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Orthogonality for $(0, -1)$ tropical normal matrices

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Abstract: We study pairs of mutually orthogonal normal matrices with respect to tropical multiplication. Minimal orthogonal pairs are characterized. The diameter and girth of three graphs arising from the orthogonality equivalence relation are computed.

Keywords: semirings, normal matrices, orthogonality relation, graphs, tropical algebra

MSC: 15B33, 14T05

1 Introduction

By tropical linear algebra we mean linear algebra done with the tropical operations $a \oplus b := \max\{a, b\}$ and $a \odot b := a + b$. The operations $\oplus$, $\odot$ are called *tropical sum*, *tropical multiplication*, respectively. These tropical operations extend, in a natural way, to matrices of any order.

We work over $R = \{0, -1\}$, where we define $(-1) \odot a = -1 = a \odot (-1)$ and $0 \odot a = 0 + a = a = a + 0 = a \odot 0$, for $a \in R$ (so, zero is the neutral element with respect to $\odot$). In particular, $(-1) \odot (-1) = (-1) + (-1) = -1$. For tropical addition, the neutral element is -1 and no opposite elements exist. To compensate this lack, tropical addition is idempotent: we have $a \oplus a = a$, for $a \in R$. Further, we have an order relation $-1 < 0$ compatible with the operations. Summing up, $(R, \oplus, \odot)$ is an *ordered semiring,* which is additively idempotent. An additively idempotent semiring is called a dioid in [12].

Note that in the definition of semiring some authors impose the condition that the neutral elements for addition and multiplication are mutually different, but we do not. Some other authors impose that the neutral element for addition e is absorbing, i.e., multiplication by this element is trivial $ea = e = ae$, for any a, but we do not. Why? Citing Pouly, *ordered idempotent semirings are essentially different from fields* and *this is one reason why mathematicians are interested in semirings*; see [21]. Another reason is that we want to produce new semirings from a given semiring, such as the semiring of square matrices and the semiring of polynomials over the initial semiring.

We refer to $(R, \oplus, \odot)$ as the *tropical semiring* or *max–plus semiring*; see [1] for a summary on max–plus properties. The so called *normal matrices*, i.e., matrices $[a_{ij}]$ satisfying $a_{ij} \le 0$ and $a_{ii} = 0$ over the tropical semiring R are the protagonists of this paper. For any $n \in \mathbb{N}$, the set of such square matrices over R is denoted by M_n^N and $(M_n^N, \oplus, \odot)$ happens to be a semiring. There exist two distinguished matrices: the all zero matrix Z_n and the identity matrix $I_n = (b_{ij})$, with $b_{ii} = 0$, $b_{ij} = -1$, if $i \neq j$.

Assuming $n \ge 2$, the bizarre property of M_n^N is that the same element, I_n, is neutral for both tropical operations $\oplus$ and $\odot$. Here Z_n is not the neutral element for tropical addition, but it keeps the absorbing property $AZ_n = Z_n = Z_nA$, for all $A \in M_n^N$.

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Every normal matrix A satisfies the inequalities $I_n \leq A \leq Z_n$ trivially, whence Z_n is the top element and I_n is the bottom element in M_n^N. Further, normal matrices satisfy

$$A \oplus I_n = A = I_n \oplus A, \quad AI_n = A = I_n A. \tag{1}$$

Orthogonality is a fundamental notion in mathematics. The purpose of this paper is to investigate pairs of *mutually orthogonal tropical matrices*, i.e., to find necessary and sufficient conditions for square matrices A, B to satisfy

$$AB = Z_n = BA, \tag{2}$$

where Z_n is the matrix, whose all elements are 0.[1] If $A^2 = Z_n$, then we say that A is *self–orthogonal*. To simplify, we write AB for $A \odot B$, since there is no non–tropical multiplication of matrices in this paper.

Mutual orthogonality of a pair A, B arises in classical algebra (e.g. idempotents and projectors, where the equivalence $A = A^2 \Leftrightarrow AB = 0 = BA$ holds, with $B = 1 - A$), functional analysis (e.g., families of orthogonal polynomials and orthogonal functions) and signal theory. In neighboring disciplines, such as statistics, economics, computer science and physics, orthogonal states are considered.

Combinatorial matrix theory is the investigation of matrices using combinatorial tools (see [4, 5, 14, 15]). Binary relations on associative rings and semirings, and, in particular, on the algebra of matrices can be understood with the help of graph theory. Indeed, one studies the so–called *relation graph*, whose vertices are matrices in some set, and edges show corresponding elements under this relation. Commuting graphs and zero–divisor graphs are examples of relation graphs; they have become classical concepts in algebra and combinatorics.

Orthogonality appears in combinatorial matrix theory and graph theory; see [13, 20, 23] and references therein. In the paper [2] the notion of the graph generated by the mutual orthogonality relation for elements of an associative ring was introduced. In the paper [17], the structure of the centralizer $\{B : AB = BA\}$ (with tropical multiplication) of a given normal matrix A was studied. In fact, mutual orthogonality of a pair A, B is a very special case of commutativity $AB = BA$. Observe that different properties of commutativity relation of matrices over semirings were intensively studied, see [26] and references therein.

Semirings are widely used in discrete event systems, dynamic programming and linguistics [10, 11, 21]. Recent attention has been paid to the semiring of normal matrices over $\mathbb{R} \cup \{-\infty\}$ with tropical operations, in [28].

This paper introduces mutual orthogonality in the semiring of square normal matrices over $(R, \oplus, \odot)$.[2] Our goal is to find necessary and sufficient conditions on (A, B) for orthogonality. The main results are gathered in sections 4 and 6: these are Theorems 4.33 and 4.35 and Corollaries 4.18 and 4.31 concerning minimality, as well as Propositions 6.10 to 6.13, concerning graphs. We depart from an easy–to–check sufficient condition for mutual orthogonality, namely, the existence of $p, q \in [n]$, such that the p–th row and the q–th column of A and the p–th column and the q–th row of B are zero (Lemma 3.13). Then Theorem 4.33 characterizes minimal pairs (A, B) as the members of the set $\mathfrak{M}_{km}$ (Notation 4.13), for some $k, m \in [n]$ with $k \neq m$. Corollary 4.31 shows that the minimal number of off–diagonal zeros in mutually orthogonal pairs (A, B) is $\Theta_n = 4n - 6$, for different matrices A, B of size $n \geq 2$, $n \neq 4$. The key concepts are the *indicator matrix C of a pair* (A, B) as well as three kinds of off–diagonal zeros in C: *propagation, cost* and *gift zeros*, introduced in Definitions 3.3 and 4.4. It is quite obvious that zeros propagate from A and B to the products AB and BA and thus, to the indicator matrix C. However, other zeros (called cost zeros and gift zeros) pop up in C. It happens that carefully placed

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1 In the case addressed in this paper, of normal matrices, since the neutral element for addition and the absorbing property are not attributes of a single element, to call orthogonality to the relation $AB = Z_n = BA$ is slightly questionable (but not atrocious). Another possible definition is: given matrices $A, B \in M_n$, say that they are *mutually perpendicular* if, for every row r in A and every column c in B, the maximum $\max_{i \in [n]}(r_i + c_i)$ is attained at least twice, and symmetrically, for every row r in B and every column c in A, the maximum $\max_{i \in [n]}(r_i + c_i)$ is attained at least twice. We do not explore this definition in the present paper.

2 We might have decided to work over the ordered idempotent semiring of extended non–positive real numbers $\mathbb{R}_{\leq 0} \cup \{-\infty\}$. Our choice of simpler semiring $\{0, -1\}$ is due to the fact that we only mind whether elements vanish or not.

zeros in A and B produce gift zeros in C, while not so carefully placed zeros in A and B produce cost zeros in C. Further, gift zeros in C are the ones to maximize in number, for minimality. This is the pivotal idea in the paper. In the special case $A = B$ when no gift zeros exist, minimality is attained by maximizing the number of cost zeros. In section 6, we study a natural graph, denoted by $\mathcal{ORTHO}$, arising from the orthogonality relation between normal matrices, as well as two subgraphs. In Propositions 6.10 to 6.13 we find their diameters and girths.

The paper is organized as follows. In Section 2, normal matrices are defined. In Section 3, general properties of mutually orthogonal tropical normal matrix pairs are collected. In Section 4 we compute Θ_n, the minimal number of off–diagonal zeros in mutually orthogonal pairs, as well as Θ_n^Δ, the minimal number of off–diagonal zeros in self–orthogonal matrices. Section 5 is devoted to the construction of orthogonal pairs of big size from smaller ones, by means of bordered matrices. In section 6 we compute the girth and the diameter of three graphs related to orthogonal pairs. One graph, denoted $\mathcal{ORTHO}$, studies the relation $AB = Z = BA$. Another graph, denoted $\mathcal{VNL}$, studies the (p, q)–sufficient condition stated in the paragraph above. The third graph, denoted $\mathcal{WNL}$, studies three other sufficient conditions for orthogonality. Altogether, these are the four sufficient conditions found in Corollary 4.18.

2 Normal matrices

Normal matrices (and a slightly weaker notion called *definite matrices*) over different sets of numbers have been studied for more than fifty years, under different names, beginning with Yoeli in [27]. The notion appears in connection with tropical algebra and geometry [6–8, 18, 22, 24, 25, 28]. In computer science they have been called DBM (*difference bound matrices*). Introduced by Bellman in the 50's, DBM matrices are widely used in software modeling [3, 9, 16, 19].

Over the semiring $(R, \oplus, \odot)$ with $R = \mathbb{R}_{\le 0} \cup \{-\infty\}$, normal matrices have a direct geometric interpretation in terms of complexes of alcoved convex sets in $\mathbb{R}^n$, see [22]. Then mutual orthogonality reflects how two such complexes annihilate each other. However, to explain this is beyond the scope of this paper.

Definition 2.1 (Normal, strictly normal and abnormal). A square matrix $A = [a_{ij}]$ is *normal* if $a_{ij} \le 0$ and all its diagonal entries equal 0. The set of order n normal matrices is denoted by M_n^N. A normal matrix $A = [a_{ij}]$ is *strictly normal* if $a_{ij} < 0$ for all $i \ne j$. The set of order n strictly normal matrices is denoted by M_n^{SN}. A matrix is *abnormal* if it is not normal.

Clearly, M_n^N and M_n^{SN} are closed under $\oplus$ and $\odot$, and I_n is the identity element for both operations.[3] We use classical addition and substraction of matrices, occasionally.

Notation 2.2 (Elementary matrices). In the set M_n,

1. let E_{ij} denote the matrix with the element -1 in the (i, j) position, and 0 elsewhere,
2. let U_{ij} denote the matrix with 0 in the (i, j) position, all diagonal entries equal to 0, and -1 elsewhere.
3. let U_n denote the matrix where every entry is equal to -1. We write U if n is understood.
4. let Z_n be the all zero matrix, and I_n be the identity matrix, with zeros on the diagonal, and -1 elsewhere. We write Z and I if n is understood.

Remark 2.3. *Although Z_n is not neutral for $\oplus$, the zero matrix Z_n is an absorbing element in M_n^N, i.e.,*

$$AZ_n = Z_n = Z_nA, \quad A \in M_n^N. \tag{3}$$

3 $(M_n^N, \oplus)$ is a semilattice (with associative, commutative and idempotent properties).

Remark 2.4. *The equality* (3) *does not hold without normality. For example, if* $A = -E_{12}$, *then* $AZ_n = -(E_{11} + \cdots + E_{1n})$ *and* $Z_n A = -(E_{12} + \cdots + E_{n2})$ *(classical addition and substraction here).*

3 Pairs of mutually orthogonal matrices

In this section, we assume $A, B \in M_n^N$. Recall that the definition of mutual orthogonality is given by the expression (2). Our goal is to find necessary and sufficient conditions on (A, B) for orthogonality.

Lemma 3.1 (Orthogonality for $n = 1,\ 2$). *Let* $A, B \in M_n^N$. *Then*[4]

1. *If* $n = 1$ *then* A *and* B *are orthogonal if and only if* $A = B = 0$.
2. *If* $n = 2$ *then*
   (a) $AB = A \oplus B = BA$,
   (b) $A = A^2$, *i.e., every normal matrix of size 2 is multiplicatively idempotent,*
   (c) A *and* B *are orthogonal if and only if* $A \oplus B = Z_2$.

Proof. For Item 1 we have $Z_1 = 0 \in R$. By the rule of multiplication, orthogonality $AB = 0 = BA$ is equivalent to $A + B = 0 = B + A$, which holds if and only if $A = 0 = B$, for $A, B \in R$.

Item 2a is a straightforward computation. The remaining two items follow directly from Item 2a. □

Neither of the former statements holds true for abnormal matrices.

Example 3.2. *1.* $1 \odot (-1) = 0 = (-1) \odot 1$, *i.e.,* 1 *and* -1 *are orthogonal.*

2. (a) *Let* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. *Then* $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $A^2 \neq A$ *in this case.*
   (b) $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

In the rest of the paper, all matrices are assumed to be normal.

Definition 3.3 (Indicator matrix). For matrices $A, B \in M_n^N$, define product matrices $L := AB = [l_{ij}]$ and $R := BA = [r_{ij}] \in M_n^N$. The matrix $C := [c_{ij}] \in M_n^N$ given by

$$c_{ij} = \begin{cases} 0, & \text{if } l_{ij} = r_{ij} = 0, \\ -1, & \text{otherwise} \end{cases}$$

is called the *indicator matrix* of the pair (A, B).

Obviously, the matrices A, B are mutually orthogonal if and only if the indicator matrix C is zero.

The next Lemma shows how easily zeros are propagated from A or B to C, by tropical multiplication.

Lemma 3.4 (Propagation of zeros). *Let* $A, B \in M_n^N$ *and* $p, q \in [n]$. *If* $a_{pq} = 0$, *then* $l_{pq} = r_{pq} = c_{pq} = 0$.

Proof. If $p = q$, the statement is true, by normality. Assume now that $p \neq q$. Then $l_{pq} = \max_{s \in [n]}(a_{ps} + b_{sq}) \leq 0$, and this maximum is attained at $s = q$, giving $l_{pq} = 0 + 0 = 0$. Similarly true for r_{pq} and, as a consequence, true for c_{pq}. □

4 Compare tropical vs. classical linear algebra: in the classical setting, it is easy to prove that for all $n \in \mathbb{N}$ and $A, B \in M_n$, the equality $AB = A + B$ implies $AB = BA$.

Corollary 3.5 (Orthogonality by propagation). *Let $A, B \in M_n^N$. If $A \oplus B = Z_n$, then $C = Z_n$.* □

Example 3.6 (Easy orthogonality 1). *Let $A, B \in M_n^N$ be matrices such that the number of zeros in every row and every column of each matrix is strictly greater than $n/2$. Then A and B are mutually orthogonal.* Indeed, by symmetry, it is enough to prove $L = AB = Z$. Let $r = [r_i]$ (resp. $l = [l_j]$) be an arbitrary row of A (resp. an arbitrary column of B). Since multiplication in the tropical semiring is the sum, addition is the maximum, and all r_i, l_j are non-positive, one obtains that $rl = 0$ if and only if there exists $i \in [n]$ with $r_i = l_i = 0$. Since more than the half of the entries of r and l are zero, such i always does exist.

Below we show that the hypotheses of Example 3.6 are indispensable.

Example 3.7.
1. The condition that the number of zeros in any row of A is strictly greater than $n/2$ (only for the rows) is not sufficient for the orthogonality. The same holds only for the columns. *Indeed, with Notations 2.2, for the matrix $A := E_{1n} + \cdots + E_{n-1,n} \in M_n^N$, the entry $(1, n)$ of A^2 is equal to -1, so A is not self–orthogonal.*
2. The condition that exactly the half of the entries of all rows and columns of A are zero is not sufficient for the orthogonality. *Indeed, for $n = 2k$ consider the matrix $A := \begin{bmatrix} Z & U \\ U & Z \end{bmatrix} \in M_n^N$. Then the entry $(1, n)$ of A^2 is -1, so A is not self–orthogonal.*

Example 3.8 (Easy orthogonality 2). *Let $n \geq 4$, $A = [a_{ij}]$, $B = [b_{ij}] \in M_n^N$ and*

$$a_{ij} = \begin{cases} 0, & \text{if } i = j \text{ or } i + j \equiv 2 \mod 3, \\ -1, & \text{otherwise,} \end{cases} \qquad b_{ij} = \begin{cases} 0, & \text{if } i + j \equiv 0 \mod 2, \\ -1, & \text{otherwise.} \end{cases}$$

Then A and B are orthogonal. Notice that $AB = Z$ follows from the facts:

1. *in each row of A there is zero in an odd position and there is zero in an even position,*
2. *if j is even (resp. odd), then $b_{kj} = 0$ for all even (resp. odd) k.*

Similarly $BA = Z$.

Notation 3.9 ($\nu(A)$). For $A \in M_n^N$, let $\nu(A)$ denote the number of zero entries in A. Since the diagonal of A vanishes, we have $\nu(A) \geq n$.

Here we use the big O notation. In Corollary 3.5 and Examples 3.6 and 3.8, the number of zeros in A and B is $O(n^2)$. Indeed, $\nu(A) \geq \frac{n^2}{2}$ and $\nu(B) \geq \frac{n^2}{2}$ in Example 3.6, and $\nu(A)$ is $O(n^2)$ and $\nu(B) = \frac{n^2}{2}$ in Example 3.8. We want to achieve orthogonality with fewer zeros, only $O(n)$. Our starting point is the following Remark (see Lemma 3.13 for a proof).

Remark 3.10 (A sufficient condition for orthogonality). *Let there exist $p, q \in [n]$, such that the p–th row and the q–th column of A and the p–th column and the q–th row of B are zero. Then $AB = Z_n = BA$.*

Below we introduce the convenient notation $V(p; q)$ to express Remark 3.10 in short.

Notation 3.11 ($V(p; q)$, $W(p; q)$ and $Z(p; q)$). For $p, q \in [n]$, we consider some subsets of M_n^N

1. $V(p; q) := \{A \in M_n^N : a_{pi} = 0 = a_{iq},\ i \in [n]\}$,
2. $W(p; q) := \{A \in M_n^N : a_{pi} = 0,\ i \in [n] \setminus \{q\}\} \cap \{A \in M_n^N : a_{iq} = 0,\ i \in [n] \setminus \{p\}\}$,
3. $Z(p; q) := \{A \in M_n^N : a_{pq} = 0\}$,
4. Let $V(p_1, \ldots, p_s; q_1, \ldots, q_t)$ be the intersection of all $V(p; q)$, with $p \in \{p_1, \ldots, p_s\} \subseteq [n]$ and $q \in \{q_1, \ldots, q_t\} \subseteq [n]$.

Remark 3.12. *It is straightforward from the definitions that*

$$V(p;q)=W(p;q)\cap Z(p;q),\qquad Z(p;p)=M_n^N\qquad \text{and}\qquad V(p;p)=W(p;p). \tag{4}$$

Next we restate and prove Remark 3.10.

Lemma 3.13 (A sufficient condition for orthogonality). *Let $A, B \in M_n^N$ and $p, q \in [n]$. If $A \in V(p;q)$ and $B \in V(q;p)$, then $AB = Z_n = BA$.*

Proof. We will prove that the indicator matrix C of the pair (A, B) is equal to zero. By Lemma 3.4, we have $C \in V(p;q) \cap V(q;p)$.

Consider $i, j \in [n] \setminus \{p, q\}$ with $i \neq j$. Using $A \in V(p;q)$, we get

$$0 \geq l_{ij} = \max_{t\in[n]}(a_{it}+b_{tj}) \geq \max_{t=q,t=i}(a_{it}+b_{tj}) = \max\{b_{ij}, b_{qj}\} = b_{ij}\oplus b_{qj}, \tag{5}$$

$$0 \geq r_{ij} = \max_{t\in[n]}(b_{it}+a_{tj}) \geq \max_{t=j,t=p}(b_{it}+a_{tj}) = \max\{b_{ij}, b_{ip}\} = b_{ij}\oplus b_{ip}. \tag{6}$$

Using $B \in V(q;p)$ we get $b_{qj} = b_{ip} = 0$, whence the right hand sides of (5) and (6) are zero. Thus $l_{ij} = r_{ij} = 0$ whence $c_{ij} = 0$. □

Corollary 3.14. *If $A \in V(p;p)$ for some $p \in [n]$, then $A^2 = Z_n$.* □

Definition 3.15 (Genericity). Let $S \subseteq M_n^N$ be a subset. The matrix $A \in S$ is called *S–generic* if no entry of A is zero, unless it is required by the structure of S.

Recall that $R = \{0, -1\}$.

Definition 3.16. For $p, q \in [n]$, the *indicator function of the pair (p, q) of indices* is

$$c(p,q)=\begin{cases}0, & \text{if } p\neq q\\ -1, & \text{otherwise.}\end{cases}$$

Lemma 3.17. *With $p, q \in [n]$ and Notation 3.9,*

1. *there exists a unique $V(p;q)$–generic matrix $A \in M_n^N$ and $\nu(A) - n = 2n - 3 - c(p,q)$,*
2. *there exists a unique $W(p;q)$–generic matrix $A \in M_n^N$ and $\nu(A) - n = 2n - 4 - 2c(p,q)$,*
3. *there exists a unique $W(p;q) \cap Z(p;q) \cap Z(q;p)$–generic matrix $A \in M_n^N$ and $\nu(A) - n = 2n - 2$.*

Proof. 1. We have a zero row, a zero column and the zero main diagonal in A, and the remaining entries are -1.

Items 2 and 3 are similar. □

Lemma 3.18. *Let $A, B \in M_n^N$ and $p, q \in [n]$ with $p \neq q$. If A is $V(p;q)$–generic and $AB = Z_n = BA$, then (5) and (6) are equalities. In particular, $b_{ij} = 0$ or $b_{qj} = 0 = b_{ip}$, all $i, j \in [n]$.*

Proof. By hypothesis, the indicator matrix C of the pair (A, B) is zero, which implies $L = AB = Z = BA = R$. The thesis is trivial, if i, j, p, q are not pairwise different, by normality.

Suppose now that $i, j, p, q \in [n]$ are pairwise different. The equality $L = Z$ implies $0 = l_{ij}$. By genericity, we know that $A \in V(p;q)$ and $a_{ij} < 0$, whenever $i \neq j$ and $i \neq p$ and $j \neq q$. Then, in $0 = l_{ij} = \max_{t\in[n]}(a_{it} + b_{tj})$, every term on the right hand side is strictly negative except, perhaps, for $t = i$ or $t = q$. It follows that the maximum, which is zero, is attained at $t = i$ or $t = q$ and, furthermore, $b_{ij} = 0$ or $b_{qj} = 0$. We conclude that $b_{ij} \oplus b_{qj} = 0$ and (5) is a chain of equalities. Similarly, we prove $b_{ij} \oplus b_{ip} = 0$. □

In Lemma 3.18 we have proved that mutual orthogonality of A and B together with $O(n)$ *aligned zeros* in A force some entries in B to vanish. This key observation leads us to the notions of cost and gift zeros given below.

4 Minimal number of zeros in pairs

In this section we assume $A, B \in M_n^N$. Our goal is to find necessary and sufficient conditions on the pair (A, B) for minimal orthogonality, i.e., the orthogonality with matrices A and B having minimal number of zeros.

Definition 4.1. Let Θ_n be the minimal number of off–diagonal zero entries among all pairs of mutually orthogonal matrices in M_n^N. With Notation 3.9,

$$\Theta_n = \min_{A,B\in M_n^N} \{\nu(A) + \nu(B) - 2n \ : \ AB = Z_n = BA\}.$$

Let $A, B \in M_n^N$. The pair (A, B) is called *minimal* if it realizes the value of Θ_n, i.e., if $\Theta_n = \nu(A) + \nu(B) - 2n$ and $AB = Z_n = BA$.

Remark 4.2. *We have $\Theta_n \leq n^2 - n$, since the pair (Z_n, B), with B strictly normal, satisfies $\nu(Z_n) = n^2$ and $\nu(B) = n$.*

Lemma 4.3. *Let $A = [a_{ij}], B = [b_{ij}] \in M_n^N$ and let $C = [c_{ij}]$ be the indicator matrix of (A, B). If $s, t, k, m \in [n]$ are pairwise different integers such that $a_{sk} = b_{kt} = b_{sm} = a_{mt} = 0$, then $c_{sk} = c_{kt} = c_{sm} = c_{mt} = c_{st} = 0$.*

Proof. We get $c_{sk} = c_{kt} = c_{sm} = c_{mt} = 0$, by Lemma 3.4, and get $c_{st} = 0$, by the definition of tropical multiplication. □

The next definition classifies the entries of indicator matrices.

Definition 4.4 (Propagation, cost and gift zeros)**.** Let $A = [a_{ij}], B = [b_{ij}] \in M_n^N$ and $C = [c_{ij}]$ be the indicator matrix of (A, B). Let $s, t \in [n]$ with $s \neq t$.

1. If $a_{st} = 0$ or $b_{st} = 0$, then $c_{st} = 0$ is called a *propagation zero*. Let prop(C) denote the number of propagation zeros in C.
2. If there exists $k \in [n]$ such that s, t, k are pairwise different integers, and $a_{st} \neq 0 \neq b_{st}$, and $a_{sk} = b_{kt} = b_{sk} = a_{kt} = 0$, then $c_{st} = 0$ is called a *cost zero*. For each such k we can use the notation
$$c_{st} := \phi_{st}^{kk}. \tag{7}$$
3. If c_{st} is not a cost zero and there exist $k, m \in [n]$ such that s, t, k, m are pairwise different integers, and $a_{st} \neq 0 \neq b_{st}$, and $a_{sk} = b_{kt} = b_{sm} = a_{mt} = 0$, then $c_{st} = 0$ is called a *gift zero*. For each such k, m we can use the notation
$$c_{st} := \phi_{st}^{km}. \tag{8}$$
Let gift(C) denote the number of gift zeros in C.

In plain words, assume $a_{st} \neq 0 \neq b_{st}$. Then *two* zero entries ($a_{sk} = a_{mt} = 0$) in A, together with *two* zero entries ($b_{kt} = b_{sm} = 0$) in B provide *five* zero entries in C: a *gift zero* in the (s, t) position and four *propagation zeros* in the (s, k), (k, t), (s, m) and (m, t) positions. In particular, *carefully placed propagation zeros are attached to gift zeros, and conversely.*

Remark 4.5. *The only difference between Items 2 and 3 in Definition 4.4 is whether $k = m$ or not. Moreover, the notation $c_{st} = \phi_{st}^{km}$ is general and, in the case $k = m$, it means a cost zero.*

Definition 4.6 (Duplicates). If $A \neq B$, a *duplicate* in the pair (A, B) is a position (s, t) with $a_{st} = 0 = b_{st}$ and $s \neq t$.

Notation 4.7 ($\Sigma(A, B, i)$ and $\Sigma(A, B)$). Denote by $\operatorname{row}(A, i)$ the i–th row of the matrix A, and by $\nu(\operatorname{row}(A, i))$ the number of zeros in the i–th row of A. For a given pair (A, B) with $A \neq B$, denote by $\Sigma(A, B, i)$ the sum $\nu(\operatorname{row}(A, i)) + \nu(\operatorname{row}(B, i)) - 2$ and by $\Sigma(A, B)$ the sum $\nu(A) + \nu(B) - 2n$. We simply write $\Sigma(i)$ or Σ, whenever the pair (A, B) is understood from the context. Note that $\Sigma(A, B, i)$ stands for the number of off–diagonal zeros in the i–th row of A and B. With these notations Θ_n from Definition 4.1 transforms into

$$\Theta_n = \min_{A,B \in M_n^N} \{\Sigma(A, B) \ : \ AB = Z_n = BA\}.$$

Remark 4.8.
1. *Propagation, cost and gift zeros are zeros in the indicator matrix C of a pair (A, B).*
2. *The indicator matrix C of a pair of normal matrices is normal and, therefore, C has zero diagonal. So zeros in C can be either diagonal, propagation, cost or gift, and these are mutually exclusive variants.*
3. *It follows from Definition 4.4 that the number of cost zeros in any row of C is less than or equal to $n - 2$, and the number of propagation zeros in any row of C is at least 1, if a cost zero exists in that row. We say that* $\operatorname{row}(C, i)$ *is a cost row if it contains $n - 2$ cost zeros and one propagation zero. Similar for columns.*
4. *It follows from Definition 4.4 that the number of gift zeros in any row of C is less than or equal to $n - 3$, and the number of propagation zeros in any row of C is at least 2, if a gift zero exists in that row. We say that* $\operatorname{row}(C, i)$ *is a gift row if it contains $n - 3$ gift zeros and 2 propagation zeros and $\Sigma(i) = 2$. Similar for columns.*
5. *Gift zeros do not exist when $A = B$. Gift zeros do not exist when $n \le 3$.*
6. *A cost zero $c_{st} = \phi_{st}^{kk}$ requires 2 duplicates (s, k) and (k, t) in the pair (A, B). Note that $\Sigma(s) \ge 2$ and $\Sigma(k) \ge 2$.*
7. *It is possible to have $c_{st} = \phi_{st}^{km} = \phi_{st}^{k'm'}$, with $(k, m) \neq (k', m')$ (valid for cost and gift zeros.)*
8. *If* $\operatorname{row}(C, s)$ *is a cost row, then there exists $k \in [n] \setminus \{s\}$ such that $c_{st} = \phi_{st}^{kk}$, for all $t \in [n] \setminus \{s, k\}$. (Indeed, if for some $t_1, t_2, k_1, k_2 \in [n]$, with $k_1 \neq k_2$, we have $c_{st_1} = \phi_{st_1}^{k_1k_1}$ and $c_{st_2} = \phi_{st_2}^{k_2k_2}$, then c_{sk_1}, c_{sk_2} are propagation zeros, but* $\operatorname{row}(C, s)$ *contains only one propagation zero, contradiction.) We say that the row is k–cost to indicate this dependence on k.*
9. *If* $\operatorname{row}(C, s)$ *is a gift row, then there exist $k, m \in [n] \setminus \{s\}$ with $k \neq m$ such that $c_{st} = \phi_{st}^{km}$, for all $t \in [n] \setminus \{s, k, m\}$. (Similar proof to Item 8). We say that the row is km–gift to indicate this dependence on k and m.*

Notation 4.9. Let propagation zeros be marked blue and diagonal zeros be marked red, for a better visualization. The symbol $\star$ denotes any element in $R = \{0, -1\}$.

Example 4.10. *Consider any pair (A, B) (not necessarily orthogonal) in M_6^N. Taking $(k, m) = (3, 4)$ and assuming $a_{s3} = b_{3t} = b_{s4} = a_{4t} = 0$ for different values of $s, t \in [6]$, we get different indicator matrices C, C', C'' of the pair (A, B), where C corresponds to the choice $s = 1$ and $t \in \{2, 5\}$, C' corresponds to $s = 1$ and $t \in \{2, 5, 6\}$, and C'' corresponds to $s \in \{1, 5, 6\}$ and $t = 2$.*

$$C = \begin{bmatrix} 0 & \phi_{12}^{34} & 0 & 0 & \phi_{15}^{34} & \star \\ \star & 0 & \star & \star & \star & \star \\ \star & 0 & 0 & \star & 0 & \star \\ \star & 0 & \star & 0 & 0 & \star \\ \star & \star & \star & \star & 0 & \star \\ \star & \star & \star & \star & \star & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} 0 & \phi_{12}^{34} & 0 & 0 & \phi_{15}^{34} & \phi_{16}^{34} \\ \star & 0 & \star & \star & \star & \star \\ \star & 0 & 0 & \star & 0 & 0 \\ \star & 0 & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & \star \\ \star & \star & \star & \star & \star & 0 \end{bmatrix}, \quad C'' = \begin{bmatrix} 0 & \phi_{12}^{34} & 0 & 0 & \star & \star \\ \star & 0 & \star & \star & \star & \star \\ \star & 0 & 0 & \star & \star & \star \\ \star & 0 & \star & 0 & \star & \star \\ \star & \phi_{52}^{34} & 0 & 0 & 0 & \star \\ \star & \phi_{62}^{34} & 0 & 0 & \star & 0 \end{bmatrix}.$$

Remark 4.11 (Bounds). *If $A \neq B$ then $\max\{\nu(A), \nu(B)\} - n \le \operatorname{prop}(C) \le \Sigma(A, B)$ holds for arbitrary pairs (A, B), the second inequality being an equality if and only if no duplicates exist. This follows from Item 1 in Definition 4.4. To get minimal pairs one has to avoid duplicates, as much as possible, i.e., one has to minimize the gap between $\operatorname{prop}(C)$ and $\Sigma(A, B)$. If $A = B$ then $\nu(A) - n = \operatorname{prop}(C)$.*

Corollary 4.12. *If $n \geq 2$ then $\Theta_n \leq 4n - 6$.*

Proof. Consider a $V(p;q)$–generic matrix A and a $V(q;p)$–generic matrix B, with different $p, q \in [n]$, and apply Lemmas 3.13 and 3.17. □

Below we introduce the set $\mathfrak{M}_{km}$ which we need for the later description of minimal pairs.

Notation 4.13 ($\mathfrak{M}_{km}$)**.** For $k, m \in [n]$, a pair (A, B) belongs to $\mathfrak{M}_{km}$ if

0. A is $V(m;k)$–generic and B is $V(k;m)$–generic, or
1. A is $W(m;k) \cap Z(k;m)$–generic and B is $W(k;m) \cap Z(m;k)$–generic, or
2. A is $W(m;k)$–generic and B is $W(k;m) \cap Z(m;k) \cap Z(k;m)$–generic, or
3. A is $W(m;k) \cap Z(m;k) \cap Z(k;m)$–generic and B is $W(k;m)$–generic.

Remark 4.14. *Two special cases arise:*

1. *If $k = m$, then, by expressions (4) in Remark 3.12, the four cases in 4.13 reduce to one case: A, B are $V(k;k)$–generic,*
2. *If $A = B$, then, by expressions (4) in Remark 3.12 and Item 4 of Notation 3.11, the four cases in 4.13 reduce to one case: A is $V(k, m; k, m)$–generic.*

In the following Lemma the necessity is trivial, however, sufficiency is crucial because it tells us how to recover the pair (A, B) from the indicator matrix C.

Lemma 4.15 (Characterization of $\mathfrak{M}_{km}$)**.** *Let $C = [c_{ij}] \in M_n^N$ be the indicator matrix of a pair (A, B) with $A \neq B$. Then $(A, B) \in \mathfrak{M}_{km}$ for some $k \neq m$, if and only if the following conditions hold:*

I *for each $s, t \in [n] \setminus \{k, m\}$ with $s \neq t$, the (s, t) entry is a gift zero with $c_{st} = \phi_{st}^{km}$,*
II *c_{km} and c_{mk} are propagation zeros,*
III *there are no duplicates in the pair (A, B) (or, equivalently, $\Sigma(A, B) = 4n - 6$).*

Proof. Necessity follows from Definition 4.4, Notation 4.13 and Item 6 in Remark 4.8. To prove sufficiency, notice that Items II and III initialize four cases (both zeros come from B and $b_{km} = 0 = b_{mk}$, or $a_{km} = 0 = b_{mk}$ or $a_{mk} = 0 = b_{km}$). Then, by Item I, we get $a_{sk} = b_{kt} = b_{sm} = a_{mt} = 0$ for all $s, t \in [n] \setminus \{k, m\}, s \neq t$, and no further off–diagonal zeros appear in A or B, by Item III. Thus, we get cases 0 to 3 of Notation 4.13. An illustration of how the argument works is found in Example 4.17. □

Corollary 4.16. *For each pair $(A, B) \in \mathfrak{M}_{km}$ with $A \neq B$ and $k \neq m$, the indicator matrix C satisfies*

1. *$C = Z_n$,*
2. prop$(C) = \Sigma(A, B) = 4n - 6$ *and* gift$(C) = (n - 3)(n - 2)$.

Proof. For Item 1, we must prove orthogonality of (A, B). If the pair (A, B) is in case 0 of Notation 4.13, then orthogonality holds true, by Lemma 3.13. If the pair (A, B) is in cases 1 to 3, then orthogonality is checked, similarly. Now, to prove Item 2, we count gift zeros using Item I and then count propagation zeros using Items II and III of Lemma 4.15. □

Example 4.17. *The following four items correspond to the 4 items from Notation 4.13 for $(k, m) = (4, 3)$. Notice that the differences between the items occur only in entries (k, m) and (m, k). That is why the indicator matrix*

C is the same in all cases. Below the symbol "–" denotes the element $-1 \in R$.

$$C = \begin{bmatrix} 0 & \phi^{43}_{12} & 0 & 0 & \phi^{43}_{15} & \phi^{43}_{16} \\ \phi^{43}_{21} & 0 & 0 & 0 & \phi^{43}_{25} & \phi^{43}_{26} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \phi^{43}_{51} & \phi^{43}_{52} & 0 & 0 & 0 & \phi^{43}_{56} \\ \phi^{43}_{61} & \phi^{43}_{62} & 0 & 0 & \phi^{43}_{65} & 0 \end{bmatrix}.$$

$$0.\ A_0 = \begin{bmatrix} 0 & - & - & 0 & - & - \\ - & 0 & - & 0 & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & 0 & - & - \\ - & - & - & 0 & 0 & - \\ - & - & - & 0 & - & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 & - & 0 & - & - & - \\ - & 0 & 0 & - & - & \\ - & - & 0 & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & 0 & - & 0 & - \\ - & - & 0 & - & - & 0 \end{bmatrix}$$

$$1.\ A_1 = \begin{bmatrix} 0 & - & - & 0 & - & - \\ - & 0 & - & 0 & - & - \\ 0 & 0 & 0 & - & 0 & 0 \\ - & - & 0 & 0 & - & - \\ - & - & - & 0 & 0 & - \\ - & - & - & 0 & - & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & - & 0 & - & - & - \\ - & 0 & 0 & - & - & - \\ - & - & 0 & 0 & - & - \\ 0 & 0 & - & 0 & 0 & 0 \\ - & - & 0 & - & 0 & - \\ - & - & 0 & - & - & 0 \end{bmatrix}$$

$$2.\ A_2 = \begin{bmatrix} 0 & - & - & 0 & - & - \\ - & 0 & - & 0 & - & - \\ 0 & 0 & 0 & - & 0 & 0 \\ - & - & - & 0 & - & - \\ - & - & - & 0 & 0 & - \\ - & - & - & 0 & - & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & - & 0 & - & - & - \\ - & 0 & 0 & - & - & - \\ - & - & 0 & 0 & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & 0 & - & 0 & - \\ - & - & 0 & - & - & 0 \end{bmatrix}$$

$$3.\ A_3 = \begin{bmatrix} 0 & - & - & 0 & - & - \\ - & 0 & - & 0 & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & 0 & 0 & - & - \\ - & - & - & 0 & 0 & - \\ - & - & - & 0 & - & 0 \end{bmatrix},\ B_3 = \begin{bmatrix} 0 & - & 0 & - & - & - \\ - & 0 & 0 & - & - & - \\ - & - & 0 & - & - & - \\ 0 & 0 & - & 0 & 0 & 0 \\ - & - & 0 & - & 0 & - \\ - & - & 0 & - & - & 0 \end{bmatrix}.$$

Corollary 4.18 (Four sufficient conditions for the existence of orthogonal pairs)**.** *Let* $n \geq 2$ *and* $A, B \in M^N_n$. *Then the pair* (A, B) *is orthogonal if there exist* $k, m \in [n]$ *such that one of the following holds:*

0. $A \in V(k; m)$ *and* $B \in V(m; k)$.
1. $A \in W(k; m) \cap Z(k; m) \cap Z(m; k)$ *and* $B \in W(m; k)$.
2. $A \in W(k; m) \cap Z(m; k)$ *and* $B \in W(m; k) \cap Z(k; m)$.
3. $A \in W(k; m)$ *and* $B \in W(m; k) \cap Z(k; m) \cap Z(m; k)$.

Proof. If $k = m$, then it follows from Item 1 of Remark 4.14 and Lemma 3.13. If $k \neq m$, then it follows from Item 1 of Corollary 4.16, by allowing A or B to have more zeros than strictly required by the structure of sets $V(k; m)$, $W(k; m)$, $Z(k; m)$. Note that Item 0 here is just Lemma 3.13, the sufficient condition we started with. □

Lemma 4.19. *Let* $n \geq 3$. *If the pair* (A, B) *is orthogonal, then* $\Sigma(i) \geq 2$, *for all* $i \in [n]$.

Proof. Suppose that $\nu(\text{row}(A, i)) = 1$, i.e., we have only one diagonal zero in row(A, i), then row(B, i) is zero, by definition of tropical multiplication, hence $\Sigma(i) = n - 1 \geq 2$. Similarly if $\nu(\text{row}(B, i)) = 1$. If $\nu(\text{row}(A, i)) > 1$ and $\nu(\text{row}(B, i)) > 1$, then also $\Sigma(i) \geq 2$ and the proof is complete. □

Lemma 4.20. *If the pair (A, B) is minimal and $n \geq 3$, then there exist at least three mutually different indices $i \in [n]$ such that $2 \leq \Sigma(i) \leq 3$.*

Proof. The first inequality $2 \leq \Sigma(i)$ holds for all $i \in [n]$ by Lemma 4.19. Suppose that for $n-2$ rows i we have $\Sigma(i) \geq 4$. Then $\Sigma \geq 4(n-2) + 2 \cdot 2 = 4n - 4 > 4n - 6$, which contradicts with Corollary 4.12 and the proof is complete. □

4.1 Arbitrary pairs

The aim of this subsection is to prove that $\Theta_n = 4n - 6$, for $n \geq 2$, $n \neq 4$, as well as to *construct minimal pairs*.

In the following we assume that $C \in M_n^N$ is the indicator matrix of the pair (A, B).

Lemma 4.21. *Let $n \geq 6$. If the pair (A, B) is minimal, then no row in C is a cost row.*

Proof. By hypothesis, the pair (A, B) is orthogonal. Suppose there exists $s \in [n]$ such that $\operatorname{row}(C, s)$ contains $n-2$ cost zeros. Then, by Item 8 of Remark 4.8, there exists $k \in [n]\backslash\{s\}$ such that $c_{st} = \phi_{st}^{kk}$, for all $t \in [n]\backslash\{s, k\}$. Then, by Item 6 of Remark 4.8, we have $n-2$ duplicates (k, t), $t \in [n] \setminus \{s, k\}$, hence $\Sigma(k) \geq 2(n-2)$. Using Lemma 4.19 we get $\Sigma \geq 2(n-1) + 2(n-2) = 4n - 6$. Hence $\Sigma = 4n - 6$, by Corollary 4.12. So $\Sigma(k) = 2(n-2)$ and $\Sigma(i) = 2$, for all $i \neq k$, i.e., (A, B) has the following structure based on the number of zeros: the number of off–diagonal zeros in $\operatorname{row}(A, k)$ and $\operatorname{row}(B, k)$ is exactly $2(n-2)$ and the number of off–diagonal zeros in $\operatorname{row}(A, i)$ and $\operatorname{row}(B, i)$ is exactly 2, for all $i \neq k$. We have four cases.

1. (A duplicate exists.) If there exists $s' \in [n] \setminus \{s, k\}$ such that $a_{s'k'} = b_{s'k'} = 0$ for some k' with $k' \neq s'$, then there is only one propagation zero in $\operatorname{row}(C, s')$. Then by Item 4 of Remark 4.8 there is no gift zero in $\operatorname{row}(C, s')$. Since $\operatorname{row}(C, s')$ is zero, then by Items 2 and 8 of Remark 4.8, $\operatorname{row}(C, s')$ is k'–cost and $c_{s't} = \phi_{s't}^{k'k'}$, for all $t \in [n] \setminus \{s', k'\}$. Hence $\Sigma(k') \geq 2(n-2)$. Since $2(n-2) > 2$ we get that $k' = k$. Since for $t = s$ we have $c_{s's} = \phi_{s's}^{kk}$, then, by Item 6 of Remark 4.8, (k, s) is also a duplicate alongside with other $n-2$ duplicates (k, t), $t \in [n] \setminus \{s, k\}$, hence $\Sigma(k) \geq 2(n-1) > 2(n-2)$, which is a contradiction with the structure of (A, B).
2. If there exists $s' \in [n] \setminus \{s, k\}$ such that $a_{s'k'} = b_{s'm'} = 0$ for some k', m', where s', k', m' are pairwise different, then there are only two propagation zeros in $\operatorname{row}(C, s')$. Moreover, by Item 6 of Remark 4.8, there is no cost zero in $\operatorname{row}(C, s')$, since $b_{s'k'} \neq 0 \neq a_{s'm'}$ (no duplicates). Since $\operatorname{row}(C, s')$ is zero, then by Items 2 and 9 of Remark 4.8 the $\operatorname{row}(C, s')$ is $k'm'$–gift and so $c_{s't} = \phi_{s't}^{k'm'}$, for all $t \in [n] \setminus \{s', k', m'\}$. Then, by Item 3 of Definition 4.4, $\nu(\operatorname{row}(A, m')) - 1 \geq n - 3$ and $\nu(\operatorname{row}(B, k')) - 1 \geq n - 3$. Since $n - 3 > 2$, we have 2 rows with more than 2 off–diagonal zeros, which contradicts with the structure of (A, B).
3. If there exists $s' \in [n] \setminus \{s, k\}$ such that $\operatorname{row}(A, s')$ has no off–diagonal zeros, then $\operatorname{row}(B, s')$ is zero and $\Sigma(s') = n - 1$, which is a contradiction with the structure of (A, B).
4. Similarly, if there exists $s' \in [n] \setminus \{s, k\}$ such that $\operatorname{row}(B, s')$ has no off–diagonal zeros, we also get a contradiction with the structure of (A, B).

Thus, there is no row in C with $n-2$ cost zeros, and the proof is complete. □

Lemma 4.22. *Let $n \geq 4$, the pair (A, B) be orthogonal, and $\operatorname{row}(C, s)$ be km–gift. Then the following holds:*

1. *$\Sigma(k) \geq n - 2$ and $\Sigma(m) \geq n - 2$.*
2. *If there exists $s' \in [n] \setminus \{s, k\}$ such that $c_{s's} = \phi_{s's}^{km'}$ for some $m' \in [n] \setminus \{s, s'\}$, then $\Sigma(k) \geq n - 1$.*
3. *If there exists $s' \in [n] \setminus \{s, k\}$ such that $c_{s's} = \phi_{s's}^{k'm}$ for some $k' \in [n] \setminus \{s, s'\}$, then $\Sigma(m) \geq n - 1$.*

Proof. 1. Since $\operatorname{row}(C, s)$ is km–gift, then s, k, m are mutually different and, by Item 9 of Remark 4.8, it has $n-3$ gift zeros $c_{st} = \phi_{st}^{km}$, for all $t \in [n] \setminus \{s, k, m\}$. Thus, by Item 3 of Definition 4.4, $a_{mt} = b_{kt} = 0$, $t \in [n] \setminus \{s, k, m\}$, hence $\nu(\operatorname{row}(A, m)) - 1 \geq n - 3$ and $\nu(\operatorname{row}(B, k)) - 1 \geq n - 3$. Consider c_{km}. Since (A, B) is an orthogonal pair, then $C = Z$, hence $c_{km} = 0$. By Item 2 of Remark 4.8, we have one of the following cases:

i. If c_{km} is a propagation zero, then at least one of a_{km}, b_{km} is zero, hence $\Sigma(k) \geq n-3+1=n-2$.
ii. If c_{km} is a cost zero $\phi_{km}^{k'k'}$ for some k' then, by Item 2 of Definition 4.4, $a_{kk'}=0$ (a zero which has not been counted previously) and $\Sigma(k) \geq n-3+1=n-2$.
iii. If c_{km} is a gift zero $\phi_{km}^{k'm'}$ for some k', m' then, by Item 3 of Definition 4.4, $a_{kk'}=0$ (a zero which has not been counted previously) and $\Sigma(k) \geq n-3+1=n-2$.

Thus, in each case we get $\Sigma(k) \geq n-2$. Reasoning similarly for $c_{mk}=0$, we also get $\Sigma(m) \geq n-2$.
2. By Item 1, we have $\Sigma(k) \geq n-2$. Since $c_{s's}=\phi_{s's}^{km'}$ (this can be a cost or a gift zero), then, by Items 2, 3 of Definition 4.4 we have $b_{ks}=0$. Note that b_{ks} does not coincide with the other $n-2$ zero entries in row(A,k) and row(B,k) mentioned above. Hence $\Sigma(k) \geq n-2+1=n-1$.
3. By Item 1.i, we have $\Sigma(m) \geq n-2$. Since $c_{s's}=\phi_{s's}^{k'm}$, then by Items 2, 3 of Definition 4.4 we have $a_{ms}=0$. Note that a_{ms} does not coincide with other $n-2$ zero entries in row(A,m) and row(B,m) mentioned above. Hence $\Sigma(m) \geq n-2+1=n-1$. □

Lemma 4.23. *Let $n \geq 5$, (A,B) be an orthogonal pair, and $\Sigma(s)=3$, for some $s \in [n]$. Then either $\nu(\text{row}(A,s))-1=1$ and $\nu(\text{row}(B,s))-1=2$ or $\nu(\text{row}(A,s))-1=2$ and $\nu(\text{row}(B,s))-1=1$.*

Proof. Indeed, if $\nu(\text{row}(A,s))-1=0$, then row(B,s) must be zero, by tropical multiplication, and then $\Sigma(s)=n-1$, which contradicts with $\Sigma(s)=3$. Similarly, if $\nu(\text{row}(B,s))-1=0$. □

Lemma 4.24. *Let $n \geq 5$ and (A,B) be an orthogonal pair. If $\Sigma(s)=3$, for some $s \in [n]$, then either*

1. *$a_{sk}=b_{sl}=b_{sm}=0$ for some $k,l,m \in [n] \setminus \{s\}$, with $l \neq m$, and for each $t \in [n] \setminus \{s,k,l,m\}$ we have $c_{st}=\phi_{st}^{km}$ or $c_{st}=\phi_{st}^{kl}$, or*
2. *$b_{sk}=a_{sl}=a_{sm}=0$ for some $k,l,m \in [n] \setminus \{s\}$, with $l \neq m$, and for each $t \in [n] \setminus \{s,k,l,m\}$ we have $c_{st}=\phi_{st}^{mk}$ or $c_{st}=\phi_{st}^{lk}$.*

In any case, there exists $k \in [n] \setminus \{s\}$ with $\Sigma(k) \geq n-3$.

Proof. By Lemma 4.23 we have two cases:

1. If $\nu(\text{row}(A,s))-1=1$ and $\nu(\text{row}(B,s))-1=2$, then $a_{sk}=b_{sl}=b_{sm}=0$, for some $k,l,m \in [n] \setminus \{s\}$, with $l \neq m$. Then we have at most 3 propagation zeros in row(C,s) and, by Item 2 of Remark 4.8 and Items 2, 3 of Definition 4.4, for each $t \in [n] \setminus \{s,k,l,m\}$ we have $c_{st}=\phi_{st}^{km}$ or $c_{st}=\phi_{st}^{kl}$. Thus, $b_{kt}=0$ for all $t \in [n] \setminus \{s,k,l,m\}$ and so $\nu(\text{row}(B,k))-1 \geq n-4$. If $\nu(\text{row}(A,k))-1=0$, then row(B,k) must be zero and $\Sigma(k)=n-1 \geq n-3$. If $\nu(\text{row}(A,k))-1>0$, then also $\Sigma(k) \geq n-3$, and Item 1 is proved.
2. If $\nu(\text{row}(A,s))-1=2$ and $\nu(\text{row}(B,s))-1=1$, then $b_{sk}=a_{sl}=a_{sm}=0$, for some $k,l,m \in [n] \setminus \{s\}$, with $l \neq m$. The rest of the proof is similar to the proof of the previous item.

□

Lemma 4.25. *Let $n \geq 6$. If the pair (A,B) is minimal and there are at least two different gift rows s,s' in C, then both rows are km–gift, for the same $k,m \in [n] \setminus \{s,s'\}$, with $k \neq m$.*

Proof. Using Item 9 of Remark 4.8, denote the gift zeros of row(C,s) by $c_{st}=\phi_{st}^{km}$, for all $t \in [n] \setminus \{s,k,m\}$, and the gift zeros of row(C,s') by $c_{s't}=\phi_{s't}^{k'm'}$, for all $t \in [n] \setminus \{l,k',m'\}$. We have three cases:

1. Suppose that $\{k,m\} \cap \{k',m'\}=\emptyset$. Then, by Item 1 of Lemma 4.22 for rows s,s' and by Lemma 4.19, we get $\Sigma \geq 4(n-2)+2(n-4)=6n-16>4n-6$, which contradicts with Corollary 4.12.
2. Suppose that $|\{k,m\} \cap \{k',m'\}|=1$. Without loss of generality consider two cases: $k=k'$ and $k=m'$. If $k=k'$, then by Lemma 4.22 for rows s,s' and by Lemma 4.19 we get $\Sigma \geq 2(n-2)+(n-1)+2(n-3)=5n-11>4n-6$, which contradicts with Corollary 4.12. If $k=m'$, then, by Item 1 of Lemma 4.22 for rows s,s', by Item 3 of Definition 4.4, and by Lemma 4.19, we get $\Sigma \geq 2(n-2)+2(n-3)+2(n-3)=6n-16>4n-6$, because $\nu(\text{row}(A,k))-1 \geq n-3$ and $\nu(\text{row}(B,k))-1 \geq n-3$, which contradicts with Corollary 4.12.

3. Suppose that $\{k, m\} = \{k', m'\}$. If $k = k'$ and $m = m'$ then the Lemma is proved. If $k = m'$ and $m = k'$, then by Item 3 of Definition 4.4 and by Lemma 4.19 we get $\Sigma \geq 4(n-3) + 2(n-2) = 6n - 16 > 4n - 6$, because $\nu(\text{row}(A, i)) - 1 \geq n - 3$ and $\nu(\text{row}(B, i)) - 1 \geq n - 3$ for $i = k, m$, which contradicts with Corollary 4.12.

Thus, $k = k'$ and $m = m'$ and the proof is complete. □

Lemma 4.26. *Let $n \geq 6$ and the pair (A, B) be minimal. If $\Sigma(i) = 2$ for some $i \in [n]$, then* row(C, i) *is gift. In particular, $A \neq B$.*

Proof. There are three cases:

1. If there exist $j, j' \in [n] \setminus \{i\}$ with $j \neq j'$ such that $a_{ij} = a_{ij'} = 0$, then $\nu(\text{row}(B, i)) = 1$, since $\Sigma(i) = 2$. Then, by tropical multiplication, row(A, i) must be zero, and this contradicts $\Sigma(i) = 2$. If there exist $j, j' \in [n] \setminus \{i\}$ with $j \neq j'$ such that $b_{ij} = b_{ij'} = 0$, it is similar.
2. If there exist $j, j' \in [n] \setminus \{i\}$ with $j \neq j'$ such that $a_{ij} = b_{ij'} = 0$, then we have a pair of propagation zeros and no duplicates in the i–th row, hence, by Item 2 of Remark 4.8, we have $n - 3$ gift zeros in the i–th row of C, providing a gift row and proving the Lemma.
3. The remaining case is $j = j'$ and $a_{ij} = b_{ij} = 0$ (a duplicate), in which we have exactly one propagation zero in row(C, i), then, by Item 4 of Remark 4.8, there is no gift zero in row(C, p). Hence, by Item 2 of Remark 4.8, we have $n - 2$ cost zeros in row(C, i), so that the row is cost, contradicting Lemma 4.21.

In every case we have A different from B, due to the existence of gift zeros. □

The following Lemma uses Notation 4.13.

Lemma 4.27. *Let $n \geq 6$ and the pair (A, B) be minimal. If there exist at least two different $p, p' \in [n]$ with $\Sigma(p) = \Sigma(p') = 2$, then $A \neq B$ and $(A, B) \in \mathfrak{M}_{km}$ for some k and m with $k \neq m$.*

Proof. By Lemma 4.26, $A \neq B$ and rows p and p' of C are gift rows, and by Lemma 4.25, there exist $k, m \in [n] \setminus \{p, p'\}$ with $k \neq m$ such that $c_{qt} = \phi_{qt}^{km}$, for all $t \in [n] \setminus \{q, k, m\}$ and $q = p, p'$. Then, by Items 2, 3 of Lemma 4.22 for row p, and, by Lemma 4.19, we get $\Sigma \geq 2(n-1) + 2(n-2) = 4n - 6$. By Corollary 4.12 $\Sigma = 4n - 6$. So (A, B) has the following structure based on the number of zeros: $\Sigma(q) = n - 1$ for $q = k, m$ and $\Sigma(q) = 2$ for $q \neq k, m$. Then similarly we get for all $q \neq k, m$ that $c_{qt} = \phi_{qt}^{km}$, $t \in [n] \setminus \{q, k, m\}$. Show that c_{km} and c_{mk} are propagation zeros. Indeed, if $c_{km} = \phi_{km}^{k'm'}$ for some $k', m' \in [n] \setminus \{k, m\}$ (not necessarily $k' \neq m'$), then $a_{m'm} = 0$, but $b_{m'm}$ is also zero because of gift zeros in row(C, m'), hence $\Sigma(m') = 3 > 2$, which is a contradiction with the structure of (A, B). Similarly for c_{mk}. Hence, by Lemma 4.15, $(A, B) \in \mathfrak{M}_{km}$ and the proof is complete. □

Lemma 4.28. *Let $n \geq 6$ and the pair (A, B) be minimal. If there exists $p \in [n]$ with $\Sigma(p) = 2$, then there exists $p' \in [n]$ with $p \neq p'$ and $\Sigma(p') = 2$.*

Proof. By contradiction, assume that $\Sigma(p) = 2$ holds only for $p \in [n]$. Then, by Lemma 4.19, $\Sigma(i) \geq 3$, for all $i \in [n] \setminus \{p\}$. By Lemma 4.26, row$(C, p)$ is a gift row. Moreover, by Item 1 of Lemma 4.22 there are 2 rows with at least $(n-2)$ off–diagonal zeros. Thus, as a whole we get $\Sigma \geq 2(n-2) + 3(n-3) + 2 = 5n - 11 > 4n - 6$, which contradicts with Corollary 4.12, and the proof is complete. □

Lemma 4.29. *Let $n \geq 7$ and the pair (A, B) be minimal. Then there exists $p \in [n]$ with $\Sigma(p) = 2$.*

Proof. By contradiction, assume that there is no row p with $\Sigma(p) = 2$. By Lemma 4.20, there exists $s \in [n]$ such that $2 \leq \Sigma(s) \leq 3$. Hence, by Lemma 4.19, $\Sigma(i) \geq 3$ for $i \in [n]$ and $\Sigma(s) = 3$. By Lemma 4.24 for row s, we have two cases, without loss of generality suppose that $a_{sk} = b_{sl} = b_{sm} = 0$ for some $k, l, m \in [n] \setminus \{s\}$, with $l \neq m$, and for each $t \in [n] \setminus \{s, k, l, m\}$ we have $c_{st} = \phi_{st}^{km}$ or $c_{st} = \phi_{st}^{kl}$, and $\Sigma(k) \geq n - 3$. Using $\Sigma(i) \geq 3$ for $i \in [n] \setminus \{k\}$, we get $\Sigma \geq (n-3) + 3(n-1) = 4n - 6$, hence, by Corollary 4.12, $\Sigma = 4n - 6$. Then $\Sigma(k) = n - 3$ and

$\Sigma(i) = 3$ for $i \in [n]\backslash\{k\}$. Take row $s' \notin \{s, k, l, m\}$ (it exists, because $n \geq 7$). Since $s' \neq k$, then, by Lemma 4.24, we have two cases:

1. If $a_{s'k'} = b_{s'l'} = b_{s'm'} = 0$ for some $k', l', m' \in [n] \setminus \{s'\}$, with $l' \neq m'$, and for each $t \in [n] \setminus \{s', k', l', m'\}$ we have $c_{s't} = \phi_{s't}^{k'm'}$ or $c_{s't} = \phi_{s't}^{k'l'}$, and $\Sigma(k') \geq n - 3$.

   (a) If $k \neq k'$, then $\Sigma(k') \geq n - 3 \geq 4$. It is a contradiction with $\Sigma(q) = 3$ for $q \neq k$.
   (b) If $k = k'$, then $c_{s't} = \phi_{s't}^{kl'}$ or $c_{s't} = \phi_{s't}^{km'}$, for all $t \in [n] \setminus \{s', k, l', m'\}$. Since $s' \in [n] \setminus \{s, k, l, m\}$ and $c_{st} = \phi_{st}^{kl}$ or $c_{st} = \phi_{st}^{km}$, for all $t \in [n] \setminus \{s, k, l, m\}$, then $c_{ss'} = \phi_{ss'}^{kl}$ or $c_{ss'} = \phi_{ss'}^{km}$. Hence by Items 2, 3 of Definition 4.4 $b_{kt} = 0$ for all $t \in [n]\setminus\{k, l', m'\}$, whence $\nu(\text{row}(B, k)) - 1 \geq n - 3$. If $\nu(\text{row}(A, k)) - 1 = 0$, then $\text{row}(B, k)$ must be zero and $\Sigma(k) \geq n - 1 > n - 3$. If $\nu(\text{row}(A, k)) - 1 > 0$, then also $\Sigma(k) > n - 3$, which contradicts with $\Sigma(k) = n - 3$.

2. If $b_{s'k'} = a_{s'l'} = a_{s'm'} = 0$ for some $k', l', m' \in [n] \setminus \{s'\}$, with $l' \neq m'$, and for each $t \in [n] \setminus \{s', k', l', m'\}$ we have $c_{s't} = \phi_{s't}^{m'k'}$ or $c_{s't} = \phi_{s't}^{l'k'}$, and $\Sigma(k') \geq n - 3$.

   (a) If $k \neq k'$, then $\Sigma(k') \geq n - 3 \geq 4$ contradicts with $\Sigma(q) = 3$ for $q \neq k$.
   (b) If $k = k'$, then $c_{s't} = \phi_{s't}^{l'k}$ or $c_{s't} = \phi_{s't}^{m'k}$, for all $t \in [n]\backslash\{s', k, l', m'\}$. Hence by Items 2, 3 of Definition 4.4 $a_{kt} = 0$ for all $t \in [n] \setminus \{s', k, l', m'\}$, whence $\nu(\text{row}(A, k)) - 1 \geq n - 4$. Since $c_{st} = \phi_{st}^{kl}$ or $c_{st} = \phi_{st}^{km}$, for all $t \in [n] \setminus \{s, k, l, m\}$, then by Items 2, 3 of Definition 4.4 $b_{kt} = 0$ for all $t \in [n] \setminus \{s, k, l, m\}$, whence $\nu(\text{row}(B, k)) - 1 \geq n - 4$. Hence $\Sigma(k) \geq 2(n - 4) > n - 3$, which contradicts with $\Sigma(k) = n - 3$.

Thus, (A, B) cannot be minimal and the proof is complete. □

Lemma 4.30. *Let $n = 2$ or $n \geq 7$. If the pair (A, B) is minimal, then $A \neq B$ and $(A, B) \in \mathfrak{M}_{km}$ for some $k, m \in [n]$ with $k \neq m$.*

Proof. Let $n = 2$. Then, by Item 2c of Lemma 3.1, using Notation 2.2, we find all minimal pairs (A, B): these are (Z, I), (I, Z), (U_{12}, U_{21}), (U_{21}, U_{12}). Note that $A \neq B$ for each of these pairs. By Notation 3.11, U_{12} is $V(1; 2)$–generic, U_{21} is $V(2; 1)$–generic, I is $W(2; 1)$–generic, and Z is $W(1; 2) \cap Z(2; 1) \cap Z(1; 2)$–generic. Hence, by Notation 4.13, if (A, B) is a minimal pair, then $(A, B) \in \mathfrak{M}_{12}$ or $(A, B) \in \mathfrak{M}_{21}$.

Now let $n \geq 7$. If (A, B) is a minimal pair, then by Lemma 4.29 there exists $p \in [n]$ with $\Sigma(p) = 2$. Hence by Lemma 4.28 there exists $p' \in [n]$ with $p \neq p'$ and $\Sigma(p') = 2$. Therefore, Lemma 4.27 is applicable which guarantees that $A \neq B$ and $(A, B) \in \mathfrak{M}_{km}$, for some $k, m \in [n]$ with $k \neq m$. The proof is complete. □

Corollary 4.31. *If $n \geq 2$, $n \neq 4$, then $\Theta_n = 4n - 6$. If $n = 4$, then $\Theta_n = 8$.*

Proof. If $n = 2$ or $n \geq 7$, then the statement follows from Lemma 4.30 and Item 2 of Corollary 4.16.

Let $n = 3$ and let (A, B) be minimal. By Lemma 4.19, $\Sigma(i) \geq 2$ for all $i \in [n]$. Then, using Corollary 4.12, we get $6 = 2n \leq \Sigma(A, B) = \Theta_3 \leq 4n - 6 = 6$, hence $\Theta_3 = 4n - 6$.

Let $n = 4$ and let (A, B) be minimal. By Lemma 4.19, $\Sigma(i) \geq 2$ for all $i \in [n]$. In addition, by Example 4.34, $\Theta_4 \leq \Sigma(A_4, B_4) = 8$. Then, $8 = 2n \leq \Sigma(A, B) = \Theta_4 \leq 8$, hence $\Theta_4 = 8$.

Let $n = 5$ and let (A, B) be minimal. Suppose that $\Theta_5 = \Sigma(A, B) < 4n - 6 = 14$. Then by Lemma 4.19, $\Sigma(i) \geq 2$ for all $i \in [n]$. Since $3(n - 1) + 2 \cdot 1 = 14 > \Theta_5$, there exist at least two rows $p, p' \in [n]$ with $\Sigma(p) = \Sigma(p') = 2$. Also there is no cost row in C (indeed: if a cost row exists, then, by Item 6 of Remark 4.8 and Lemma 4.19, we get $\Sigma(A, B) \geq 2(n - 2) + 2(n - 1) = 4n - 6 > \Theta_5$, a contradiction). Hence p and p' are gift rows, by the proof of Lemma 4.26. Then, by the proof of Lemma 4.25, using $5n - 11 = 6n - 16 = 14 > \Theta_5$, we get that there exist $k, m \in [n] \setminus \{p, p'\}$ with $k \neq m$ such that $c_{qt} = \phi_{qt}^{km}$, for all $t \in [n] \setminus \{q, k, m\}$ and $q = p, p'$. Then, by Items 2, 3 of Lemma 4.22 for row p, and, by Lemma 4.19, we get $\Sigma(A, B) \geq 2(n - 1) + 2(n - 2) = 4n - 6$, which is a contradiction with $\Sigma(A, B) < 4n - 6$. Hence, $\Theta_5 \geq 4n - 6$, and Corollary 4.12 completes the proof.

Let $n = 6$ and let (A, B) be minimal. Suppose that $\Theta_6 = \Sigma(A, B) < 4n - 6 = 18$. By Lemma 4.19, $\Sigma(i) \geq 2$ for all $i \in [n]$. Since $3n = 18 > \Theta_6$, there exists at least one row $p \in [n]$ with $\Sigma(p) = 2$. Then, by Lemmas 4.27, 4.28, we get $A \neq B$ and $(A, B) \in \mathfrak{M}_{km}$, for some $k, m \in [n]$ with $k \neq m$. Hence $\Sigma(A, B) = 4n - 6$, by Item 2 of Corollary 4.16, which is a contradiction with $\Sigma(A, B) < 4n-6$. Hence, $\Theta_6 \geq 4n-6$, and Corollary 4.12 completes the proof. □

Remark 4.32. *Comparing* $\Theta_n = \operatorname{prop}(C) = 4n-6$ *and* $\operatorname{gift}(C) = (n-2)(n-3)$, *we notice that* $\operatorname{prop}(C) \leq \operatorname{gift}(C)$ *if and only if* $n \geq 8$, *the case* $n = 7$ *giving* $4n-6 = 22 > 20 = (n-2)(n-3)$. *Asymptotically, the ratio* $\operatorname{gift}(C)/\operatorname{prop}(C)$ *is* $n/4$.

Theorem 4.33. *Let* $n = 2$ *or* $n \geq 7$. *Then the pair* (A, B) *is minimal if and only if* $A \neq B$ *and* $(A, B) \in \mathfrak{M}_{km}$ *for some* $k, m \in [n]$ *with* $k \neq m$.

Proof. The necessity follows from Lemma 4.30. Let us prove the sufficiency.

Assume, $n = 2$. Then, by Item 2c of Lemma 3.1, using Notation 2.2, we find all minimal pairs (A, B): (Z, I), (I, Z), (U_{12}, U_{21}), (U_{21}, U_{12}). Using Notation 3.11, we get that for $k \neq m$, the sets of $V(k; m)$–generic matrices and of $W(m; k) \cap Z(k; m)$–generic matrices are both equal to $\{U_{km}\}$, the set of $W(k; m)$–generic matrices equals $\{I\}$, and the set of $W(k; m) \cap Z(m; k) \cap Z(k; m)$–generic matrices equals $\{Z\}$. Hence, by Notation 4.13, if $A \neq B$ and $(A, B) \in \mathfrak{M}_{km}$ for some $k, m \in [2]$ with $k \neq m$, then (A, B) is minimal.

Now let $n \geq 7$. Then $\Theta_n = 4n - 6$, by Corollary 4.31. If $A \neq B$ and $(A, B) \in \mathfrak{M}_{km}$, for some $k, m \in [n]$ with $k \neq m$, then, by Item 2 of Corollary 4.16, we get $\Sigma(A, B) = 4n - 6 = \Theta_n$, hence (A, B) is minimal. □

The following example shows that Theorem 4.33 does not hold for $n = 3, 4, 5, 6$. It also shows that few gift rows or no gift rows is possible for $n \leq 6$.

Example 4.34. *The following orthogonal pairs* (A_n, B_n), $n = 3, 4, 5, 6$, *are minimal, but* $(A_n, B_n) \notin \mathfrak{M}_{km}$ *for all* $k, m \in [n]$ *with* $k \neq m$. *The minimality of the pairs follows from Corollary 4.31.*

$$A_3 = \begin{bmatrix} 0 & - & - \\ - & 0 & - \\ - & - & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$A_4 = \begin{bmatrix} 0 & - & - & 0 \\ - & 0 & 0 & - \\ - & 0 & 0 & - \\ 0 & - & - & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & - & 0 & - \\ - & 0 & - & 0 \\ 0 & - & 0 & - \\ - & 0 & - & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & \phi_{12}^{43} & 0 & 0 \\ \phi_{21}^{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_{34}^{21} \\ 0 & 0 & \phi_{43}^{12} & 0 \end{bmatrix}.$$

$$A_5 = \begin{bmatrix} 0 & - & - & 0 & - \\ - & 0 & 0 & - & 0 \\ - & 0 & 0 & - & 0 \\ 0 & - & - & 0 & - \\ - & 0 & 0 & - & 0 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 & - & 0 & - & - \\ - & 0 & - & 0 & - \\ 0 & - & 0 & - & - \\ - & 0 & - & 0 & 0 \\ - & - & - & 0 & 0 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 0 & \phi_{12}^{43} & 0 & 0 & \phi_{15}^{43} \\ \phi_{21}^{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_{34}^{21} & 0 \\ 0 & 0 & \phi_{43}^{12} & 0 & 0 \\ \phi_{51}^{34} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$A_6 = \begin{bmatrix} 0 & - & - & 0 & - & - \\ - & 0 & - & - & 0 & - \\ - & - & 0 & - & - & 0 \\ 0 & - & - & 0 & - & - \\ - & 0 & - & - & 0 & - \\ - & - & 0 & - & - & 0 \end{bmatrix}, \quad B_6 = \begin{bmatrix} 0 & - & - & - & 0 & 0 \\ - & 0 & - & 0 & - & 0 \\ - & - & 0 & 0 & 0 & - \\ - & 0 & 0 & 0 & - & - \\ 0 & - & 0 & - & 0 & - \\ 0 & 0 & - & - & - & 0 \end{bmatrix}, \quad C_6 = \begin{bmatrix} 0 & \phi_{12}^{45} & \phi_{13}^{46} & 0 & 0 & 0 \\ \phi_{21}^{54} & 0 & \phi_{23}^{56} & 0 & 0 & 0 \\ \phi_{31}^{64} & \phi_{32}^{65} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{45}^{12} & \phi_{46}^{13} \\ 0 & 0 & 0 & \phi_{54}^{21} & 0 & \phi_{56}^{23} \\ 0 & 0 & 0 & \phi_{64}^{31} & \phi_{65}^{32} & 0 \end{bmatrix}.$$

4.2 Self–orthogonal matrices

If $A^2 = Z_n$, then A is called *self–orthogonal*. Now we set $A = B$ and let C be the indicator matrix of the pair (A, A). Then, $a_{sk} = a_{kt} = 0$ with $s, t, k \in [n]$ pairwise different and $a_{st} \neq 0$ yield a cost zero $c_{st} = \phi_{st}^{kk}$.

Notice that $\Theta_n \leq 2\Theta_n^{\Delta}$ where Θ_n^{Δ} is the minimum over the diagonal Δ of $M_n^N \times M_n^N$

$$\Theta_n^{\Delta} := \min_{A \in M_n^N} \{\nu(A) - n : A^2 = Z_n\}. \tag{9}$$

By Corollary 3.14 and Lemma 3.17, we know that $\Theta_n^{\Delta} \leq 2n-2$. The aim of this subsection is to prove that $\Theta_n^{\Delta} = 2n-2$ and so $\Theta_n = 2\Theta_n^{\Delta} - 2$ (this agrees with the minimal values of $\nu(A) - n$ found in Lemma 3.17). The following theorem is an analogue of Theorem 4.33 for the case $A = B$. Observe that gift zeros do not exist, as remarked in Item 5 of Remark 4.8, and we cannot use Lemmas 4.15, 4.22, 4.25, 4.26 and 4.29.

Theorem 4.35. *Let $n \geq 5$. The matrix $A \in M_n^N$ is self–orthogonal with the minimal number of off–diagonal zeros Θ_n^{Δ} if and only if there exists $k \in [n]$ such that A is $V(k;k)$–generic.*

Proof. Let A be self–orthogonal with minimal number of off–diagonal zeros, i.e., $\nu(A) - n = \Theta_n^{\Delta}$. By Lemma 4.19, $\nu(\text{row}(A,i)) - 1 \geq 1$ for all $i \in [n]$. Show that there exist at least two rows s, s' with $\nu(\text{row}(A,s)) - 1 = 1$ and $\nu(\text{row}(A,s')) - 1 = 1$ (indeed, if $\nu(\text{row}(A,i)) - 1 \geq 2$ for $n-1$ rows, then $\nu(A) - n \geq 2(n-1) + 1 = 2n-1 > 2n-2$, which contradicts with $\Theta_n^{\Delta} = \nu(A) - n \leq 2n-2$). Let $a_{sk} = 0$ for some $k \in [n] \setminus \{s\}$ and $a_{s'k'} = 0$ for some $k' \in [n] \setminus \{s'\}$. Since $\text{row}(C,s)$ contains only one propagation zero, C does not contain gift zeros, and $\text{row}(C,s)$ is zero then, by Items 2 and 8 of Remark 4.8, $\text{row}(C,s)$ is k–cost and $c_{st} = \phi_{st}^{kk}$, for all $t \in [n] \setminus \{s,k\}$. Similarly, $\text{row}(C,s')$ is k'–cost and $c_{s't} = \phi_{s't}^{k'k'}$, for all $t \in [n] \setminus \{s',k'\}$. By Item 2 of Definition 4.4, $a_{kt} = 0$ for all $t \in [n] \setminus \{s,k\}$, and $a_{k't} = 0$ for all $t \in [n] \setminus \{s',k'\}$. If $k \neq k'$, then $\nu(\text{row}(A,k)) - 1 \geq n-2$ and $\nu(\text{row}(A,k')) - 1 \geq n-2$, whence using $\nu(\text{row}(A,i)) - 1 \geq 1$ for $i \in [n] \setminus \{k,k'\}$ we get $\Theta_n^{\Delta} = \nu(A) - n \geq 2(n-2) + (n-2) = 3n-6 > 2n-2$, a contradiction. Thus, $k = k'$ and $a_{kt} = 0$ for all $t \in [n] \setminus \{k\}$. Hence $\nu(\text{row}(A,k)) - 1 \geq n-1$. Then $\nu(A) - n \geq (n-1) + (n-1) = 2n-2$, hence $\nu(A) - n = 2n-2$. Then $\nu(\text{row}(A,k)) - 1 = n-1$ and $\nu(\text{row}(A,i)) - 1 = 1$ for all $i \in [n] \setminus \{k\}$. Since for all $i \in [n] \setminus \{k\}$ $\text{row}(C,i)$ contains only one propagation zero, similarly we get that $\text{row}(C,i)$ is k–cost and $c_{it} = \phi_{it}^{kk}$, for all $i \in [n] \setminus \{k\}$ and for all $t \in [n] \setminus \{i,k\}$, which completes the proof.

For the sufficiency, let the matrix A be $V(k;k)$–generic. Then, by Corollary 3.14 and Lemma 3.17 A is self–orthogonal and $\nu(A) - n = 2n-2$. Using the proved necessity we get the desired result. □

The following example shows that Theorem 4.35 is not true for $n = 3$.

Example 4.36. $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$ *is self–orthogonal, by Example 3.6, but $A \notin V(k;k)$, for $k \in [3]$.*

5 Orthogonality by bordering

In this section we study what happens with orthogonality after adding a row and a column, which enables us to construct orthogonal pairs of arbitrary sizes. We assume $n \geq 2$.

Let the matrix $A = \begin{bmatrix} B & v \\ w^T & 0 \end{bmatrix} \in M_n^N$ be decomposed into blocks, with $B \in M_{n-1}^N$, and v, w non–positive vectors.

Proposition 5.1 (Orthogonality by bordering). *Let $A_k = \begin{bmatrix} B_k & v_k \\ w_k^T & 0 \end{bmatrix} \in M_n^N$ be as above, with $k = 1, 2$. If $B_1B_2 = Z_{n-1} = B_2B_1$, then $A_1A_2 = Z_n = A_2A_1$ if and only if $B_1v_2 \oplus v_1 = B_2v_1 \oplus v_2$ and $w_1^TB_2 \oplus w_2^T = w_2^TB_1 \oplus w_1^T$ are zero vectors.*

Proof. Easy computations show that

$$A_1A_2 = \begin{bmatrix} Z_{n-1} & B_1v_2 \oplus v_1 \\ w_1^TB_2 \oplus w_2^T & 0 \end{bmatrix}, \; A_2A_1 = \begin{bmatrix} Z_{n-1} & B_2v_1 \oplus v_2 \\ w_2^TB_1 \oplus w_1^T & 0 \end{bmatrix}.$$

The rest is immediate. □

Notation 5.2. For $i, j \in [n]$ let $P^{ij} = [p_{kl}]$ be the *permutation matrix* corresponding to the transposition (ij), i.e., $p_{kl} = 0$ if $(k, l) = (i, j)$ or (j, i) or $k = l \in [n] \setminus \{i, j\}$, $p_{kl} = -1$, otherwise.[5]

Definition 5.3 (Orthogonal set of a matrix). For any subset $S \subseteq M_n^N$, define the set $\mathrm{Or}(A)_S := \{B \in S : AB = Z_n = BA\}$. Write $\mathrm{Or}(A)$ if S is the ambient space.

Lemma 5.4 (Decreasing size). *If $A \in M_n^N$ and there exists $i \in [n]$ such that both the i–th row and the i–th column of A have no zero entries except on the main diagonal, then $P^{ni}AP^{ni} = \begin{bmatrix} B & v \\ w^T & 0 \end{bmatrix}$, with $v, w \in \mathbb{R}_{\le 0}^{n-1}$ without zero entries, and*

$$\mathrm{Or}(P^{ni}AP^{ni}) = \left\{ \begin{bmatrix} D & Z_{(n-1)\times 1} \\ Z_{1\times(n-1)} & 0 \end{bmatrix} : D \in \mathrm{Or}(B) \right\}.$$

In particular, $|\,\mathrm{Or}(A)| = |\,\mathrm{Or}(B)|$. □

Proof. First, it is easy to check that $P^{ni}\,\mathrm{Or}(A)P^{ni} = \mathrm{Or}(P^{ni}AP^{ni})$. Now, set $A_1 = P^{ni}AP^{ni}$, $B_1 = B$, $v_1 = v$, $w_1 = w$ in Lemma 5.1 and notice that, by hypothesis, the vectors v_1 and w_1 never vanish. If $A_1A_2 = Z_n = A_2A_1$, then the vector $B_1v_2 \oplus v_1 = B_1v_2$ is zero, by Lemma 5.1. From normality of B_1 and monotonicity of $\odot$, it follows that $I_{n-1} \le B_1 \le Z_{n-1}$ and $v_2 \le B_1v_2 \le Z_{n-1}v_2$. But the vector $Z_{n-1}v_2$ is constant, so it vanishes if and only if v_2 vanishes. Similarly, w_2 vanishes. Then the last row and column in A_2 are zero and, if we write $D = B_2$, we get the statement. □

Corollary 5.5. *The only matrix orthogonal to a strictly normal matrix of order n is Z_n.*

Proof. Let $A \in M_n^{SN}$ and, for $j \in [n]$, let A_j denote the principal submatrix of A of order j. Then applying Lemma 5.4 repeatedly for $j \in [n]$, $i = j$, $A = A_j$ and $B = A_{j-1}$, we obtain that $|\,\mathrm{Or}(A_n)| = |\,\mathrm{Or}(A_{n-1})| = \cdots = |\,\mathrm{Or}(A_2)| = 1$. Since $Z \in \mathrm{Or}(X)$ for any $X \in M_n^N$, the result follows. □

Corollary 5.6 (Self–orthogonality by bordering). *Let $A = \begin{bmatrix} B & v \\ w^T & 0 \end{bmatrix} \in M_n^N$ be decomposed into blocks, with $B \in M_{n-1}^N$. If B is self–orthogonal, then A is self–orthogonal if and only if Bv and w^TB are zero vectors.*

Proof. We have $A^2 = \begin{bmatrix} B^2 \oplus vw^T & Bv \oplus v \\ w^TB \oplus w^T & 0 \end{bmatrix}$, where the matrix vw^T is non–positive, but not necessarily normal. By hypothesis, we have $B^2 = Z$, whence $B^2 \oplus vw^T = Z \oplus vw^T = Z$ and $Bv \oplus v = (B \oplus I)v = Bv$. □

6 Three orthogonality graphs

In this section we compute the diameter and girth of three types of graphs related to the orthogonality relation. Graphs can have loops but no multiple edges. We assume $n \ge 3$ since otherwise the graphs under consideration are disconnected and more or less trivial. In the first and most intuitive graph, denoted $\mathcal{ORTHO}$, vertices are matrices and an edge between two matrices A, B means that A, B are mutually orthogonal. A loop stands for a self–orthogonal matrix.

Let $\Gamma = (V, E)$ be a graph with the vertex set V and edges $E \subseteq V \times V$. We consider three different sets of vertices.

5 Properties of P^{ij} are analogous in classical and tropical linear algebra. In general, P^{ij} is not a normal matrix.

Definition 6.1. A *path* (or *walk)* is a sequence $v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k$ of vertices $v_0, \dots, v_k \in V$ and edges $e_1, \dots, e_k \in E$ where $e_i = (v_{i-1}, v_i)$ for all $i = 1, \dots, k$. If $v_0 = v_k$, then the path is *closed*. The *length of the former path* is k. A path is *elementary* if all the edges are distinct. A *cycle* is a closed elementary path.

Definition 6.2. The *girth* of a graph Γ is the length of the shortest cycle in Γ which is not a loop.

Definition 6.3. The graph Γ is said to be *connected* if it is possible to establish a path from any vertex to any other vertex of Γ.

Definition 6.4. The *distance* $\operatorname{dist}(u, v)$ between two vertices u and v in a graph Γ is the length of the shortest path between them. If u and v are unreachable from each other, then we set $\operatorname{dist}(u, v) = \infty$. It is assumed that $\operatorname{dist}(u, u) = 0$ for any vertex u.

Definition 6.5. The *diameter* $\operatorname{diam}(\Gamma)$ of a graph Γ is the maximum of distances between vertices, for all pairs of vertices in Γ.

Recall that the absorbing property (3) in Remark 2.3 says that Z is orthogonal to every matrix. On the other hand, if A is strictly normal, then the only matrix orthogonal to A is Z, by Corollary 5.5. So, it is reasonable to consider

$$(M_n^N)^* := M_n^N \setminus \left(M_n^{SN} \cup \{Z\}\right)$$

as a set of vertices. Namely, we delete the vertex which is connected with all other vertices as well as the set of isolated vertices.

Recall Notation 3.11. In view of Corollary 4.18, other interesting sets of matrices are

$$VNL := \bigcup_{p,q \in [n],\, p \neq q} V(p;q), \qquad VNL^* = VNL \setminus \{Z\},$$

$$WNL := \bigcup_{p,q \in [n],\, p \neq q} W(p;q), \qquad WNL^* = WNL \setminus \{Z\}.$$

Now we define the corresponding graphs.

Definition 6.6. The vertex set of the graph $\mathcal{ORTHO}$ is $(M_n^N)^*$. Matrices $A, B \in (M_n^N)^*$ are joined by an edge in $\mathcal{ORTHO}$ if and only if $AB = Z = BA$. In particular, loops in $\mathcal{ORTHO}$ correspond to self–orthogonal matrices.

Lemma 3.13 is motivation for the following Definition.

Definition 6.7. The vertex set of the graph $\mathcal{VNL}$ is VNL^*. Matrices $A, B \in VNL^*$ are joined by an edge in $\mathcal{VNL}$ if and only if there exist $p, q \in [n]$ with $A \in V(p;q)$ and $B \in V(q;p)$.

Definition 6.8. The vertex set of the graph $\mathcal{WNL}$ is WNL^*. Matrices $A, B \in WNL^*$ are joined by an edge in $\mathcal{WNL}$ if and only if there exist $k, m \in [n]$ with A and B satisfying one of the conditions $1, 2, 3$ in Corollary 4.18.

Remark 6.9. *Notice that $VNL^*, WNL^* \subseteq (M_n^N)^*$ and, by Corollary 4.18, the graphs $\mathcal{VNL}, \mathcal{WNL}$ are subgraphs of $\mathcal{ORTHO}$.*

Proposition 6.10. *Let $n \geq 3$. Then* $\operatorname{girth}(\mathcal{ORTHO}) = \operatorname{girth}(\mathcal{VNL}) = 3$.

Proof. Recall Notation 3.11 and take $A_1 \in V(1, 2; 3)$, $A_2 \in V(2, 3; 1)$ and $A_3 \in V(3, 1; 2)$. Then A_1–A_2–A_3–A_1 make a cycle in $\mathcal{VNL}$, so $\operatorname{girth}(\mathcal{ORTHO}) \leq \operatorname{girth}(\mathcal{VNL}) \leq 3$. Since the graphs under consideration have no multiple edges, the proof is complete. □

Proposition 6.11. *Let $n \geq 3$. Then $\mathcal{ORTHO}$ is connected and* $\operatorname{diam}(\mathcal{ORTHO}) = 3$.

Proof. Let A, B be matrices corresponding to distinct vertices in the graph $\mathcal{ORTHO}$. Each of them has at least one off–diagonal zero. Assume $a_{ij} = 0 = b_{km}$, with $i \neq j$ and $k \neq m$. We can assume that $(i, j) \neq (k, m)$. Then, by Notations 2.2 and Corollary 3.5, $A \in \mathrm{Or}(E_{ij})_{(M_n^N)^*}$, $B \in \mathrm{Or}(E_{km})_{(M_n^N)^*}$, and also $E_{km} \in \mathrm{Or}(E_{ij})_{(M_n^N)^*}$, since $(i, j) \neq (k, m)$. So the path $A - (E_{ij}) - (E_{km}) - B$ shows that $\mathrm{diam}(\mathcal{ORTHO}) \leq 3$.

Now, consider the matrices U_{ij}, U_{km}, with $i, j \in [n]$, $i \neq j$, $k, m \in [n]$, $k \neq m$ and $(i, j) \neq (k, m)$. Since $n \geq 3$, there exists $l \in [n] \setminus \{i, k\}$, and the l–th row of the product $U_{ij}U_{km}$ is non–zero. Hence, U_{ij} and U_{km} are not orthogonal and $\mathrm{dist}(U_{ij}, U_{km}) > 2$, which completes the proof. □

Proposition 6.12. *Let $n \geq 3$. Then* $\mathrm{diam}(\mathcal{VNL}) = 2$.

Proof. Let A, B be matrices corresponding to two distinct vertices in the graph $\mathcal{VNL}$. If for some $p, q \in [n]$ both $A, B \in V(p; q)$ then we have a path $A - C - B$, where $C \in V(q; p)$ is non–zero. Now assume $A \in V(p; q)$, $p \neq q$, $B \in V(c; d)$, $c \neq d$, and $p \neq c$. If $|\{p, q, c, d\}| = 2$, then $(p, q) = (d, c)$ and we have a path $A - B$. Otherwise the set $\{p, q, c, d\}$ contains at least 3 different numbers. Consider the set $S = V(q; p) \cap V(d; c)$ and an S–generic matrix C. We show that $C \neq Z$. Indeed, in C the q–th and the d–th rows are zero, and also the p–th and the c–th columns are zero. The number of zeros in C satisfies $\nu(C) \leq 2n + 2(n - 2) + n = 5n - 4$. But since $|\{p, q, c, d\}| \geq 3$, then at least 3 diagonal zeros (among $c_{qq}, c_{pp}, c_{dd}, c_{cc}$) intersect with zero rows and columns of C. So $\nu(C) \leq 5n - 4 - 3 = 5n - 7$. It is clear that $5n - 7 < n^2$, so C is not the zero matrix. So we have a path $A - C - B$, by Definition 6.7. We have shown that $\mathrm{diam}(\mathcal{VNL}) \leq 2$.

Now, consider a $V(i; j)$–generic matrix L, $i, j \in [n]$, $i \neq j$, and a $V(k; m)$–generic matrix M, $k, m \in [n]$, $k \neq m$, $(i, j) \neq (k, m)$, $(i, j) \neq (m, k)$. Then, by Definition 6.7, L and M are not joined by an edge in $\mathcal{VNL}$ and $\mathrm{dist}(L, M) > 1$, which completes the proof. □

Proposition 6.13. *If $n \geq 4$, then* $\mathrm{diam}(\mathcal{WNL}) = 2$.

Proof. Let A, B be matrices corresponding to two distinct arbitrary vertices in the graph $\mathcal{WNL}$. If, for some $p, q \in [n]$ both $A, B \in W(p; q)$ then we have a path $A - C - B$, where $C \in W(q; p) \cap Z(q; p) \cap Z(p; q)$ is non–zero. Assume $A \in W(p; q)$, $p \neq q$, $B \in W(c; d)$, $c \neq d$, and $p \neq c$. If $|\{p, q, c, d\}| = 2$, then $(p, q) = (d, c)$ and we have a path $A - C - B$, where $C \in V(p; q) \cap V(q; p)$ is non–zero. Otherwise the set $\{p, q, c, d\}$ contains at least 3 different numbers. Consider the set $S = V(q; p) \cap Z(p; q) \cap V(d; c) \cap Z(c; d)$ and an S–generic matrix C. We show that $C \neq Z$. Indeed, in C the q–th and the d–th rows are zero, and also the p–th and the c–th columns are zero, besides $c_{pq} = c_{cd} = 0$. The number of zeros in C satisfies $\nu(C) \leq 2n + 2(n - 2) + 2 + n = 5n - 2$. But since $|\{p, q, c, d\}| \geq 3$, then at least 3 diagonal zeros (among $c_{qq}, c_{pp}, c_{dd}, c_{cc}$) have been counted twice. So $\nu(C) \leq 5n - 2 - 3 = 5n - 5$. We have $5n - 5 < n^2$, so C is not the zero matrix. So we have a path $A - C - B$, by Definition 6.8. We have shown that $\mathrm{diam}(\mathcal{WNL}) \leq 2$.

Now, consider a $W(i; j)$–generic matrix L, with $i, j \in [n]$, $i \neq j$, and a $W(k; m)$–generic matrix M, with $k, m \in [n]$, $k \neq m$, $(i, j) \neq (k, m)$. Then, by Definition 6.8, L and M are not joined by an edge in $\mathcal{WNL}$ and $\mathrm{dist}(L, M) > 1$, which completes the proof. □

The following example shows that the above Lemma is not true if $n = 3$.

Example 6.14. *For* $A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \in W(1; 2)$, $B = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix} \in W(1; 3)$ *we get that* $\mathrm{dist}(A, B) = 3$ *in* $\mathcal{WNL}$.

Indeed, the following path shows that $\mathrm{dist}(A, B) \leq 3$ in $\mathcal{WNL}$

$$A - \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} - B.$$

Note that matrices A and B are not mutually orthogonal. Let us show that $\mathrm{dist}(A, B) > 2$ in $\mathcal{WNL}$. Indeed, suppose that we have a path $A - D - B$ in $\mathcal{WNL}$. By definition, A and D are connected if for some k and m one of the following items holds:

1. $A \in W(k; m) \cap Z(k; m) \cap Z(m; k)$ and $D \in W(m; k)$,
2. $A \in W(k; m) \cap Z(m; k)$ and $D \in W(m; k) \cap Z(k; m)$,
3. $A \in W(k; m)$ and $D \in W(m; k) \cap Z(k; m) \cap Z(m; k)$.

The structure of A implies that the only case left is the item 3 with $k = 1, m = 2$. Similarly, for B with $k = 1, m = 3$. But $D \in W(2; 1) \cap Z(1; 2) \cap Z(2; 1)$ and $D \in W(3; 1) \cap Z(1; 3) \cap Z(3; 1)$ only if D is zero and the proof is complete.

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