IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 63-3, 67-1, 68-1, 68-3. Solutions are invited to Problem 63-1; to all of the problems from issue 65; to parts (a) and (b) of Problem 66-2; to Problem 66-4; Problems 68-2 and 68-4 and to all of the new problems from the present issue 69.

Problem 63-3: Products of Rectangular Circulant Matrices

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An $n\times n$ matrix C is said to be circulant if $c_{ij}=c_{kl}$ whenever $j-i=l-k \mod n$. Let m, n and q be positive integers. We can define a rectangular circulant matrix as follows: An $m\times n$ matrix C is said to be a rectangular circulant if $c_{ij}=c_{kl}$ whenever $j-i=l-k \mod \gcd(m,n)$. As an example, a 6×9 rectangular circulant looks like

$$\begin{bmatrix} a & b & c & a & b & c & a & b & c\\ c & a & b & c & a & b & c & a & b\\ b & c & a & b & c & a & b & c & a\\ a & b & c & a & b & c & a & b & c\\ c & a & b & c & a & b & c & a & b\\ b & c & a & b & c & a & b & c & a \end{bmatrix}.$$

Let A be an $m\times n$ rectangular circulant and B be an $n\times q$ rectangular circulant.

(a) Show that AB is also a rectangular circulant.

(b) What is the maximum rank that AB can have?

(c) Show that the Moore-Penrose inverse of a rectangular circulant matrix is a rectangular circulant matrix.

Solution 63-3 by M.J. de la Puente, *Universidad Complutense de Madrid, Spain*, mpuente@ucm.es

Let $\mathbb{K}$ be a field and let m and n be positive integers. Let I_m be the identity matrix of size m and $J_{m\times n}$ be the all-ones matrix of size $m\times n$ defined over $\mathbb{K}$. Let $\otimes$ denote the Kronecker product of matrices.

The following characterization of the rectangular circulant as a Kronecker product will be used in all three parts of the solution.

Proposition 1 (Characterization of rectangular circulants). *Let $m,n\in\mathbb{N}$ and let $d=\gcd(m,n)$, $m'=\frac{m}{d}$, and $n'=\frac{n}{d}$. If $C\in M_{m\times n}(\mathbb{K})$ is a rectangular circulant, then $C=J_{m'\times n'}\otimes C'$, with C' a square circulant of size d. Conversely, for any all-ones matrix J and any square circulant matrix S, the matrix $J\otimes S$ is a rectangular circulant.*

Proof. For the first statement, take C' to be the $d\times d$ leading principal submatrix of C. The definition of a rectangular circulant implies that C can expressed as a block matrix

$$C=\begin{bmatrix} C' & \overset{n'}{\cdots} & C'\\ {}^{m'}\vdots & & \vdots\\ C' & \cdots & C' \end{bmatrix}.$$

The second statement is straightforward. □

(a) We begin with the following lemma:

Lemma 2 (Refinement of square circulant is block square circulant). *Let S be square circulant of size n, let d divide n, and let $n'=\frac{n}{d}$. Then S is a block circulant square matrix with square blocks of uniform size, i.e.,*

$$S=\begin{bmatrix} S_{11} & \cdots & S_{1n'}\\ \vdots & & \vdots\\ S_{n'1} & \cdots & S_{n'n'} \end{bmatrix},$$

with square blocks S_{ij} of size d such that $S_{ij} = S_{kl}$ whenever $j - i = l - k \mod n'$. Moreover, the sum $\overline{S} := \sum_{i=1}^{n'} S_{ij}$ is the same for all $j \in \{1, 2, \ldots, n'\}$ and $\overline{S}$ is square circulant. Besides, $\overline{S} = \sum_{j=1}^{n'} S_{ij}$ is the same for all $i \in \{1, 2, \ldots, n'\}$.

Proof. The (k, l)-entry of $\overline{S}$ is $\sum_r s_{kr}$, where the sum is extended to all $r \equiv l \mod d$. □

Notice that the blocks S_{ij} are Toeplitz but fail to be square circulant, in general.

Example 3. Let $S = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ s_6 & s_1 & s_2 & s_3 & s_4 & s_5 \\ s_5 & s_6 & s_1 & s_2 & s_3 & s_4 \\ s_4 & s_5 & s_6 & s_1 & s_2 & s_3 \\ s_3 & s_4 & s_5 & s_6 & s_1 & s_2 \\ s_2 & s_3 & s_4 & s_5 & s_6 & s_1 \end{bmatrix}$, so that $n = 6$. Then if we take $d = 2$ in Lemma 2, we have $\overline{S} = \begin{bmatrix} s_1 + s_3 + s_5 & s_2 + s_4 + s_6 \\ s_2 + s_4 + s_6 & s_1 + s_3 + s_5 \end{bmatrix}$, while taking $d = 3$ gives $\overline{S} = \begin{bmatrix} s_1 + s_4 & s_2 + s_5 & s_3 + s_6 \\ s_3 + s_6 & s_1 + s_4 & s_2 + s_5 \\ s_2 + s_5 & s_3 + s_6 & s_1 + s_4 \end{bmatrix}$.

Let $d_A = \gcd(m, n)$ and $d_B = \gcd(n, q)$ and let A' and B' be as given for A and B, respectively, by Proposition 1. Then, letting $d = \gcd(d_A, d_B)$, we have $d = \gcd(m, n, q)$, and we let $e_A = \frac{d_A}{d}$, $e_B = \frac{d_B}{d}$, $m' = \frac{m}{d}$, $n' = \frac{n}{d}$ and $q' = \frac{q}{d}$. Note that $\gcd(e_A, e_B) = 1$. By Lemma 2, A' and B' can be refined to block square circulants, namely as

$$A' = \begin{bmatrix} A'_{11} & \cdots & A'_{1e_A} \\ \vdots & & \vdots \\ A'_{e_A1} & \cdots & A'_{e_Ae_A} \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} B'_{11} & \cdots & B'_{1e_B} \\ \vdots & & \vdots \\ B'_{e_B1} & \cdots & B'_{e_Be_B} \end{bmatrix},$$

each with square blocks A'_{ij} and B'_{ij}, respectively, all of size d. Hence we may define n_A, m_A, n_B and m_B, as well as A_{ij} and B_{ij} for $i, j \in \{1, 2, \ldots, n'\}$, such that

$$A = \begin{bmatrix} A' & \overset{n_A}{\cdots} & A' \\ {}_{m_A}\vdots & & \vdots \\ A' & \cdots & A' \end{bmatrix} = \begin{bmatrix} A'_{11} & \cdots & A'_{1e_A} & \cdots & A'_{11} & \cdots & A'_{1e_A} \\ \vdots & & \vdots & & \vdots & & \vdots \\ A'_{e_A1} & \cdots & A'_{e_Ae_A} & \cdots & A'_{e_A1} & \cdots & A'_{e_Ae_A} \\ \vdots & & \vdots & & \vdots & & \vdots \\ A'_{11} & \cdots & A'_{1e_A} & \cdots & A'_{11} & \cdots & A'_{1e_A} \\ \vdots & & \vdots & & \vdots & & \vdots \\ A'_{e_A1} & \cdots & A'_{e_Ae_A} & \cdots & A'_{e_A1} & \cdots & A'_{e_Ae_A} \end{bmatrix} = [A_{ij}]_{i,j=1,2,\ldots,n'}$$

and

$$B = \begin{bmatrix} B' & \overset{n_B}{\cdots} & B' \\ {}_{m_B}\vdots & & \vdots \\ B' & \cdots & B' \end{bmatrix} = \begin{bmatrix} B'_{11} & \cdots & B'_{1e_B} & \cdots & B'_{11} & \cdots & B'_{1e_B} \\ \vdots & & \vdots & & \vdots & & \vdots \\ B'_{e_B1} & \cdots & B'_{e_Be_B} & \cdots & B'_{e_B1} & \cdots & B'_{e_Be_B} \\ \vdots & & \vdots & & \vdots & & \vdots \\ B'_{11} & \cdots & B'_{1e_B} & \cdots & B'_{11} & \cdots & B'_{1e_B} \\ \vdots & & \vdots & & \vdots & & \vdots \\ B'_{e_B1} & \cdots & B'_{e_Be_B} & \cdots & B'_{e_B1} & \cdots & B'_{e_Be_B} \end{bmatrix} = [B_{ij}]_{i,j=1,2,\ldots,n'}.$$

The right-most expressions in the two equations above give block decompositions of A and B, showing that A and B are block circulant square matrices, with all blocks square of size d. The number of such blocks in any row or column of A or B is $n' = n_Ae_A = n_Be_B$. Now block multiplication of A and B can be performed and we get

$$(AB)_{ij} = \sum_{k=1}^{n'} A_{ik}B_{kj} = \left(\sum_{l=1}^{e_A} A'_{il}\right)\left(\sum_{k=1}^{e_B} B'_{kj}\right), \quad i, j = 1, 2, \ldots, n',$$

where the last equality is due to the fact that e_A and e_B are coprime. Further, take the square circulant matrices $\overline{A'} := \sum_{l=1}^{e_A} A'_{il}$ and $\overline{B'} := \sum_{k=1}^{e_B} B'_{kj} \in M_d(\mathbb{K})$ (which do not depend on i and j) and get $(AB)_{ij} = (AB)_{11} = \overline{A'}\,\overline{B'} \in M_d(\mathbb{K})$. Therefore

$$AB = J_{m'\times q'} \otimes \overline{A'}\,\overline{B'} \tag{1}$$

proving that AB is rectangular circulant, by Proposition 1.

(b) In the proof of part (a), we took $d = \gcd(m, n, q)$ and derived the identity (1) where $\overline{A'}\,\overline{B'}$ is a square circulant of size d. There exist invertible matrices P, Q such that $AB = PUQ$, with

$$U = \begin{bmatrix} \overline{A'}\,\overline{B'} & 0_{d\times(n-d)} \\ 0_{(m-d)\times d} & 0_{(m-d)\times(n-d)} \end{bmatrix},$$

whence $\operatorname{rk}(AB) = \operatorname{rk}(\overline{A'}\,\overline{B'}) \le d$. Since one can choose A and B so that $\overline{A'}\,\overline{B'}$ is an invertible circulant the maximum rank that AB can be is $d = \gcd(m, n, q)$.

(c) Let $A \in M_{m\times n}(\mathbb{K})$ be rectangular circulant. Write $d = \gcd(m, n)$, $m' = \frac{m}{d}$, $n' = \frac{n}{d}$ and express $A = J_{m'\times n'} \otimes A'$, with A' square circulant of size d, as in proposition 1. We first state some easily proved statements about the Moore-Penrose inverse of $J_{m\times n}$, Kronecker products and circulant matrices respectively. It is easy to see that $J^+_{m\times n} = \frac{1}{mn} J_{n\times m}$. In section 2.6 in [2], it is shown that $(A \otimes B)^+ = A^+ \otimes B^+$. In p. 90 in [1] it is shown that the Moore-Penrose inverse of a square circulant is square circulant. Hence we have

$$A^+ = (J_{m'\times n'} \otimes A')^+ = \left(\frac{1}{m'n'} J_{n'\times m'}\right) \otimes (A')^+ = J_{n'\times m'} \otimes \frac{1}{m'n'}(A')^+$$

with $\frac{1}{m'n'}(A')^+$ square circulant, whence A^+ is rectangular circulant, by Proposition 1.

References

[1] P. J. Davis. *Circulant Matrices*. Chelsea Publishing, New York, 2nd edition, 1994.

[2] A. N. Langville and W. J. Stewart. The Kronecker product and stochastic automata networks. *J. Comput. Appl. Math.*, 167(2):429–447, 2004.

Problem 67-1: Integer Solutions of a Matrix Equation

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Let n be a positive integer, and let J_n be the $n \times n$ matrix all of whose entries are equal to 1.

(a) Show that there exists a matrix $X \in M_n(\mathbb{Z})$ such that $X^2 + X = J_n$ if and only if $n = m^2 + m$ for some $m \in \mathbb{Z}$.

(b) When n is of the form $m^2 + m$, find the number of $n \times n$ zero-one matrices X which solve $X^2 + X = J_n$.

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For each positive integer m, let $\mathcal{A}_m = \{0, 1, \ldots, m\}$. The Kautz digraph of degree m and dimension n is the directed graph whose vertex set consists of all n-tuples $(a_1, a_2, \ldots, a_n)$ of elements in $\mathcal{A}_m$ that satisfy the condition $a_i \ne a_{i+1}$ for all i in the range $1 \le i \le n-1$ and where there is a directed edge from $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_n)$ if and only if $b_i = a_{i+1}$, for all i in the range $1 \le i \le n-1$.

A key elementary property of the Kautz digraph of degree m and dimension 2 is that, given any two not necessarily distinct vertices (a_1, a_2) and (b_1, b_2), there is exactly one directed path of length less than or equal to two from (a_1, a_2) to (b_1, b_2). If $b_1 = a_2$, this path is of length one and if $b_1 \ne a_2$, this path is of length two and has (a_2, b_1) as its intermediate vertex.