm{or}(A) \subseteq \top(F(A))$.
2. If $F(A) \in \mathrm{or}(A)$, then, by Corollary 2.22, $\top(F(A)) \subseteq \mathrm{or}(A)$. Using Item 1, we get $\mathrm{or}(A) = \top(F(A))$. The sufficiency is obvious.

□

Lemma 3.7. *$F(A) \leq \neg A$, for each $A \in M_n^N$.*

Proof. Write $F(A) = [f_{ij}]$ and $\neg A = [b_{ij}]$. Let $\mathcal{I}, \mathcal{J} \subseteq [n]$ correspond to almost nowhere zero rows and columns of A, respectively. Consider $i, j \in [n]$.

1. If $a_{ij} = -1$, then $b_{ij} = 0$ and $f_{ij} \leq b_{ij}$.
2. If $a_{ij} = 0$ and $i = j$, then $f_{ij} = 0 = b_{ij}$.
3. If $a_{ij} = 0$ and $i \neq j$, then $i \notin \mathcal{I}$ and $j \notin \mathcal{J}$, whence $f_{ij} = -1 = b_{ij}$.

□

4 Orthogonal set and basis for elementary matrices

Definition 4.1 (Elementary matrices)**.** Matrices E_{ij}, U_{ij}, U_{ij}^{kl}, R_i, C_j introduced in Notations 2.7 and 2.11 are called *elementary matrices*.

Definition 4.2 (Independence)**.** Given $\mathcal{K} \subseteq [n] \times [n] \setminus \Delta$ (see page 2), we say that $\mathcal{K}$ is *independent* if no two different elements in $\mathcal{K}$ have the same first entry and no two different elements in $\mathcal{K}$ have the same second entry. The family $\mathcal{S} = \{E_{ij} : (i, j) \in \mathcal{K}\}$ is *independent* if $\mathcal{K}$ is.

Lemma 4.3 (Orthogonal set of E_{ij})**.** *Let $i, j, k, l \in [n]$, $i \neq j$ and $k \neq l$.*

1. $\text{or}(E_{ij}) = \{X \in M_n^N : \text{nr}(X, i, \emptyset) \geq 1$ *and* $\text{nc}(X, j, \emptyset) \geq 1\}$, *i.e., X has at least one off–diagonal zero entry in the i–th row and at least one off–diagonal zero entry in the j–th column.*

2. $\text{or}(E_{ij} + E_{kl}) = \text{or}(E_{ij}) \cap \text{or}(E_{kl})$ *if $i \neq k$ and $j \neq l$.*

3. $\text{or}(E_{ij} + E_{il}) = \{X \in M_n^N : \text{nr}(X, i, \emptyset) \geq 1, \text{nc}(X, j, \{l\}) \geq 1$ *and* $\text{nc}(X, l, \{j\}) \geq 1\}$ *if $j \neq l$.*

Proof. 1. It is straightforward to check that $E_{ij} \leq XE_{ij}$ and $E_{ij} \leq E_{ij}X$ for all $X \in M_n^N$. Thus $X \in \text{or}(E_{ij})$ if and only if $(E_{ij}X)_{ij} = 0 = (XE_{ij})_{ij}$. Since $(XE_{ij})_{ij} = (\max_{k \neq i} x_{ik}) \oplus (-1)$, we have that $(XE_{ij})_{ij} = 0$ if and only if there exists $k \in [n]$, $k \neq i$, with $x_{ik} = 0$, i.e., $\text{nr}(X, i, \emptyset) \geq 1$. Similarly we get $\text{nc}(X, j, \emptyset) \geq 1$.

Items 2, 3 are similar.

□

Corollary 4.4 (Basis for E_{ij}, U_{ij} and U_{ij}^{kl}).

1. $\text{b}(E_{ij}) = \{U_{kj}^{il} : k, l \notin \{i, j\}\} \cup \{U_{ij}\}$ *and* $|\text{b}(E_{ij})| = (n-2)^2 + 1$.
2. $\text{b}(U_{ij}) = \{E_{ij}\}$.
3. $\text{b}(U_{ij}^{kl}) = \{\sum_{s \in \{i,k\}, t \in \{j,l\}, s \neq t} E_{st}\}$, *if $i \neq k$ and $j \neq l$.*

Proof. Item 1 follows from Item 1 of Lemma 4.3. Items 2 and 3 follow from Lemma 2.15 and Item 2 of Corollary 3.6. □

The following lemma shows that the complement of the orthogonal set of some matrices is a finite union of sublattices.

Lemma 4.5 (Some complement sets). *For $i, j, k, l \in [n]$ with $i \neq j$, $k \neq l$ and $(i, j) \neq (k, l)$ we have:*

1. $R_i, C_j \notin \text{or}(E_{ij})$, *and $A \in \text{or}(E_{ij})$, for all $A \in \top(R_i) \cup \top(C_j)$, $R_i \neq A \neq C_j$,*

2. $\text{or}(E_{ij})^{\complement} = \bot(R_i) \cup \bot(C_j)$,

3. $\text{or}(E_{ij} + E_{kl})^{\complement} = \text{or}(E_{ij})^{\complement} \cup \text{or}(E_{kl})^{\complement}$, *if $i \neq k$ and $j \neq l$.*

4. *If A is a finite sum of independent matrices E_{ij}, then $\text{or}(A)^{\complement}$ is a finite union of lattices of the form $\bot(R_i)$ and $\bot(C_j)$.*

Proof. Item 1 follows from Item 1 of Lemma 4.3 and also from Item 1 of Lemma 2.12. In fact Item 2 is equivalent to Item 1 of Lemma 4.3. Item 3 follows from Item 2 of Lemma 4.3. Item 4 follows from Items 2 and 3. □

Example 4.6. *The condition $i \neq k$ and $j \neq l$ for the matrices E_{ij}, E_{kl} is important in Item 3 of Lemma 4.5. Indeed, let $n = 3$, then $\text{or}(E_{12} + E_{13})^{\complement} \nsubseteq \text{or}(E_{12})^{\complement} \cup \text{or}(E_{13})^{\complement}$, because $E_{13} \in \text{or}(E_{12} + E_{13})^{\complement}$, but $E_{13} \notin \text{or}(E_{12})^{\complement} \cup \text{or}(E_{13})^{\complement}$.*

Example 4.7. *Generalization of Item 3 in the last lemma for more summands is not always possible, because of the dependency of matrices E_{ij}. Indeed, let $n = 4$, then $\text{or}(E_{12}+E_{13}+E_{14})^{\complement} \nsubseteq \text{or}(E_{12})^{\complement} \cup \text{or}(E_{13})^{\complement} \cup \text{or}(E_{14})^{\complement}$, because $E_{14} \in \text{or}(E_{12} + E_{13} + E_{14})^{\complement}$, but $E_{14} \notin \text{or}(E_{12})^{\complement} \cup \text{or}(E_{13})^{\complement} \cup \text{or}(E_{14})^{\complement}$.*

Remark 4.8. *The set $\text{or}(U)^{\complement} = M_n^N \setminus \{Z\}$ cannot be a union of lattices of the form $\bot(R_i)$ and $\bot(C_j)$.*

5 Orthogonality in the language of graphs

We can also study orthogonality using graph theory. First recall a result from [2] which will be used later. P^{1i} stands for a permutation matrix.

Corollary 5.1 ([2, Corollary 5.4])**.** *Let $A \in M_n^N$ and there exists $i \in [n]$ such that both the i–th row and the i–th column of A are almost nowhere zero. If $P^{1i}AP^{1i} = \begin{bmatrix} 0 & v^t \\ w & X \end{bmatrix}$, where v and w are almost nowhere zero, then*

$$\operatorname{or}(P^{1i}AP^{1i}) = \left\{ \begin{bmatrix} 0 & 0_{1\times n-1} \\ 0_{n-1\times 1} & Y \end{bmatrix},\ Y \in \operatorname{or}(X) \right\}.$$

It means that $|\operatorname{or}(A)| = |\operatorname{or}(X)|$.

Definition 5.2. With any $A = [a_{ij}] \in M_n^N$ we can associate a directed graph $\Gamma(A)$ with vertex set $\mathrm{V}(\Gamma(A)) = [n]$, and we have a directed edge $(i, j) \in \mathrm{E}(\Gamma(A))$ if and only if $i \neq j$ and $a_{ij} = 0$.

Remark 5.3. *Our definition forbids loops. In terms of graphs Corollary 5.1 implies that the indegree and outdegree of the specified in Corollary 5.1 vertex i are zero, so i is an isolated vertex. It follows that while studying orthogonality we can consider the cut graph $\Gamma'(A)$ instead of $\Gamma(A)$. In $\Gamma'(A)$ we exclude all isolated vertices.*

Definition 5.4. A union graph $\Gamma(A, B)$ of the graphs $\Gamma(A)$ and $\Gamma(B)$, where $A, B \in M_n^N$, is a directed two–colored graph with the vertex set $\mathrm{V}(\Gamma(A, B)) = [n]$ and the edge set $\mathrm{E}(\Gamma(A, B)) = \mathrm{E}(\Gamma(A)) \cup \mathrm{E}(\Gamma(B))$. In other words, we may say that the edges of $\mathrm{E}(\Gamma(A))$ and $\mathrm{E}(\Gamma(B))$ are red and black, respectively. Further we denote red and black edges by $(i, j)_r$ and $(i, j)_b$, respectively. The condition $(i, j) \in \mathrm{E}(\Gamma(A, B))$ reflects that at least one of $(i, j)_r$ and $(i, j)_b$ is in $\mathrm{E}(\Gamma(A, B))$.

Definition 5.5. A union graph $\Gamma(A, B)$, $A, B \in M_n^N$, is called *orthogonally complete* if $B \in \operatorname{or}(A)$. By Definition 2.13 this is equivalent to $A \in \operatorname{or}(B)$.

Lemma 5.6. *A union graph $\Gamma(A, B)$, $A, B \in M_n^N$, is orthogonally complete if and only if for each (i, j) either $(i, j) \in \mathrm{E}(\Gamma(A, B))$ or there exist k, m such that $(i, k)_r, (k, j)_b \in \mathrm{E}(\Gamma(A, B))$ and $(i, m)_b, (m, j)_r \in \mathrm{E}(\Gamma(A, B))$.*

Proof. It is exactly the definition of tropical multiplication of normal matrices in terms of union graphs. □

Corollary 5.7. *An orthogonally complete graph $\Gamma(A, B)$, $A, B \in M_n^N$, is strongly connected and* $\operatorname{diam}(\Gamma(A, B)) \in \{1, 2\}$.

Corollary 5.8. *The union graph $\Gamma(A, A)$ is orthogonally complete if and only if for each (i, j) either $(i, j) \in E(\Gamma(A))$ or there exists k such that $(i, k) \in E(\Gamma(A))$ and $(k, j) \in E(\Gamma(A))$.*

Proof. The graph $\Gamma(A, A)$ is a two–colored duplicate of $\Gamma(A)$, so that $(i, k)_b \in \Gamma(A)$ if and only if $(i, k)_r \in \Gamma(A)$. □

Corollary 5.9 (Self–orthogonality)**.** *A matrix A is self–orthogonal if and only if $\Gamma(A)$ is strongly connected and* $\operatorname{diam}\Gamma(A) \in \{1, 2\}$. □

Let $A \in M_n^N$ and consider the set of indices $\mathcal{I} = \{i_1, i_2, \ldots, i_k\}$ such that for each $i_l \in \mathcal{I}$ the i_l–th row and i_l–th column of A are almost nowhere zero. Then, by Corollary 5.1, instead of studying order n normal matrices that are orthogonal to A we can study order $n - k$ normal matrices that are orthogonal to the matrix A' obtained from A by deleting the rows and columns corresponding to $\mathcal{I}$. Below we provide the interpretation of decreasing of matrix order in terms of graph theory.

Definition 5.10. Let $\Gamma(A)$, $A \in M_n^N$, have isolated vertices $\{i_1, i_2, \ldots, i_k\}$, $k \leq n$. Then a cut union graph $\Gamma'(A, B)$ of two graphs $\Gamma(A)$ and $\Gamma(B)$, $B \in M_n^N$, is a directed two–colored graph that is obtained from $\Gamma(A, B)$ by excluding vertices $\{i_1, i_2, \ldots, i_k\}$ with their edges.

Remark 5.11. *Let $B \in \operatorname{or}(A)$. Then by Corollary 5.1, the indegree and outdegree of each of the vertices $\{i_1, i_2, \ldots, i_k\}$ in the graph $\Gamma(A, B)$ is equal to $n - 1$ and, moreover, all these edges are black.*

Lemma 5.12. *The union graph $\Gamma(A,B)$, $A,B \in M_n^N$, is orthogonally complete if and only if the cut union graph $\Gamma'(A,B)$ is orthogonally complete.*

Proof. Follows from Corollary 5.1. □

Definition 5.13. A graph $\Gamma(B)$, $B \in M_n^N$, is called a *minimal cover* of $\Gamma(A)$, $A \in M_n^N$, if $\Gamma(A,B)$ is orthogonally complete and there is no proper orthogonally complete subgraph $\Gamma(A,X)$, $X \in M_n^N$.

It turns out that the set of minimal covers stands for the set of minimally orthogonal matrices.

Lemma 5.14. *A graph $\Gamma(B)$, $B \in M_n^N$, is a minimal cover of graph $\Gamma(A)$, $A \in M_n^N$, if and only if $B \in \mathrm{b}(A)$.*

Proof. Follows from Definition 2.23. □

6 From the basis to the orthogonal set of a matrix

For $A \in M_n^N$, the set $\mathrm{b}(A)$ of matrices minimally orthogonal to A is useful since $\mathrm{or}(A) = \bigcup_{B\in \mathrm{b}(A)} \top(B)$, by Corollary 2.25. Note that $\mathrm{b}(A)$ can have more than one element, as the following example shows.

Example 6.1. *For* $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$ *all elements of* $\mathrm{b}(A)$ *are listed below:*

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

For $\mathcal{S} \subseteq M_n^N$ the set $\mathrm{b}(\mathcal{S})$ helps us find $|\mathrm{or}(\mathcal{S})|$.

Lemma 6.2. *Let $\mathcal{S} \subseteq M_n^N$ and $\mathrm{b}(\mathcal{S}) = \{B_1, B_2, \ldots, B_m\}$. If we write $T_i = \top(B_i)$, then*

1) $$|\mathrm{or}(\mathcal{S})| = \sum_{i=1}^{m} |T_i| - \sum_{i<j} |T_i \cap T_j| + \sum_{i<j<k} |T_i \cap T_j \cap T_k| - \ldots$$
$$\ldots + (-1)^{m-1} |T_1 \cap T_2 \cap \ldots \cap T_m|.$$

2) $T_{i_1} \cap T_{i_2} \cap \ldots \cap T_{i_l} = \top(B)$*, where* $B = B_{i_1} \oplus B_{i_2} \oplus \ldots \oplus B_{i_l}$.

Proof. By Corollary 2.25 $\mathrm{or}(\mathcal{S}) = \bigcup_{i=1}^{m} T_i$. Then the first statement is just the inclusion–exclusion principle for the union of m sets. The second item follows directly from the definition of a tropical sum, which is maximum. □

Denote the number of off–diagonal zeros in A by $\nu(A)$.

Remark 6.3. $|\top(A)| = 2^{n^2-n-\nu(A)}$ *for* $A \in M_n^N$.

We can describe the relation between the orthogonal sets of normal matrices, using the sets of minimally orthogonal matrices.

Corollary 6.4. *Let $\mathcal{S}, \mathcal{S}' \subseteq M_n^N$ and $\mathrm{b}(\mathcal{S}) \subseteq \mathrm{b}(\mathcal{S}')$. Then $\mathrm{or}(\mathcal{S}) \subseteq \mathrm{or}(\mathcal{S}')$.*

Proof. Follows from Corollary 2.25. □

Lemma 6.5 (Connection between or and b)**.** *Let $\mathcal{S}, \mathcal{S}' \subseteq M_n^N$. Then $\mathrm{or}(\mathcal{S}) = \mathrm{or}(\mathcal{S}')$ if and only if $\mathrm{b}(\mathcal{S}) = \mathrm{b}(\mathcal{S}')$.*

Proof. The sufficiency follows from Corollary 6.4. Now suppose that $\mathrm{b}(\mathcal{S}) \neq \mathrm{b}(\mathcal{S}')$. Then without loss of generality there exists $A \in \mathrm{b}(\mathcal{S})$ such that $A \notin \mathrm{b}(\mathcal{S}')$. If there exists $B \in \mathrm{b}(\mathcal{S}')$ such that $B < A$, then $B \notin \mathrm{or}(\mathcal{S})$, hence $\mathrm{or}(\mathcal{S}) \neq \mathrm{or}(\mathcal{S}')$. Using $A \notin \mathrm{b}(\mathcal{S}')$, if there is no $B \in \mathrm{b}(\mathcal{S}')$ such that $B < A$, then $A \notin \mathrm{or}(\mathcal{S}')$, hence $\mathrm{or}(\mathcal{S}) \neq \mathrm{or}(\mathcal{S}')$. □

Lemma 6.6. *Let $A, B \in M_n^N$ and $C \in M_n^N$ be indicator matrix of (A, B). Then the following hold*

1. $\mathrm{b}(A) \cup \mathrm{b}(B) \subseteq \mathrm{or}(A \oplus B)$,
2. $\mathrm{b}(A) \cup \mathrm{b}(B) \subseteq \mathrm{or}(AB)$, $\mathrm{b}(A) \cup \mathrm{b}(B) \subseteq \mathrm{or}(BA)$ *and* $\mathrm{b}(A) \cup \mathrm{b}(B) \subseteq \mathrm{or}(C)$,
3. $\bigcup_i \mathrm{b}(A_i) \subseteq \mathrm{or}(A)$, *where* $A = \bigoplus_i A_i$,
4. $\bigcup_i \mathrm{b}(A_i) \subseteq \mathrm{or}(A)$, *where* $A = \bigodot_i A_i$.

Proof. Follows from Lemma 2.20 and $A, B \leq A \oplus B \leq AB, BA$. □

Remark 6.7. *In Lemma 6.6 we cannot change* or *to* b*, as the following example shows.*

Example 6.8. 1. *Put $A = Z$ and $B = U$, then* $\mathrm{b}(A) = \{U\},\ \mathrm{b}(B) = \{Z\},\ \mathrm{b}(A \oplus B) = \mathrm{b}(Z) = \{U\}$.

2. *Put $A = B = N \neq Z \in M_n^N$, $N^2 = Z$, then* $U \notin \mathrm{b}(N) = \mathrm{b}(A) = \mathrm{b}(B),\ \mathrm{b}(AB) = \mathrm{b}(Z) = \{U\}$.

An interesting case is $|\,\mathrm{b}(A)| = 1$, because then $\mathrm{or}(A) = \top(B)$ is a sublattice, where $\mathrm{b}(A) = \{B\}$.

Theorem 6.9. *Let $A = [a_{ij}] \in M_n^N$. Then the following statements are equivalent:*
1) $|\,\mathrm{b}(A)| = 1$, i.e., there exists $B \in M_n^N$ such that $\mathrm{or}(A) = \top(B)$.
2) there exists only one minimal cover of graph $\Gamma(A)$.
3) for each (i, j) at least one of the following holds:
a) $(i, j) \in \mathrm{E}(\Gamma(A))$.
b) the outdegree of the vertex i is zero in $\Gamma(A)$.
c) the indegree of the vertex j is zero in $\Gamma(A)$.
d) there exist k, m such that $(i, k),\ (m, j) \in \mathrm{E}(\Gamma(A))$, and the outdegree of vertex k and the indegree of vertex m are zero in $\Gamma(A)$.
4) for each (i, j) at least one of the following holds:
a) $a_{ij} = 0$.
b) the i–th row of A is almost nowhere zero.
c) the j–th column of A is almost nowhere zero.
d) there exist $k, m \in [n]$ such that $a_{ik} = a_{mj} = 0$, and the k–th row and m–th column of A are almost nowhere zero.

Proof. Note that 1) is equivalent to 2) by Lemma 5.14, and 3) is equivalent to 4) by Definition 5.2. It is enough to prove that 1) is equivalent to 4).

Assume that 4) holds for A. By Item 2 of Corollary 3.6, it is sufficient to prove that $F(A) = (f_{ij}) \in \mathrm{or}(A)$. Let $C = [c_{ij}] \in M_n^N$ be the indicator matrix of the pair $(A, F(A))$. Show that for all (i, j) we have $c_{ij} = 0$, i.e. $C = Z$. Fix (i, j). By the hypothesis of 4), at least one of a), b), c), d) holds for (i, j).

a. If $a_{ij} = 0$, then by Lemma 2.15 $c_{ij} = 0$.

b. If the i–th row of A is almost nowhere zero, then $f_{ij} = 0$, hence by Lemma 2.15 $c_{ij} = 0$.

c. Similarly if the j–th column of A is almost nowhere zero, then $c_{ij} = 0$.

d. If there exist $k, m \in [n]$ such that $a_{ik} = a_{mj} = 0$, and the k–th row and the m–th column of A are almost nowhere zero, then $f_{kj} = f_{im} = 0$, so the definition of tropical multiplication yields $c_{ij} = 0$.

Thus, $\mathrm{b}(A) = \{F(A)\}$ and $|\,\mathrm{b}(A)| = 1$.

We prove now the contrapositive. Assume, there is (i, j) such that neither a), b), c) nor d) of 4) hold. Negations of a), b), c) yield $a_{ij} \neq 0, \mathrm{nr}(A, i, \emptyset) \geq 1, \mathrm{nc}(A, j, \emptyset) \geq 1$. Since $a_{ij} \neq 0$, then $i \neq j$ by normality of A. Without loss of generality the negation of d) yields that for all $k \in [n]$ with $a_{ik} = 0$ the k–th row is not almost nowhere zero, i.e. $\mathrm{nr}(A, k, \emptyset) \geq 1$. Let us find at least 2 elements in $\mathrm{b}(A)$.

I. By Item 1 of Lemma 4.3, A is orthogonal to E_{ij}. Choose $B = [b_{pq}]$ to be a minimal element in $\mathrm{or}(A) \cap \perp(E_{ij})$. Then $B \in \mathrm{b}(A)$, by Lemma 2.27.

II. Since $(AB)_{ij} = 0$, then there exists $k \in [n]$ such that $a_{ik} = b_{kj} = 0$, but $a_{ij}, b_{ij} \neq 0$, hence $k \neq i, j$. Since $a_{ik} = 0$, it follows by the negation of d) that $\mathrm{nr}(A, k, \emptyset) \geq 1$. Then, by Item 1 of Lemma 4.3, A is orthogonal to E_{kj}. Choose $B' = [b'_{pq}]$ to be a minimal element in $\mathrm{or}(A) \cap \perp(E_{kj})$. Then $B' \in \mathrm{b}(A)$, by Lemma 2.27. Since $b_{kj} = 0$ and $b'_{kj} \neq 0$, then $B \neq B'$ and $|\,\mathrm{b}(A)| > 1$, which completes the proof. □

Corollary 6.10. *For $A \in M_n^N$, we have $|\,\mathrm{b}(A)| = 1$ if and only if $\mathrm{b}(A) = \{F(A)\}$.* □

Definition 6.11. Let $A \in M_n^N, i, j \in [n]$. Position (i,j) is called *fuzzy* if none of conditions a), b), c), d) in 4) of Theorem 6.9 holds.

The fuzzy positions are the source of existing more than one minimally orthogonal matrix to A. We can estimate the cardinality of $\mathrm{b}(A)$.

Corollary 6.12. *Let $A \in M_n^N$. Then*

$$|\,\mathrm{b}(A)| \leq \prod_{(i,j)^*} (1 + \mathrm{nr}(A, i, \emptyset)\,\mathrm{nc}(A, j, \emptyset)),$$

where $(i,j)^*$ are fuzzy positions.

Example 6.13 (Lack of reciprocity for $\mathrm{b}(S)$)**.** *Let $n \geq 4$. $\mathrm{b}(U_{1n}^{2,n-1}) = \{E_{1\,n-1} + E_{1\,n} + E_{2\,n-1} + E_{2\,n}\}$, by Item 3 of Corollary 4.4.*

On the other hand, by Item 1 of Corollary 4.4 we have $U_{1n}^{2,n-1} \in \mathrm{b}(E_{2n})$, but $E_{2n} \notin \mathrm{b}(U_{1n}^{2,n-1})$.

Remark 6.14. *Note that Item 2 of Lemma 3.4 does not hold for each $B \in \mathrm{b}(A)$. Indeed, according to Example 6.13, $U_{1n}^{2,n-1} \in \mathrm{b}(E_{2n})$, but $\mathrm{row}(1, U_{1n}^{2,n-1}) \neq \mathrm{row}(1, U)$.*

Lemma 6.15. *For $A \in M_n^N$, we have*

1. *$\ulcorner A \in \mathrm{or}(A)$ and $\top(\ulcorner A) \subseteq \mathrm{or}(A)$,*
2. *there exists $B \in \mathrm{b}(A)$ such that $B \in \bot(\ulcorner A)$. In particular, if $|\,\mathrm{b}(A)| = 1$, then $\mathrm{b}(A) \subseteq \bot(\ulcorner A)$.*

Proof. Item 1 follows from Corollaries 2.16, 2.22. Next prove Item 2. By Item 1, $\ulcorner A \in \mathrm{or}(A)$. Choose B to be a minimal element in $\mathrm{or}(A) \cap \bot(\ulcorner A)$. Then $B \in \mathrm{b}(A)$, by Lemma 2.27. □

Lemma 6.16 (Basis for R_i and C_j)**.** $\mathrm{b}(R_i) = \{\ulcorner R_i\}$ *and* $\mathrm{b}(C_j) = \{\ulcorner C_j\}$.

Proof. Follows from Item 1 of Lemma 6.15 and Item 2 of Corollary 3.6. □

Corollary 6.17. *The basis for the matrices R_i, C_j, U_{ij} and U_{ij}^{kl} (with $i \neq k$ and $j \neq l$) is a singleton.*

Proof. Follows from Lemma 6.16 and Items 2, 3 of Corollary 4.4. □

7 Orthogonal set and basis for matrix subsets

We work in $(\mathcal{P}(M_n^N), \cup, \cap)$ where $\oplus, \mathrm{or}, \mathrm{Min}$ and $\mathrm{b} = \mathrm{Min} \circ \mathrm{or}$ make sense (see Notation 2.28).

Notation 7.1. For subsets $\mathcal{S}, \mathcal{S}' \subseteq M_n^N$, consider the tropical sum $\mathcal{S} \oplus \mathcal{S}' := \{A \oplus B \mid A \in \mathcal{S},\ B \in \mathcal{S}'\}$.

Lemma 7.2. *For $A, B \in M_n^N$, we have* $\mathrm{Min}(\mathrm{or}(A) \cap \mathrm{or}(B)) = \mathrm{b}(\{A, B\}) = \mathrm{Min}(\mathrm{b}(A) \oplus \mathrm{b}(B))$.

Proof. By Corollary 2.25,

$$\mathrm{or}(A) \cap \mathrm{or}(B) = \left(\bigcup_{A_i \in \mathrm{b}(A)} \top(A_i) \right) \cap \left(\bigcup_{B_i \in \mathrm{b}(B)} \top(B_i) \right) = \bigcup_{\substack{A_i \in \mathrm{b}(A) \\ B_j \in \mathrm{b}(B)}} \top(A_i) \cap \top(B_j) = \bigcup_{i,\,j} \top(A_i \oplus B_j).$$

Hence $\mathrm{Min}(\mathrm{or}(A) \cap \mathrm{or}(B)) = \mathrm{Min}(\{A_i \oplus B_j \mid A_i \in \mathrm{b}(A),\ B_j \in \mathrm{b}(B)\})$. □

Corollary 7.3. $\mathrm{Min}(\bigcap_i \mathrm{or}(A_i)) = \mathrm{b}(\{A_i\}_i) = \mathrm{Min}\left(\bigoplus_i \mathrm{b}(A_i)\right)$.

Now we can get some information about iterated orthogonal sets of a matrix.

Corollary 7.4. *For $A \in M_n^N$, $j \in \mathbb{N}$, we have* $\mathrm{b}(\mathrm{or}^j(A)) = \mathrm{Min}\left(\bigoplus_{B \in \mathrm{or}^{j-1}(A)} \mathrm{b}(B) \right)$.

Proof. By definition $\mathrm{or}^j(A) = \mathrm{or}(\mathrm{or}^{j-1}(A)) = \bigcap_{B \in \mathrm{or}^{j-1}(A)} \mathrm{or}(B)$. □

8 Two notions of minimality compared

Recall a Definition and a Corollary from [2].

Definition 8.1 ([2, Definition 4.1]). Let Θ_n be the minimal number of off–diagonal zero entries among all pairs of orthogonal matrices in M_n^N, i.e.,

$$\Theta_n = \min_{A,B \in M_n^N} \{\nu(A) + \nu(B) \ : \ AB = Z = BA\}.$$

Let $A, B \in M_n^N$. The pair (A, B) is called *minimal* if it realizes the value of Θ_n, i.e., if $\nu(A) + \nu(B) = \Theta_n$ and $AB = Z = BA$.

Corollary 8.2 ([2, Corollary 4.31]). *If $n \geq 2$, $n \neq 4$, then $\Theta_n = 4n - 6$. Besides, $\Theta_4 = 8$.*

The question arises: what is the relation between $A \in \mathrm{b}(B)$ and $B \in \mathrm{b}(A)$ and (A, B) is a minimal pair? We have two notions of minimality: do they coincide?

Lemma 8.3. *If (A, B) is a minimal pair, then $A \in \mathrm{b}(B)$ and $B \in \mathrm{b}(A)$.*

Proof. Suppose the pair (A, B) is minimal and $A \notin \mathrm{b}(B)$. Then, there exists $A' < A$ such that $A'B = Z = BA'$ and, clearly, $\nu(A') + \nu(B) < \nu(A) + \nu(B)$, a contradiction. Similarly for $B \notin \mathrm{b}(A)$. □

The following example shows that the converse is not true.

Example 8.4. *Let $n = 6$, $A = R_1 + R_2 + R_3$, $B = R_4 + R_5 + R_6$. Then, by Item 1 of Lemma 6.15 and Item 2 of Corollary 3.6, $A \in \mathrm{b}(B)$ and $B \in \mathrm{b}(A)$, but the pair (A, B) is not minimal by Corollary 8.2 since $\nu(A) + \nu(B) = n^2 - n > 4n - 6$.*

If C is the indicator matrix of the pair (A, B), then it can happen that $\mathrm{b}(A) \cap \mathrm{b}(B) \not\subseteq \mathrm{b}(C)$, as Item 2 of Example 6.8 shows. Several other properties of intersections for orthogonal sets are presented below.

Example 8.5. *The following 5×5 matrices were considered previously in [2]:*

$$A_1 = \begin{bmatrix} 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix}, \ A_5 = \begin{bmatrix} 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 & -1 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & -1 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \end{bmatrix}, \ B_3 = \begin{bmatrix} 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \end{bmatrix}, \ B_4 = \begin{bmatrix} 0 & -1 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

Note that

1. $\{A_3\} = \mathrm{b}(B_1)$, $\{B_1\} = \mathrm{b}(A_3)$, (A_3, B_1) *is minimal pair and* $\nu(A_3) = \nu(B_1) = 2n - 3$,
2. $A_1, A_5 \in \mathrm{b}(B_4)$, $\{B_4\} = \mathrm{b}(A_1)$, $|\,\mathrm{b}(B_4)| > 1$, (A_1, B_4) *is minimal pair and* $\nu(A_1) = 2n - 2$, $\nu(B_4) = 2n - 4$,
3. $A_2 \in \mathrm{b}(B_2)$, $B_2, B_3 \in \mathrm{b}(A_2)$, $|\,\mathrm{b}(A_2)| > 1$, $|\,\mathrm{b}(B_2)| > 1$, (A_2, B_2) *is minimal pair and* $\nu(A_2) = \nu(B_2) = 2n - 3$.

It holds that $\mathrm{or}(A_4) \subset \mathrm{or}(A_3) \subset \mathrm{or}(A_1)$ *and* $\mathrm{or}(A_4) \subset \mathrm{or}(A_2) \subset \mathrm{or}(A_1)$. *Then* $\mathrm{or}(A_2) \bigcap \mathrm{or}(A_3) \neq \emptyset$, *but neither* $\mathrm{or}(A_2) \subset \mathrm{or}(A_3)$ *nor* $\mathrm{or}(A_3) \subset \mathrm{or}(A_2)$. *Moreover,* $\nu(B_1) = 2n - 3$, $|\,\mathrm{or}(A_3)| = 2^{n^2 - n - \nu(B_1)} = 2^{n^2 - 3n + 3}$ *and* $|\,\mathrm{or}(A_2)| > 2^{n^2 - 3n + 3}$, *because* $B_2, B_3 \in \mathrm{b}(A_2)$. *Also,* $A_2 \notin \mathrm{b}(B_3)$.

9 Geometric interpretation of normal matrices

Why do we care about normal matrices? One recent application of normal matrices to the theory of polytopes is the following: convex alcoved polytopes in Euclidean n–space can be represented (uniquely up to translations) by idempotent (with respect to tropical multiplication) normal matrices, see [10, 11, 14]. Alcoved polytopes in $\mathbb{R}^n$ are those polytopes with only two types of facet equations: $x_i = constant$ or $x_i - x_j = constant, i, j \in [n]$, $i \neq j$. Slightly more general than normal matrices are Kleene stars. These are used in the study of max–cones, see [12, 13].

The reader should notice that the essential orthogonality questions of a matrix $\mathbf{A} = [\mathbf{a}_{ij}] \in M_n^N(\mathbb{R}_{\leq 0} \cup \{-\infty\})$ can be studied via its image $A = [a_{ij}] \in M_n^N(\{0, -1\})$, where $a_{ij} = \begin{cases} 0, \text{if } \mathbf{a}_{ij} = 0, \\ -1, \text{otherwise.} \end{cases}$ It is known that a normal matrix $\mathbf{A}$ describes a complex t-conv$(\mathbf{A})$ of convex alcoved polytopes of impure dimension, in general (see [6]). The set t-conv$(\mathbf{A})$ is, by definition, the subset of $\mathbb{R}^n$ consisting of all tropical linear combinations of the columns of $\mathbf{A}$. Consider two normal matrices $\mathbf{A}$ and $\mathbf{B}$ of equal size n. The product $\mathbf{AB}$ is normal, but different from $\mathbf{BA}$ (in general) so we have two complexes: t-conv$(\mathbf{AB})$ and t-conv$(\mathbf{BA})$. As in classical linear algebra, a matrix $\mathbf{A}$ represents a map $f_{\mathbf{A}}$ (see [4, 9]). The complex t-conv$(\mathbf{AB})$ is nothing but the image $f_{\mathbf{A}}(\text{t-conv}(\mathbf{B}))$. To ask whether $\mathbf{A}$ and $\mathbf{B}$ annihilate each other, i.e., $\mathbf{AB} = Z = \mathbf{BA}$, means to ask whether $f_{\mathbf{A}}(\text{t-conv}(\mathbf{B})) = \{0\} = f_{\mathbf{B}}(\text{t-conv}(\mathbf{A}))$.

Below we present this geometric look in three examples, with $n = 3$. Notice that the figures are 2–dimensional, because we only represent the intersection of sets with the plane $\{z = 0\}$. This is so because the set t-conv$(\mathbf{A})$ is closed under tropical multiplication by scalars and, thus, the intersection of t-conv$(\mathbf{A})$ with $\{z = 0\}$ determines the whole set t-conv$(\mathbf{A})$. To draw pictures in $\{z = 0\}$, we use matrices $\mathbf{A}_0$ (denoted with a zero subscript) with zero last row, obtained from the given matrices $\mathbf{A}$ by scalar multiplication of columns, whence t-conv$(\mathbf{A}_0) =$ t-conv$(\mathbf{A})$. It is well known (and can be seen in the figures) that, if $\mathbf{A}$ is normal, then t-conv$(\mathbf{A})$ convex if and only if $\mathbf{A}$ normal idempotent.

Explanation for the figures: a square marks the origin in $\mathbb{R}^2 = \{z = 0\}$, round dots mark generators of t-conv$(\mathbf{A})$ and t-conv$(\mathbf{B})$, i.e., they are the columns of $\mathbf{A_0}$ and $\mathbf{B_0}$. The three tropical lines are shown dashed, dotted and dashed–doted. Sets t-conv$(\mathbf{A}) \cap \{z = 0\}$, t-conv$(\mathbf{B}) \cap \{z = 0\}$ are shown in solid black. Overlapping rays of different tropical lines are marked with rhombi.

Example 9.1. $\mathbf{A} = \begin{bmatrix} 0 & 0 & -3 \\ -1 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix}$. *Since* $\mathbf{A} \oplus \mathbf{B} = Z$*, then we know that* $\mathbf{A}, \mathbf{B}$ *are mutually orthogonal, by Corollary 2.16.*

The set t-conv$(\mathbf{A})$ *is the image of the map* $f_{\mathbf{A}} : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \mathbf{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \max\{x, y, z-3\} \\ \max\{x-1, y, z-4\} \\ \max\{x, y, z\} \end{bmatrix}.$$

We consider the intersection of t-conv$(\mathbf{A})$ *with* $\{z = 0\}$ *(see Figure 1) and we study the map (also denoted* $f_{\mathbf{A}}$*) induced by* $f_{\mathbf{A}}$ *on* $\{z = 0\}$*, defined as follows (see Figure 2):*

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \max\{x, y, -3\} - \max\{x, y, 0\} \\ \max\{x-1, y, -4\} - \max\{x, y, 0\} \\ 0 \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -\max\{x, y, 0\} \\ -\max\{x, y, 0\} \\ -\max\{x, y, 0\} \end{bmatrix} = (-\max\{x, y, 0\}) \mathbf{A} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The image of a point $[x, y, 0]^T$ *by* $f_{\mathbf{A}}$ *depends on the values attained by*

$$\max\{x, y, -3\} = x \oplus y \oplus (-3)$$

$$\max\{x-1, y, -4\} = (-1) \odot x \oplus y \oplus (-4)$$

$$\max\{x, y, 0\} = x \oplus y \oplus 0.$$

These expressions represent three tropical lines. These lines split $\{z = 0\}$ *in regions* $R_1, R_2, \ldots R_s$*, with* $s \leq 10$ *(see Figure 2). A tropical line is the union of three rays. Clearly, there exist* 10 *regions if and only if (a) no two* *lines have overlapping rays and (b)* $\dim \text{t-conv}(\mathbf{A} \cap \{z=0\}) = 2$*. The image of a point* $[x,y,0]^T$ *depends on the region* R_j *it belongs to. We have*

$$[x,y,0]^T \mapsto \begin{cases} [0,-1,0]^T, \ \textit{if } 0 \le x \textit{ and } y \le x-1; \textit{region } R_1 \\ [0,y-x,0]^T, \ \textit{if } 0 \le x \textit{ and } x-1 \le y \le x; \textit{region } R_2 \\ [0,0,0]^T, \ \textit{if } x \le y \textit{ and } 0 \le y; \textit{region } R_3 \\ [y,y,0]^T, \ \textit{if } -3 \le y \le 0 \textit{ and } x \le y; \textit{region } R_4 \\ [-3,y,0]^T, \ \textit{if } -4 \le y \le -3 \textit{ and } x \le -3; \textit{region } R_5 \\ [-3,-4,0]^T, \ \textit{if } x \le -3 \textit{ and } y \le -4; \textit{region } R_6 \\ [x,x,0]^T, \ \textit{if } -3 \le x \le 0 \textit{ and } y \le x-1; \textit{region } R_7 \\ [x,y,0]^T, \ \textit{if } -3 \le x \le 0 \textit{ and } x-1 \le y \le x; \textit{region } R_8 = \text{t-conv}(\mathbf{A}) \cap \{z=0\}. \end{cases}$$

We can do analogously for $\mathbf{B}$ *and the map* $f_{\mathbf{B}}$ *(see Figures 1 and 2). In order to picture* $\text{t-conv}(\mathbf{B}) \cap \{z=0\}$*, we use* $\mathbf{B}_0 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *because* $\text{t-conv}(\mathbf{B}_0) = \text{t-conv}(\mathbf{B})$*. Notice* $\mathbf{A}_0 = \mathbf{A}$*.*

Since $\text{t-conv}(\mathbf{B}) \cap \{z=0\}$ *is contained in region* R_3 *above, then* $f_{\mathbf{A}}(\text{t-conv}(\mathbf{B}) \cap \{z=0\}) = [0,0,0]^T$*. Since* $\text{t-conv}(\mathbf{A}) \cap \{z=0\}$ *is contained in the region* R_7 *in Figure 2 right, and* $f_{\mathbf{B}}$ *maps every point in* R_7 *to the origin, then* $f_{\mathbf{B}}(\text{t-conv}(\mathbf{A}) \cap \{z=0\}) = [0,0,0]^T$*. See Figure 3 for* $\mathbf{AB} = Z = \mathbf{BA}$*.*

Example 9.2. $\mathbf{A} = \begin{bmatrix} 0 & 0 & -3 \\ -1 & 0 & -4 \\ 0 & a_{32} & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & b_{23} \\ -1 & -2 & 0 \end{bmatrix}$*, with* $a_{32} \le 0$*,* $b_{23} \le 0$*.*

We have $\mathbf{BA} = [r_{ij}]$ *with* $r_{ij} = \begin{cases} \max\{-3, b_{23}\}, \ \textit{if } (i,j) = (2,3), \\ \max\{-1, a_{32}\}, \ \textit{if } (i,j) = (3,2), \\ 0, \ \textit{otherwise.} \end{cases}$ *If we choose* $a_{32} < 0$ *or* $b_{23} < 0$*, then* $\mathbf{BA} \neq Z$*. For instance, take* $a_{32} < 0$ *and* $\mathbf{B}$ *as in Example 9.1. Then* $\text{t-conv}(\mathbf{A}) \cap \{z=0\}$ *is not contained in the region* R_7 *of Figure 2 right, because* $\text{t-conv}(\mathbf{A}) \cap \{z=0\}$ *contains a segment joining the origin to point* $[-a_{32}, -a_{32}, 0]^T$*. See Figure 7 for the union graph* $\Gamma(A,B)$*. The reader can produce the figure of* $\text{t-conv}(\mathbf{A} \cap \{z=0\})$ *from the matrix* $\mathbf{A}_0$*, for this example.*

Example 9.3. $\mathbf{A} = \begin{bmatrix} 0 & 0 & -3 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & b_{23} \\ -1 & -2 & 0 \end{bmatrix}$*, with* $b_{23} \le 0$*. By Corollary 2.16,* $\mathbf{A}, \mathbf{B}$ *are mutually orthogonal, because* $\mathbf{A} \oplus \mathbf{B} = Z$*.*

Notice that $\text{t-conv}(\mathbf{A}) \cap \{z=0\}$ *is contained in* $R_6 \cup (R_8 \cap R_9)$ *and it is mapped by* f_B *to the origin (see Figure 5). Also,* $\text{t-conv}(\mathbf{B}) \cap \{z=0\}$ *is contained in* $R_3 \cup (R_6 \cap R_7)$ *and it is mapped by* f_A *to the origin (see Figure 5). The result is* $\mathbf{AB} = Z = \mathbf{BA}$*, represented in Figure 6.*

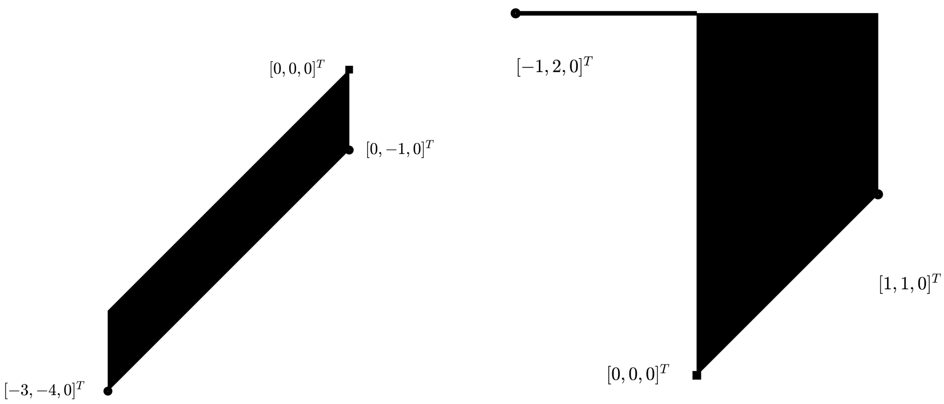

Figure 1: Example 9.1: the sets $\text{t-conv}(\mathbf{A}) \cap \{z = 0\}$ (left) and $\text{t-conv}(\mathbf{B}) \cap \{z = 0\}$ (right).

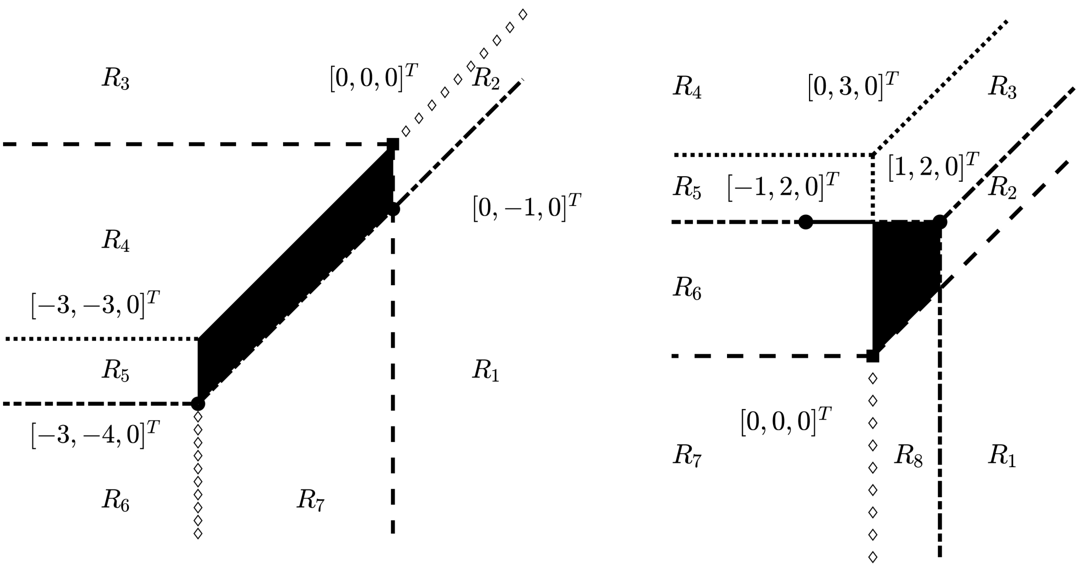

Figure 2: Example 9.1: the regions associated to $f_{\mathbf{A}}$ (left) and $f_{\mathbf{B}}$ (right). Region R_8 is the black area (left) and region R_9 is the black area (right).

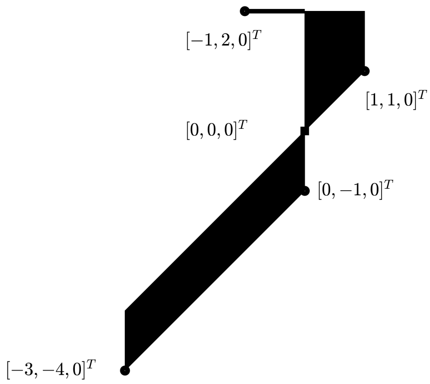

Figure 3: Example 9.1: the sets $\text{t-conv}(\mathbf{A}) \cap \{z = 0\}$ and $\text{t-conv}(\mathbf{B}) \cap \{z = 0\}$ together.

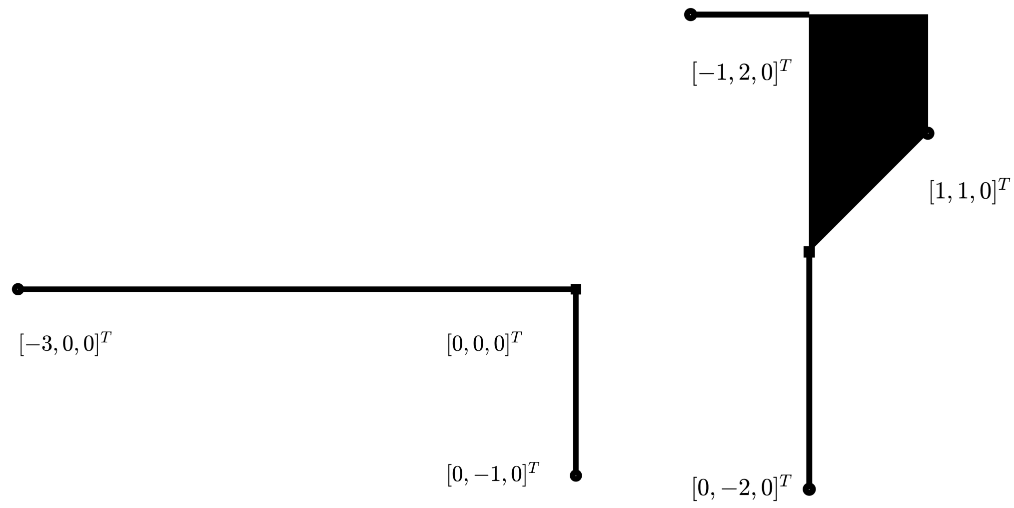

Figure 4: Example 9.3: the sets $\text{t-conv}(\mathbf{A}) \cap \{z = 0\}$ (left) and $\text{t-conv}(\mathbf{B}) \cap \{z = 0\}$ (right).

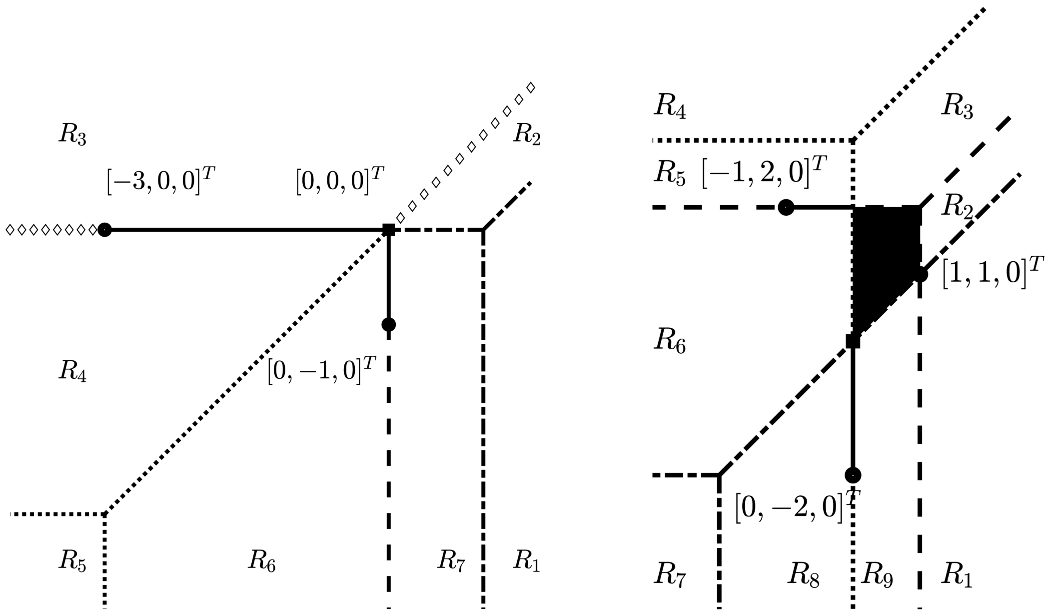

Figure 5: Example 9.3: the regions associated to $f_{\mathbf{A}}$ (left) and $f_{\mathbf{B}}$ (right). Region R_{10} is the black area (right).

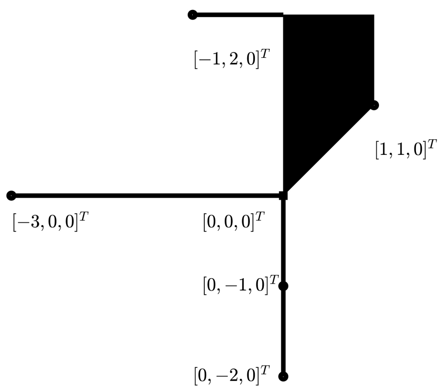

Figure 6: Example 9.3: the sets $\text{t-conv}(\mathbf{A}) \cap \{z=0\}$ and $\text{t-conv}(\mathbf{B}) \cap \{z=0\}$ together.

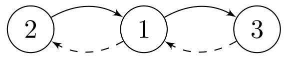

Figure 7: Example 9.2: the union graph $\Gamma(A, B)$, $A, B \in M_3^N$, $a_{32} = -1 = b_{23}$, red edges in the graph are marked by dashed lines.

References

[1] I.B. Kozhukhov, A.V. Mikhalev, *Polygons over semigroups: selected questions of structure theory*, Journal of Math. Sci. (New York), to appear.

[2] B. Bakhadly, A. Guterman and M.J. de la Puente, *Orthogonality for* $(0, -1)$ *tropical normal matrices*, Special matrices, **8** (2020), 40–60.

[3] G. Birkhoff, Lattice Theory, Providence: Rhode Island, 1967, 568 pp.

[4] P. Butkovič, *Simple image set of* $(\max, +)$ *linear mappings*, Discrete Appl. Math. **105**, (2000), 73–86.

[5] P. Butkovič, *Max–plus linear systems: theory and algorithms*, 2010, Springer.

[6] M. Develin, B. Sturmfels, *Tropical convexity*, *Doc. Math.* **9**, (2004) 1–27, Erratum in Doc. Math. **9** (electronic), (2004) 205–206.

[7] P. Crawley, R.P. Dilworth, Algebraic theory of lattices, Prentice Hall, 1973.

[8] G.L. Litvinov, S.N. Sergeev, (eds.) *Tropical and idempotent mathematics*, Moscow 2007, American Mathematical Society, Contemp. Math. **495** (2009).

[9] M.J. de la Puente, *Tropical linear maps on the plane*, Linear Algebra Appl. **435**, n.7, (2011) 1681–1710.

[10] M.J. de la Puente, *On tropical Kleene star matrices and alcoved polytopes*, Kybernetika, **49**:6 (2013), 897–910.

[11] M.J. de la Puente, *Quasi-Euclidean classification of Alcoved Convex Polyhedra, Linear and Multilinear Algebra*, **67**, (2019), doi 10.1080/03081087.2019.1572065

[12] S. Sergeev, *Multiorder, Kleene stars and cyclic projectors in the geometry of max cones*, Contemp. Math. **495** (2009) 317-342.

[13] S. Sergeev, H. Scheneider and P. Butkovič, *On visualization, subeigenvectors and Kleene stars in max algebra*, Linear Algebra Appl. **431** 2395–2406, (2009).

[14] N.M. Tran, *Enumerating polytropes*, J. Combin. Theory Ser. A, **151** (2017), 1–22.

[15] M. Yoeli, *A note on a generalization of boolean matrix theory*, Amer. Math. Monthly **68**:6 (1961) 552–557.

[16] B. Yu, X. Zhao and L. Zeng, *A congruence on the semiring of normal tropical matrices*, Linear Alg. Appl. **555** (2018), 321–335.