Extreme points of some families of non-additive measures*

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Abstract

Non-additive measures are a valuable tool to model many different problems arising in real situations. However, two important difficulties appear in their practical use: the complexity of the measures and their identification from sample data. For the first problem, additional conditions are imposed, leading to different subfamilies of non-additive measures. Related to the second point, in this paper we study the set of vertices of some families of non-additive measures, namely k-additive measures and p-symmetric measures. These extreme points are necessary in order to properly apply a new method of identification of non-additive measures based on genetic algorithms, whose cross-over operator is the convex combination. We solve the problem through techniques of Linear Programming.

Keywords: Decision analysis, genetic algorithms, multiple criteria analysis, linear programming, non-additive measures, k-additivity, p-symmetry, vertices.

1 Introduction

Non-additive measures (also called capacities or fuzzy measures) constitute a generalization of classical probability distributions in which we have removed additivity and monotonicity is imposed instead. This extension is perfectly justified in many practical situations, in which additivity is too restrictive.

For example, in the field of Decision Making, models based on Probability, as those from von Neumann and Morgenstern [42] or Anscombe and Aumann [2] to cite a few, can lead to inconsistencies due to *risk aversion* and the *certainty effect* as the well-known paradoxes of Ellsberg [14] and Allais [1], respectively. However, models based on non-additive measures [6, 36] are able to handle and interpret these problems. Moreover, in recent years, the analysis and use of non-additive measures have been enriched by different equivalent representations of a capacity [18], that are obtained through invertible linear transformations applied on the measure.

Non-additive measures have been successfully applied to model problems in Multicriteria Decision Making and Cooperative Games. In the former case, non-additive measures allow the decision maker to introduce vetoes and favors in the model [18], as well as interactions among the different criteria [19]. In the theory of Cooperative

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Games, non-additive measures represent the strength of coalitions of players; they are related to the Shapley value [38], as shown in [17]. Other fields related to non-additive measures are combinatorics [35], pseudo-Boolean functions [22], etc.

This versatility of non-additive measures has led to a huge number of related works, both from a theoretical and from a practical point of view [12, 43], and they have become a powerful tool in many fields.

However, despite the fact of the many advantages of non-additive measures, their practical use has to face with the hurdle of an increment in the complexity. In the case of finite spaces of cardinality n, just $n-1$ values suffice to define a probability measure, while $2^n - 2$ coefficients are needed for non-additive measures. This exponential complexity is the *Achilles heel* of the theory.

In an attempt to cope with the complexity involved by the use of non-additive measures, Grabisch [17] has proposed the concept of k-additive measures, and Miranda and Grabisch [30] have recently proposed a generalization of symmetric measures, the so-called p-symmetric measures. In both cases we obtain an important reduction in the number of coefficients needed to define the non-additive measure. These families provide a model which is both flexible and simple to use.

Suppose a situation that can be modelled through the Choquet integral [8] with respect to a p-symmetric or a k-additive measure (an axiomatic characterization of such a model for the k-additive case can be found in [29]). Next step should be the identification of the corresponding measure.

We will assume that some kind of experimental information is available. We also assume that this sample information is numerical; if this is not the case, it should be transformed into numerical data through a tool dealing with ordinal values, as MACBETH [13] or TOMASO [26]. The goal is to find a (k-additive, p-symmetric) measure fitting this data. It must be remarked at this point that it is possible to find several measures equally suitable [33].

If the considered proximity criterion is the squared error, different techniques exist to solve the problem. For example, Grabisch and Nicolas [21] have developed a method based on solving a quadratic problem. On the other hand, Wang *et al.* [44] have developed a method based on genetic algorithms; although in this case a suboptimal solution is found, the computational cost is greatly reduced.

In a recent paper [9], we have proposed an algorithm for the learning of k-additive and p-symmetric measures from sample data based on genetic algorithms. In that paper, the cross-over operator was the convex combination. This operator seems to be logical and simple, and has the advantage that the resulting measure keeps the k-additivity (resp. p-symmetry), so that we do not need to check the conditions at each step. However, an important drawback is that the search space is reduced in each iteration; this is solved in part through the introduction of a mutation operator, but anyway we are reduced to the region determined by the initial population. Therefore, it is needed an initial population wide enough to contain the searched measure inside the region that it determines. In this sense, the best option is to consider the set of vertices.

The aim of this paper is to determine the set of vertices or extreme points for these two families of non-additive measures. All along the paper, the results are based on linear programming methods [3].

The paper is organized as follows: In Section 2 we introduce the basic concepts on non-additive measures that will be needed throughout the paper. Section 3 is devoted to give a brief description of genetic algorithms and explain the special characteristics of ours. Vertices of the set of k-additive measures are studied in Section 4, while Section 5 deals with the same problem for p-symmetric measures. We finish with some conclusions and open problems.

2 Basic concepts on non-additive measures

Consider a finite referential set of n elements (*criteria* in Multicriteria Decision Making, *players* in Cooperative Games, ...), $X = \{x_1, ..., x_n\}$. Let us denote by $\mathcal{P}(X)$ the set of subsets of X. Subsets of X are denoted $A, B, ...$ and also by $A_1, A_2, ...$

Definition 1. *[8, 10, 40] A* **non-additive measure** *over* $(X, \mathcal{P}(X))$ *is a mapping* $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ *satisfying*

- $\mu(\emptyset) = 0$, $\mu(X) = 1$ *(boundary conditions).*
- $\forall A, B \in \mathcal{P}(X)$, *if* $A \subseteq B$, *then* $\mu(A) \leq \mu(B)$ *(monotonicity).*

We will denote the set of all non-additive measures on X by $\mathcal{FM}(X)$. Remark that $\mathcal{FM}(X)$ is a bounded convex polyhedron i.e., the intersection of a finite number of semispaces; each of these semispaces is linked to a monotonicity constraint.

A special class of non-additive measures is the set of $\{0,1\}$-valued measures.

Definition 2. *A non-additive measure is* $\{0,1\}$**-valued** *if it only takes values 0 and 1.*

Definition 3. *Given a convex subset of non-additive measures, $\mathcal{F} \subseteq \mathcal{FM}(X)$, we say that $\mu \in \mathcal{F}$ is a* **vertex** *of $\mathcal{F}$ if we cannot find two non-additive measures $\mu_1, \mu_2 \in \mathcal{F}$, both different form μ, and a scalar $\lambda \in (0,1)$ such that*

$$\mu(A) = \lambda\mu_1(A) + (1-\lambda)\mu_2(A),\ \forall A \in \mathcal{P}(X).$$

The set of vertices completely determine the convex polyhedron, as any point in it can be written as a convex combination of the vertices (see [3]).

Given a vector $\vec{x}$ in a convex polyhedron on $\mathbb{R}^p$, if it satisfies one of the constraints with equality, we say that this constraint is *active* for $\vec{x}$. If $\vec{x}$ is an extreme point or vertex, then, the system defined by its active constraints has a unique solution. Two distinct vertices in the polyhedron are said to be *adjacent* if we can find $p-1$ linearly independent constraints that are active at both of them.

As $\mathcal{FM}(X)$ is a bounded convex polyhedron, any measure μ can be put as a convex combination of the vertices. The vertices of $\mathcal{FM}(X)$ have been obtained by Radojevic in [34, Proposition 2], in which it is given a constructive way to find the coefficients of the convex combination:

Proposition 1. *The set of $\{0,1\}$-valued measures constitutes the set of vertices of $\mathcal{FM}(X)$.*

Remark 1. *The proof appearing in [34] can be adapted as an alternative proof of Theorem 2 below to show that the vertices of the set of p-symmetric measures are the $\{0,1\}$-valued measures that are also p-symmetric. Reciprocally, the proof of Theorem 2 based on unimodular matrices can be adapted to prove Proposition 1.*

A special case of $\{0,1\}$-valued measures are the so-called unanimity games:

Definition 4. *A* **unanimity game** *over $A \subseteq X$, $A \neq \emptyset$ is a non-additive measure defined by*

$$u_A(B) := \begin{cases} 1 & \text{if A} \subseteq \text{B} \\ 0 & \text{otherwise} \end{cases}$$

For $\emptyset$, we define the unanimity game by

$$u_\emptyset(B) := \begin{cases} 1 & \text{if B} \neq \emptyset \\ 0 & \text{if B} = \emptyset \end{cases}$$

There are other set functions that can be used to equivalently represent a non-additive measure. In this paper we will need the so-called Möbius transform.

Definition 5. *[35] Let μ be a set function (not necessarily a non-additive measure) on X. The* **Möbius transform (or inverse)** *of μ is another set function on X defined by*

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B),\ \forall A \subseteq X. \tag{1}$$

The Möbius transform given, the original set function can be recovered through the *Zeta transform* [7]:

$$\mu(A) = \sum_{B \subseteq A} m(B). \tag{2}$$

The value $m(A)$ represents the strength of the subset A in any coalition in which it appears. Remark that the Möbius transform can attain negative values; when it is a non-negative function, it corresponds to the *basic probability mass assignment* in Dempster-Shafer theory of evidence [37].

Related to unanimity games, the following result can be proved.

Proposition 2. *[20] The Möbius transform of the unanimity game on A, with $A \neq \emptyset$, is given by*

$$m(B) = \begin{cases} 1 & \text{if} \quad B = A \\ 0 & \text{otherwise} \end{cases}$$

For $u_\emptyset$, the Möbius inverse is given by $m(A) = (-1)^{|A|+1}, \forall A \subseteq X, A \neq \emptyset$ and $m(\emptyset) = 0$.

Unanimity games are usually taken as the *basis* of the set of non-additve measures because any non-additive measure can be written as

$$\mu(A) = \sum_{B \neq \emptyset} m(B) u_B(A), \forall A \in \mathcal{P}(X).$$

Finally, we will need the following result:

Proposition 3. *[19] Let $\mu^1, ..., \mu^p$ be a family of non-additive measures, with $m^1, ..., m^p$, their corresponding Möbius representations, and $\alpha_1, ..., \alpha_p \in [0, 1]$ so that $\sum_{i=1}^{p} \alpha_i = 1$. Then, $\mu = \sum_{i=1}^{p} \alpha_i \mu^i$ is a non-additive measure whose Möbius transform is given by*

$$m = \sum_{i=1}^{p} \alpha_i m^i.$$

In order to aggregate the different values, the so-called fuzzy integrals are used. One of the most important is the Choquet integral. This integral is a generalization of the concept of expected value.

Definition 6. *[8] The* **Choquet integral** *of a function*

$$f : X \to \mathbb{R}^+$$

with respect to a non-additive measure μ on X is defined by

$$\mathcal{C}_\mu(f) := \sum_{i=1}^{n} (f(x_{(i)}) - f(x_{(i-1)}))\mu(B_i), \tag{3}$$

where $\{x_{(1)}, \ldots, x_{(n)}\}$ is a permutation of the set $\{x_1, \ldots, x_n\}$ satisfying

$$0 =: f(x_{(0)}) \leq f(x_{(1)}) \leq ... \leq f(x_{(n)})$$

and $B_i = \{x_{(i)}, ..., x_{(n)}\}$.

In order to determine a non-additive measure, $2^n - 2$ values are necessary. The number of coefficients grows exponentially with n and so does the complexity of the problem of learning. This drawback reduces considerably the practical use of non-additive measures. Thus, some subfamilies of non-additive measures have been defined in an attempt to reduce complexity, e.g. k-additive measures [17], p-symmetric measures [30], k-intolerant measures [24, 25], λ-measures [41], or more generally decomposable measures [11].

Definition 7. *[17] A non-additive measure μ is said to be* **k-order additive** *or* **k-additive** *if its Möbius transform vanishes for any $A \subseteq X$ such that $|A| > k$ and there exists at least one subset A with exactly k elements such that $m(A) \neq 0$.*

In this sense, a probability measure is just a 1-additive measure. Thus, k-additive measures generalize probability measures, that are very restrictive in many situations. They fill the gap between probability measures and general non-additive measures. For a k-additive measure, the number of coefficients is reduced to

$$\sum_{i=1}^{k} \binom{n}{i}.$$

More about k-additive measures can be found e.g. in [19]. We will denote the set of all k'-additive measures on X with $k' \leq k$ by $\mathcal{FM}^k(X)$; we will use the fact that $\mathcal{FM}^k(X)$ is a convex polyhedron (the proof is straightforward considering the Möbius transform). Specially appealing is the 2-additive case, that provides a generalization of probability allowing interactions while keeping a reduced complexity.

Definition 8. *A non-additive measure is said to be* **symmetric** *if it satisfies for any* $A, B \in \mathcal{P}(X)$,

$$|A| = |B| \Rightarrow \mu(A) = \mu(B).$$

It can be proved [16] that the Choquet integral w.r.t. a symmetric measure is an OWA operator [45]; then, the same as for probabilities, the Choquet integral w.r.t. a symmetric measure has a low complexity. Therefore, we define p-symmetric measures in a way such that the corresponding Choquet integral is kept to a reduced complexity.

The definition of p-symmetric measure is based on the concept of indifferent elements and subsets of indifference.

Definition 9. *[30] Given two elements* x_i, x_j *of the universal set* X*, we say that* x_i *and* x_j *are* **indifferent elements** *for* μ *if*

$$\forall A \subseteq X \backslash \{x_i, x_j\},\ \mu(A \cup x_i) = \mu(A \cup x_j). \tag{4}$$

In Multicriteria Decision Making, the definition of indifferent elements reflects the fact that criteria x_i and x_j are equivalent, so that we do not care about which one is fulfilled.

This can be extended to more than two elements through subsets of indifference.

Definition 10. *[30] Given a subset* C *of* X*, we say that* C *is a* **subset of indifference** *if and only if* $\forall B_1, B_2 \subseteq C$, $|B_1| = |B_2|$ *and* $\forall A \subseteq X \backslash C$*, we have*

$$\mu(A \cup B_1) = \mu(A \cup B_2). \tag{5}$$

From this definition, two elements of the same subset of indifference are indifferent elements in the sense of Definition 9.

Indeed, the relation $\mathcal{R}$ defined by $x_i \mathcal{R} x_j$ if and only if x_i and x_j are indifferent elements for μ is an equivalent relation, and the subsets of indifference are the equivalence classes.

Definition 11. *Let* Π_1, Π_2 *be two partitions of* X*. We say that* Π_2 *is* **coarser** *than* Π_1*, denoted* $\Pi_1 \sqsubseteq \Pi_2$*, if*

$$\forall A_1 \in \Pi_1,\ \exists A_2 \in \Pi_2,\ \ s.t.\ \ A_1 \subseteq A_2.$$

The set of partitions of X with $\sqsubseteq$ has the structure of a lattice (see [39]). Then, the subset of partitions $\Pi = \{A_1, ..., A_r\}$ of X such that A_i is a subset of indifference for μ has a least upper bound. In [31], it is shown that this least upper bound is a partition whose elements are subsets of indifference for μ, too. We will refer to this least upper bound as the *indifference partition*. Now, it is straightforward to define p-symmetric measures.

Definition 12. *[30] Given a non-additive measure* μ*, we say that* μ *is a* p**-symmetric measure** *if and only if the (unique) coarsest partition of the universal set in subsets of indifference has* p *nonempty subsets.*

With these definitions, a symmetric measure is just a 1-symmetric measure. Given $\{A_1, ..., A_p\}$ a partition of X, the set of all non-additive measures μ such that A_i for $i = 1, ..., p$, are subsets of indifference for μ is denoted by $\mathcal{FM}(A_1, ..., A_p)$. Remark that in $\mathcal{FM}(A_1, ..., A_p)$ we include all non-additive measures whose indifferent partition is $(A_1, ..., A_p)$ but also the non-additive measures whose indifferent partition is coarser. It will be proved below (Proposition 12) that $\mathcal{FM}(A_1, ..., A_p)$ is a convex polyhedron. In the same spirit of k-additive measures, p-symmetric measures appear as a middle term between symmetric measures and general non-additive measures. They reduce the complexity of non-additive measures and provide a generalization of the idea of symmetry.

Remark that, when dealing with a p-symmetric measure w.r.t. $\{A_1, ..., A_p\}$, we only need to know the number of elements of each A_i belonging to a given subset B of the universal set X. Then, we can identify $B \subseteq X$ with a p-dimensional vector $(b_1, ..., b_p)$ where $b_i := |A_i \cap B|$, $\forall i = 1, ..., p$.

This property allows a reduction in the complexity of the measure:

Lemma 1. *[32] Let* μ *be a* p*-symmetric measure w.r.t. the indifference partition* $\{A_1, ..., A_p\}$*. Then, it can be represented in a* $(|A_1| + 1) \times \cdots \times (|A_p| + 1)$ *matrix whose entries are defined by*

$$M(i_1, ..., i_p) := \mu(i_1, ..., i_p),\ 0 \leq i_j \leq |A_j|.$$

Remark 2. *Lemma 1 also holds for any measure in $\mathcal{FM}(A_1, ..., A_p)$.*

In [32], some properties of p-symmetric measures have been studied, among them a decomposition of the corresponding Choquet integral.

With all these concepts, we can now state our problem as follows: Given m objects represented by the functions $f_1, \ldots, f_m$, and their assigned overall scores $y_1, \ldots, y_m$, and a family of non-additive measures $\mathcal{F}$, we look for $\mu \in \mathcal{F}$ minimizing

$$\sum_{i=1}^{m} (\mathcal{C}_\mu(f_i) - y_i)^2. \tag{6}$$

That is, we look for a non-additive measure μ in $\mathcal{F}$ that best fits our data with respect to the Choquet integral.

3 Genetic algorithms

Genetic algorithms are general optimization methods based on the theory of natural evolution [23]. The main concepts are those of *individual* and *population*, which are, respectively, a candidate solution and the set of individuals being considered at a certain step in the algorithm.

Starting from an initial population, at each iteration (or *generation*), some individuals are selected with probability proportional to their *fitness* (which is measured according to the function that we want to optimize) and new individuals are generated from them using a *cross-over operator*. These new individuals replace the old ones (their *parents*) and the process continues until an optimum is found or until the maximum number of generations is reached. Then, the best individual in the last population is returned as a possible solution of the problem.

Sometimes, the cross-over operator greatly reduces the diversity of the populations and the risk of finding only a local optimum increases. To avoid this, a *mutation operator* which randomly changes individuals is defined. With some predetermined frequency, this operator is applied.

These algorithms have been successfully used in many optimization problems (see [15]) and have a lot of good properties (robustness, few requirements on the function to optimize, low complexity,...) which make them an appealing choice for the problem of identification.

To apply the method of genetic algorithms to our problem we must first choose a cross-over operator on non-additive measures and select a suitable representation. We have chosen the *convex combination* of two non-additive measures μ_1 and μ_2, defined by

$$\lambda\mu_1 + (1 - \lambda)\mu_2,$$

with $\lambda \in [0, 1]$ chosen at random when the operator is applied.

The key advantage of this operator comes from the fact that it can also be applied to sub-families of non-additive measures which are convex, such as k-additive measures and p-symmetric measures (when the partition of subsets of indifference is fixed). Then, we can select different classes of measures for the learning problem without having to define new, specific cross-over operators.

The main drawback with this operator is that the search space is reduced in each generation (see figure 1), since the convex combinations of the new measures are always a subset of the convex combinations of the previous ones. What is more, if the initial population is not carefully selected, then it might be the case that only bad approximations can be found among the convex combinations of those initial individuals.

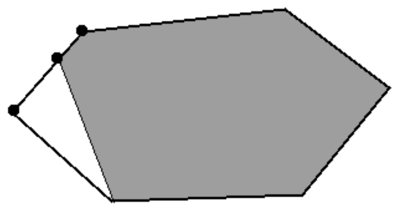
To overcome this drawback, one can use the extreme points of the family. Then, the starting population is initialized to the set of vertices and, seldom, some individuals are combined with one of the vertices chosen at random (this is the mutation operator) in order to keep the diversity high enough.

Once we know the vertices, the natural representation of the measures is by their coefficients as convex combination of the initial population. With this representation the cross-over operator has low complexity (if the number of vertices is small) and also the number of Choquet integrals that must be computed is kept to a minimum, since they are also convex combination of the integrals of the parents. Thus, we only have to integrate with respect to the vertices, which is usually easy.

Then, a measure μ will be represented as

$$(\lambda_1, \ldots, \lambda_l, z_1, \ldots, z_m),$$

Figure 1: Reduction of the search space due to convex combination



where $\lambda_1, \ldots, \lambda_l$, are coefficients of μ with respect to the l vertices and $z_i = \mathcal{C}_\mu(f_i)$. From this representation the fitness function (quadratic error) can be easily computed, too.

4 Vertices of k-additive measures

In this section we will show that, contrary to what happens with $\mathcal{FM}(X)$ (see Proposition 1), the set of $\{0,1\}$-valued measures in $\mathcal{FM}^k(X)$ does not completely determine the set of vertices for $k > 2$. The basic idea is to obtain two $(k+1)$-additive measures μ_1, μ_2, such that they are adjacent vertices in the convex polyhedron $\mathcal{FM}(X)$, and such that it is possible to find $\lambda \in (0,1)$ satisfying

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2 \in \mathcal{FM}^k. \quad (7)$$

If such a λ exists, μ is not a $\{0,1\}$-valued measure because it is not a vertex of the set $\mathcal{FM}(X)$ (see Proposition 1). Moreover, μ would be the only one k-additive measure in the segment joining μ_1 and μ_2. As μ_1 and μ_2 are adjacent vertices, the points in the segment line between μ_1 and μ_2 can only be written as a convex combination other points in this line, and this would imply that μ is in fact an extreme point of $\mathcal{FM}^k$. This result is stated in several propositions based on results of linear programming.

First, remark that for a $\{0,1\}$-valued measure μ, there are some subsets A satisfying the following conditions:

$$\begin{array}{ll} \mu(A) = 1, & \\ \mu(B) = 1, & \forall B \supseteq A, \\ \mu(C) = 0, & \forall C \subset A. \end{array} \quad (8)$$

This leads us to introduce the following concept:

Definition 13. *Consider a $\{0,1\}$-valued measure μ. We will say that a subset A of X is a* **minimal subset** *for μ if it satisfies the condition* (8).

Remark 3. *If we consider the lattice $(\mathcal{P}(X), \cup, \cap)$, then a minimal subset for a $\{0,1\}$-valued measure μ can be equivalently defined as a subset of X such that $\mu(A) = 1$ and whose principal filter $\mathcal{F}_A$ and principal ideal $\mathcal{I}_A$ (see [39]) satisfy*

$$\mu(B) = 1, \forall B \in \mathcal{F}_A, \ \mu(B) = 0, \forall B \in \mathcal{I}_A \backslash \{A\}.$$

Remark 4. *A $\{0,1\}$-valued measure is completely defined by its minimal subsets. To see this, it suffices to remark that $\mu(A) = 1$ if it contains a minimal subset and $\mu(A) = 0$ otherwise.*

We propose now an algorithm to obtain the Möbius transform of a $\{0,1\}$-valued measure from its minimal subsets.

Proposition 4. *Let μ be a $\{0,1\}$-valued measure whose minimal subsets are $B_1, ..., B_r$. If we denote by m the corresponding Möbius transform of μ, then m can be computed through:*

- **Step 0:** *Initially, set $m(A) \rightarrow 0$, $\forall A \subseteq X$.*

- **Step 1:** $m(B_i) \to 1, \forall i = 1, ..., r$.
- **Step 2:** *For any* $i, j, i \neq j$, $m(B_i \cup B_j) \to m(B_i \cup B_j) - 1$.
- **Step 3:** *For any different* i, j, k, $m(B_i \cup B_j \cup B_k) \to m(B_i \cup B_j \cup B_k) + 1$.
- ...
- **Step r:** $m(B_1 \cup ... \cup B_r) \to m(B_1 \cup ... \cup B_r) + (-1)^{r+1}$.

Proof: We will prove that the non-additive measure μ_m obtained through our algorithm is indeed μ. Consider $A \subseteq X$.

- If $B_i \nsubseteq A, \forall i$, then $\mu(A) = 0$ (Remark 4). On the other hand, applying the algorithm, we have $m(C) = 0, \forall C \subseteq A$. Therefore, by Equation (2),

$$\mu_m(A) = \sum_{C \subseteq A} m(C) = 0,$$

  and the result follows.

- Suppose that $\exists B_i \subseteq A$. In this case, $\mu(A) = 1$. Assume without loss of generality that $B_1, ..., B_p \subseteq A$ and $B_{p+1}, ..., B_r \nsubseteq A$. Notice that:

$$\bigcup_{j \in J} B_j \subseteq A \Leftrightarrow J \subseteq \{1, ..., p\}.$$

  Consider $\emptyset \neq J \subseteq \{1, ..., p\}$ such that $|J| = k$. The corresponding $C := \cup_{j \in J} B_j$ appears at step k, and then $m(C)$ turns into $m(C) + (-1)^{k+1}$. Remark also that these C's are exactly the $\binom{p}{k}$ subsets (maybe repeated!) derived from the minimal subsets appearing at step k and contained in A. Consequently,

$$\mu_m(A) = \sum_{C \subseteq A} m(C) = \sum_{i=1}^{p} \binom{p}{i} (-1)^{i+1} = (-1)(1-1)^p + 1 = 1.$$

Then, the result holds. ■

Corollary 1. *Given a* $\{0,1\}$*-valued measure whose minimal subsets are* $B_1, ..., B_r$ *and such that* $| \cup_{i=1}^{r} B_i| = k$*, the resulting* $\{0,1\}$*-valued measure is a* k*-additive measure (at most).*

Let us apply the algorithm of Proposition 4 in an example.

Example 1. *Let us consider* $X = \{1, 2, 3, 4, 5\}$ *and assume that the minimal subsets are*

$$\{1, 3\},\ \{1, 4\},\ \{1, 5\},\ \{3, 4\},\ \{3, 5\}.$$

Applying our algorithm we obtain:

- **Unions of 1 minimal subset:**

$$\{1, 3\},\ \{1, 4\},\ \{1, 5\},\ \{3, 4\},\ \{3, 5\}.$$

- **Unions of 2 minimal subsets:**

$$\{1, 3, 4\},\ \{1, 3, 5\},\ \{1, 3, 4\},\ \{1, 3, 5\},\ \{1, 4, 5\},\ \{1, 3, 4\},\ \{1, 3, 4, 5\},\ \{1, 3, 4, 5\},\ \{1, 3, 5\},\ \{3, 4, 5\}.$$

  (Notice that there are subsets appearing more than once).

- **Unions of 3 minimal subsets:**

$$\{1,3,4,5\},\ \{1,3,4\},\ \{1,3,4,5\},\ \{1,3,4,5\},\ \{1,3,5\},\ \{1,3,4,5\},\ \{1,3,4,5\},\ \{1,3,4,5\},\ \{1,3,4,5\},\ \{1,3,4,5\}.$$

- **Unions of 4 minimal subsets:**

$$\{1,3,4,5\},\ \{1,3,4,5\},\ \{1,3,4,5\},\ \{1,3,4,5\},\ \{1,3,4,5\}.$$

- **Unions of 5 minimal subsets:**

$$\{1,3,4,5\}.$$

Then, the Möbius transform is given by:

$$m(1,3)=1,\ m(1,4)=1,\ m(1,5)=1,\ m(3,4)=1,\ m(3,5)=1.$$

Consider for example $m(1,3,4)$. As $\{1,3,4\}$ appears three times as union of 2 minimal subsets and once as a union of three minimal subsets, we have

$$m(1,3,4)=-3+1=-2.$$

Similarly,

$$m(1,3,5)=-3+1=-2,\ m(1,4,5)=-1,\ m(3,4,5)=-1,\ m(1,3,4,5)=-2+8-5+1=2.$$

And finally, $m(A)=0$ otherwise.

Remark 5. *At this point, it is important to note that it is possible to find a $\{0,1\}$-valued measure whose minimal subsets are $B_1,...,B_r$, with $|\cup_{i=1}^r B_i|=k$, and such that this $\{0,1\}$-valued measure is not exactly k-additive but k'-additive, with $k'<k$:*

Let us consider $X=\{1,2,3,4,5\}$. Assume that the minimal elements are

$$\{1,2\},\ \{1,3\},\ \{1,4\},\ \{3,4,5\}.$$

Then, $\cup_{i=1}^4 B_i=\{1,2,3,4,5\}$. Let us apply the algorithm of Proposition 4:

- **Unions of 1 minimal subset:**

$$\{1,2\},\ \{1,3\},\ \{1,4\},\ \{3,4,5\}.$$

- **Unions of 2 minimal subsets:**

$$\{1,2,3\},\ \{1,2,4\},\ \{1,2,3,4,5\},\ \{1,3,4\},\ \{1,3,4,5\},\ \{1,3,4,5\}.$$

- **Unions of 3 minimal subsets:**

$$\{1,2,3,4\},\ \{1,2,3,4,5\},\ \{1,2,3,4,5\},\ \{1,3,4,5\}.$$

- **Unions of 4 minimal subsets:**

$$\{1,2,3,4,5\}.$$

Then, the Möbius transform is given by:

$$m(1,3)=1,\ m(1,4)=1,\ m(1,5)=1,\ m(3,4)=1,\ m(3,5)=1.$$

$$m(1,2,3)=-1,\ m(1,2,4)=-1,\ m(1,3,4)=-1,\ m(1,3,4,5)=-1,\ m(1,2,3,4)=1,\ m(1,2,3,4,5)=0.$$

And finally, $m(A)=0$ otherwise. Then, μ is 4-additive instead of 5-additive.

Let us now turn to the monotonicity conditions of a non-additive measure (Definition 1). These constraints can be put in the form

$$\begin{array}{rl} \mu(\emptyset)=0,\ \mu(X)=1. & \\ \mu(X\backslash\{x_i\})-1+h(X,x_i)=0. & \\ \mu(A\backslash\{x_i\})-\mu(A)+h(A,x_i)=0, & \forall A\neq X,\ \forall x_i\in A. \\ h(A,x_i)\geq 0,\ \mu(A)\geq 0, & \forall A\neq\emptyset,\ X,\ \forall x_i\in A. \end{array} \tag{9}$$

Then, we have written the monotonicity conditions as the constraints of a linear programming problem in the standard form [3] with variables $\mu(A), h(A,x_i)$, $\forall A\neq\emptyset$, $\forall x_i\in A$; variables $h(A,x_i), \forall A\neq\emptyset$, $\forall x_i\in A$ are the slack variables. Now, consider a vertex μ_0, i.e. a $\{0,1\}$-valued measure (Proposition 1). This measure is a basic feasible solution of System (9) and it must be defined through some basic variables (see [3] for a proof); the other variables (called non-basic variables) attain value 0. It must be kept in mind throughout this section that several choices of basic variables can define the same vertex, since we are dealing with a very degenerated problem (in which some of the basic variables attain value 0, too).

We first study the values of these basic variables.

Proposition 5. *Suppose we introduce the variable $\mu(A)$ in the basis and there exists a $B\supset A$ such that the corresponding $\mu(B)$ is not a basic variable. Then, $\mu(A)$ attains the value 0 in the new solution.*

Proof: The proof is straightforward. As $\mu(B)$ is not a basic variable, it follows that $\mu(B)=0$, whence $\mu(A)=0$ in order to satisfy the monotonicity constraints of non-additive measures (Definition 1), i.e. in order to obtain a feasible solution. ■

Remark 6. *It must be noted that the reciprocal of this result does not hold, i.e. it is possible to find situations in which $\mu(B)$ is a basic variable for any $B\supseteq A$ and $\mu(A)$ attains the value 0:*

Consider $|X|=3$. In this case, the monotonicity constraints are given by:

$$-1+\mu(x_i,x_j)+h(X,x_i)=0,\ \forall x_i,x_j\in X.$$

$$-\mu(x_i,x_j)+\mu(x_i)+h(\{x_i,x_j\},x_j)=0,\ \forall x_i,x_j\in X.$$

$$-\mu(x_i)+h(\{x_i\},x_i)=0,\ \forall x_i\in X.$$

Let us consider as initial basis the one whose basic variables are the slack ones. Then, $\mu(A)=0, \forall A\neq X$, as $\mu(A)$ is not a basic variable. We consider the following process:

We introduce in the basis $\mu(x_1)$; then, we can choose between $h(\{x_1,x_2\},x_2), h(\{x_1,x_3\},x_3)$ and $h(\{x_1\},x_1)$ to leave the basis. Assume $h(\{x_1,x_2\},x_2)$ leaves the basis. Then, $\mu(x_1)=0$ by monotonicity (Proposition 5).

We introduce $\mu(x_1,x_2)$. In this case, if the variable leaving the basis is $h(X,x_3)$, then $\mu(x_1)$ attains again value 0 by monotonicity ($\mu(x_1,x_3)\geq\mu(x_1)$). But this implies that $\mu(x_1,x_2)$ should attain value 0 in order to satisfy the constraint

$$-\mu(x_1,x_2)+\mu(x_1)+h(\{x_1,x_2\},x_2)=0.$$

On the other hand, it is $\mu(x_1,x_2)=1$ because of the constraint

$$-1+\mu(x_1,x_2)+h(X,x_3)=0.$$

Then, we arrive to a contradiction and $h(X,x_3)$ cannot leave the basis.

If $\mu(x_1)$ leaves the basis, then $\mu(x_1,x_2)=0$ in order to satisfy

$$-\mu(x_1,x_2)+\mu(x_1)+h(\{x_1,x_2\},x_2)=0.$$

Finally, if $h(\{x_1,x_2\},x_1)$ leaves the basis, it is $\mu(x_1,x_2)=0$ because of the constraint

$$-\mu(x_1,x_2)+\mu(x_2)+h(\{x_1,x_2\},x_1)=0,$$

as $\mu(x_2)$ is not a basic variable.

Then $\mu(x_1,x_2)=0$ and taking $A=\{x_1,x_2\}$, this shows that it is possible to find situations in which a basic variable $\mu(A)$ attains the value 0, even if any other variable $\mu(B), A\subseteq B$ is a basic variable, too.

However, it is always possible to find a way in which $\mu(A) = 1$ whenever it is a basic variable and $\mu(B), B \supseteq A$, is also a basic variable:

Proposition 6. *Suppose we introduce the variable $\mu(A)$ in the basis and assume that $\mu(B)$ is a basic variable for any $B \supset A$. Then, it is possible to obtain a basic solution such that $\mu(A) = 1$.*

Proof: It suffices to build such a solution. We will make the proof by induction on $n - |A|$.

If $n - |A| = 1$, then $A = X \backslash \{x_i\}$. Let us consider the solution in which the basic variables are the slack ones. Then, all basic variables attain value 0, except the slack variables $h(X, x_i)$, whose value is 1. The variable $\mu(X \backslash \{x_i\})$ appears in the following monotonicity constraints:

$$\mu(X \backslash \{x_i\}) + h(X, x_i) = 1.$$

$$\mu(X \backslash \{x_i, x_j\}) - \mu(X \backslash \{x_i\}) + h(X \backslash \{x_i\}, x_j) = 0.$$

Consequently, $\mu(X \backslash \{x_i\})$ can get into the basis with a value of 1 and $h(X, x_i)$ leaves the basis. Therefore, the result holds.

Assume now $n - |A| > 1$ and that the result holds until $n - |A| - 1$. Let us start again from the solution in which the basic variables are the slack ones. Then, we introduce all the variables $\mu(X \backslash \{x_i\})$ with $A \subset X \backslash \{x_i\}$. All these variables get into the basis with a value of 1. Then, we introduce the variables $\mu(X \backslash \{x_i, x_j\})$ with $A \subset X \backslash \{x_i, x_j\}$. Then, all these variables get into the basis with a value of 1 (by induction). We continue the process for $\mu(X \backslash \{x_i, x_j, x_k\})$ with $A \subset X \backslash \{x_i, x_j, x_k\}$, and so on.

When introducing $\mu(B)$ in the basis, one of the slack variables of type $h(B \cup \{x_j\}, x_j)$ gets out the basis and the other slack variables on this form stay in the basis but with value 0.

Then, when introducing $\mu(A)$ in the basis, we have a solution in which for any $B \supset A$, the variable $\mu(B)$ is in the basis and with value 1. At this point, the constraints in which $\mu(A)$ appears are:

$$\mu(A) + h(A \cup \{x_i\}, x_i) + NonBasicVariables = 1, \; x_i \notin A,$$

by the application of simplex method, and

$$\mu(A \backslash \{x_j\}) - \mu(A) + h(A, x_j) = 0, \; x_j \in A,$$

as these constraints have never been used in the previous steps.

Then, $\mu(A)$ gets into the basis with a value of 1 and one of the slack variables $h(A \cup \{x_i\}, x_i)$ gets out the basis. The other variables of this type attain value 0, though remaining in the basis.

Remark that when introducing $\mu(A)$ in the basis, if $\mu(A)$ attains value 1, then variables $h(A, x_j)$, $x_j \in A$ also attain value 1. We will use this property below. ■

Next step is to find a way to identify some adjacent solutions of a basic feasible solution. The way we obtain adjacent solutions is by introducing a non-basic variable into the set of basic variables [3].

In the following, we will assume that we are in the conditions of Proposition 6, i.e. we will suppose that we have started from the initial solution in which the basic variables are the slack ones; given the variables $\mu(A_1), ..., \mu(A_r)$, these variables have been introduced in the basis in order of decreasing cardinality, and when introducing any $\mu(A_j)$, a slack variable in the form $h(A_j \cup \{x_i\}, x_i)$ has left the basis. From a similar process of the one developed in the proof of Proposition 6, we know we can always act this way.

Proposition 7. *Let μ' be a $\{0, 1\}$-valued measure. Choose A such that $\mu'(A) = 1$. Let us define*

$$\mu_1^*(D) = \begin{cases} 1 & \text{if } \; D \nsubseteq A \; \text{ and } \; \mu'(D) = 1 \\ 0 & \qquad\qquad \text{otherwise} \end{cases}$$

Then, μ^* is an adjacent measure of μ'.

Proof: Let us consider the method explained before and suppose we have introduced in the basis any variable $\mu(B)$ such that $\mu'(B) = 1$. Thus, μ^* is obtained by making $\mu^*(A) = 0$ (this is possible if we reintroduce a suitable slack variable in the basis and $\mu(A)$ is no more a basic variable); this implies that for any $B \subset A$ such that

$\mu(B) = 1$, we necessarily have $\mu^*(B) = 0$ by the monotonicity constraints (and also by Proposition 5). For any other $C \subseteq X$, the value of $\mu(C)$ remains the same. We obtain then the $\{0,1\}$-valued measure μ^* and then the proposition holds. ■

Remark 7. *We can obtain other adjacent measures of μ' by reversing the process. In this case, we have to choose A such that $\mu'(A) = 0$ and $\mu'(B) = 1$ for any $B \supset A$; and consider $B_1, ..., B_r$ a family of non-empty subsets of A s.t. for any C satisfying that $\exists B_i \subseteq C \subset A$, it follows that $C = B_j$ for some j. If we introduce $\mu(A)$ as a basic variable, then $\mu(A) = 1$ by Proposition 6. Even more, by construction, $\mu(B_i)$ is a basic variable, too. Before introducing $\mu(A)$, we had $\mu(B_i) = 0$ by Proposition 5. After introducing $\mu(A)$, we have $\mu(B_i) = 1$ (see the proof of Proposition 6). We obtain then the $\{0,1\}$-valued measure μ^*.*

Now, we have to look for two adjacent $(k+1)$-additive measures such that their convex combination leads to a k-additive measure. We first study the relationship between the Möbius transforms of two adjacent measures obtained through Proposition 7 or Remark 7. We make the proof under the conditions of Proposition 7. Similar proofs can be done straightforwardly in the conditions of Remark 7.

Proposition 8. *Let μ be a $\{0,1\}$-valued measure whose corresponding Möbius transform is m. Consider $A \subseteq X$ such that $\mu(A) = 1$. Let us define the non-additive measure μ' given by*

$$\mu'(B) = \begin{cases} \mu(B) & \text{if } B \not\subseteq A \\ 0 & \text{otherwise} \end{cases}$$

Then, if m' is the Möbius transform of μ', we have

$$m'(C) = m(C) + (-1)^{|C \setminus (C \cap A)|+1} m(C \cap A), \forall C \subseteq X. \tag{10}$$

Proof: Note that the measure in this proposition corresponds to an adjacent measure of μ by Proposition 7. If we apply again Equation (1), we have

$$m'(C) = \sum_{B \subseteq C} (-1)^{|C \setminus B|} \mu'(B) = \sum_{B \subseteq C, B \not\subseteq A} (-1)^{|C \setminus B|} \mu(B),$$

by the definition of μ'. Consequently,

$$m'(C) = \sum_{B \subseteq C} (-1)^{|C \setminus B|} \mu(B) - \sum_{B \subseteq C, B \subseteq A} (-1)^{|C \setminus B|} \mu(B) = m(C) - \sum_{B \subseteq C \cap A} (-1)^{|C \setminus B|} \mu(B).$$

Now,

$$\sum_{B \subseteq C \cap A} (-1)^{|C \setminus B|} \mu(B) = \sum_{B \subseteq C \cap A} (-1)^{|C \setminus (C \cap A)| + |(C \cap A) \setminus B|} \mu(B) =$$

$$(-1)^{|C \setminus (C \cap A)|} \sum_{B \subseteq C \cap A} (-1)^{|(C \cap A) \setminus B|} \mu(B) = (-1)^{|C \setminus (C \cap A)|} m(C \cap A).$$

This finishes the proof. ■

We state our main result.

Theorem 1. *There are vertices of the set $\mathcal{FM}^k(X), k > 2$, that are not $\{0,1\}$-valued measures.*

Proof: It suffices to build an example.
Consider $X = \{1, 2, 3, 4\}$ and assume μ is the $\{0,1\}$-valued measure whose minimal subsets are

$$\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}.$$

Applying Proposition 4, we have that $m(1, 2, 3, 4)$ is given by:

- Unions of 2 minimal subsets:
$$\{1,4\} \cup \{2,3\}.$$

- Unions of 3 minimal subsets:
$$\{1,2\} \cup \{1,3\} \cup \{1,4\};\ \{1,2\} \cup \{2,3\} \cup \{1,4\};\ \{1,3\} \cup \{2,3\} \cup \{1,4\}.$$

- Unions of 4 minimal subsets:
$$\{1,2\} \cup \{1,3\} \cup \{2,3\} \cup \{1,4\}.$$

Consequently, $m(1,2,3,4) = 1$ and therefore μ is a 4-additive measure.

Define a new measure μ' by
$$\mu'(A) = \begin{cases} \mu(A) & \text{if } A \not\subseteq \{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$
By Proposition 7, we know that μ' is adjacent to μ.

Applying now Proposition 8, we have that
$$m'(1,2,3,4) = m(1,2,3,4) + (-1)^2 m(1,2,3).$$
Let us compute $m(1,2,3)$. Applying again Proposition 4:

- Unions of 2 minimal subsets:
$$\{1,3\} \cup \{2,3\};\ \{1,2\} \cup \{1,3\};\ \{1,2\} \cup \{2,3\}.$$

- Unions of 3 minimal subsets:
$$\{1,2\} \cup \{2,3\} \cup \{1,3\}.$$

Then, $m(1,2,3) = -2$, whence $m'(1,2,3,4) = -1$ (by Proposition 8).

Now, $\mu^* = 0.5\mu + 0.5\mu'$ is a 3-additive measure because $m^*(1,2,3,4) = 0$ and $m^*(1,2,3) \neq 0$, (applying Proposition 3).

On the other hand, as μ and μ' are adjacent vertices in $\mathcal{FM}(X)$, if μ_1, μ_2 are such that $\mu^* = \lambda\mu_1 + (1-\lambda)\mu_2$, then, necessarily both μ_1 and μ_2 are in the line joining μ and μ'. This implies that μ_1 and μ_2 are 4-additive measures (the only 3-additive measure in this line is μ^*). Consequently, μ^* is an extreme point of $\mathcal{FM}^3(X)$ but it is not a $\{0,1\}$-valued measure because it is not a vertex of $\mathcal{FM}(X)$ (Proposition 1). ■

This theorem shows that the structure of the convex polyhedron $\mathcal{FM}^k(X)$ is more complex than the one of $\mathcal{FM}(X)$. It proves that, although the set of $\{0,1\}$-valued measures in $\mathcal{FM}^k(X)$ are vertices, there are other vertices different from these. These new vertices come from k-additive measures that are in the line joining two adjacent $\{0,1\}$- valued measures that are not k-additive. Then, we have more vertices than the $\{0,1\}$-valued measures. However, even if we consider $\{0,1\}$-valued measures as our initial population, we obtain a huge number of measures. Indeed, the following can be proved:

Proposition 9. *The number of $\{0,1\}$-valued measures in $\mathcal{FM}^k(X)$ is at least*
$$\sum_{i=1}^{k} \binom{n}{i} B_i,$$
where B_j are the Bell numbers, defined recursively by
$$B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i,\ B_0 := 1.$$

Proof: Let us consider a subset of cardinality i; the number of different subsets of cardinality i is $\binom{n}{i}$. For a fixed subset of cardinality i, we consider the measure whose minimal subsets are a partition of this subset. Any of these measures is an i-additive measure (at most) by Corollary 1. The number of different partitions of a set of cardinality i is the i-th Bell number (see e.g. [5, page 42]). Consequently, the number of extreme points is at least

$$\sum_{i=1}^{k} \binom{n}{i} B_i,$$

whence the result. ■

For example, if $n = 10$ and we consider the 4-additive case, we have

$$B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15,$$

whence the number of vertices is greater than 3850.

Even more, we might be considering here just a small part of the set of vertices of $\mathcal{FM}^k(X)$. We are not considering the case of obtaining a subset not from a partition but from a non-disjoint union of subsets; the same can be said about the possibility of obtaining a k-additive measure when the union of minimal subsets has a cardinality greater than k, as pointed out in Remark 5. Finally, we are not considering the vertices that are not $\{0, 1\}$-valued measures (Theorem 1). Consequently, the actual number of vertices might be far away from the bound found in Proposition 9. Of course, we cannot consider in our algorithm as initial population so many points and thus, a reduction should be made.

For probabilities, i.e. 1-additive measures, the following result holds as a consequence of the results in [7]:

Proposition 10. *The extreme points of $\mathcal{FM}^1(X)$ are the $\{0, 1\}$-valued measures that are in $\mathcal{FM}^1(X)$. These $\{0, 1\}$-valued measures are the unanimity games on the singletons u_{x_i}, $x_i \in X$. Moreover, given P a probability distribution over X, it can be written as*

$$P = \sum_{x_i \in X} P(x_i) u_{x_i}.$$

Let us now turn to the special case of 2-additive measures. In this case, the following can be proved:

Proposition 11. *The set of extreme points of $\mathcal{FM}^2(X)$ are the $\{0, 1\}$-valued measures that are in $\mathcal{FM}^2(X)$. These $\{0, 1\}$-valued measures are given by:*

- $m(x_i) = 1,\, m(A) = 0,$ *otherwise (the extreme points of probabilities, u_{x_i}, $x_i \in X$).*
- $m(x_i) = 1,\, m(x_j) = 1,\, m(x_i, x_j) = -1,\, m(A) = 0,$ *otherwise. We will denote these measures by μ'_{x_i,x_j}.*
- $m(x_i, x_j) = 1,\, m(A) = 0,$ *otherwise (u_{x_i,x_j}, $\{x_i, x_j\} \subseteq X$).*

Moreover, given μ a 2-additive measure over X, it can be written as

$$\mu = \sum_{m(x_i,x_j)<0} -m(x_i, x_j)\mu'_{x_i,x_j} + \sum_{m(x_i,x_j)>0} m(x_i, x_j) u_{x_i,x_j} + \sum_{x_i \in X} c(x_i) u_{x_i},$$

with $c(x_i) := [m(x_i) + \sum_{m(x_i,x_j)<0} m(x_i, x_j)]$.

Proof: Let μ be a 2-additive measure. Let us denote by μ^* the measure given by

$$\mu^* = \sum_{m(x_i,x_j)<0} -m(x_i, x_j)\mu'_{x_i,x_j} + \sum_{m(x_i,x_j)>0} m(x_i, x_j) u_{x_i,x_j} + \sum_{x_i \in X} c(x_i) u_{x_i},$$

with $c(x_i) = [m(x_i) + \sum_{m(x_i,x_j)<0} m(x_i, x_j)]$. It suffices to check that the corresponding Möbius transforms m and m^* are the same. We have several cases:

- If $|A| > 2$, then both $m(A)$ and $m^*(A) = 0$, and then the result holds.

- Suppose $|A| = 2$ and $m(A) < 0$. Then, $m^*(A) = -m(A) \cdot (-1) = m(A)$.
- If $|A| = 2$ and $m(A) \geq 0$, then $m^*(A) = m(A) \cdot 1 = m(A)$.
- Finally, if $A = \{x_i\}$, it follows that

$$m^*(A) = c(x_i) \cdot 1 - \sum_{m(x_i,x_j)<0} m(x_i, x_j) = m(x_i) + \sum_{m(x_i,x_j)<0} m(x_i, x_j) - \sum_{m(x_i,x_j)<0} m(x_i, x_j) = m(x_i),$$

whence the result. ■

The reason of this result for probabilities and 2-additive measures lays in the fact that $m(x_i) \in [0, 1], m(x_i, x_j) \in [-1, 1]$ for any non-additive measure [28]. On the other hand, if μ is a $\{0, 1\}$-valued measure, then it is $m(A) \in \mathbb{Z}, \forall A \subseteq X$ as a consequence of Proposition 4. Thus, if we apply Proposition 8 removing from the basis a subset in the form $\mu(x_i, x_j)$ or $\mu(x_i)$, it is not possible to reverse the sign of $m(x_i, x_j, x_k)$. Therefore, it is not possible to find two 3-additive adjacent measures in the conditions of Theorem 1.

The number of vertices of $\mathcal{FM}^2(X)$ is given in next corollary.

Corollary 2. *The number of extreme points of $\mathcal{FM}^2(X)$ is n^2.*

Proof: It suffices to sum up situations 1, 2 and 3 in last proposition. Therefore,

$$n + \binom{n}{2} + \binom{n}{2} = n + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} = n + n(n-1) = n^2.$$

Therefore, the result holds. ■

5 The case of p-symmetric measures

Let us now turn to the p-symmetric case. First of all, we have to remark that in general, the convex combination of p-symmetric measures is not a p-symmetric measure for $p \geq 2$:

Example 2. *Assume $|X| = 3$ and let m_1 and m_2 be the Möbius transforms of two 2-symmetric measures μ_1, μ_2, respectively.*

For μ_1, the subsets of indifference are $\{x_1, x_2\}$ and $\{x_3\}$, and m_1 is defined by

$$m_1(x_1) = 0.5 = m_1(x_2),\ m_1(A) = 0, \ \ \textit{otherwise.}$$

For μ_2, the subsets of indifference are $\{x_1, x_3\}$ and $\{x_2\}$, and m_2 is defined by

$$m_2(x_1) = 0.5 = m_2(x_3),\ m_2(A) = 0, \ \ \textit{otherwise.}$$

It is straightforward to check that both μ_1 and μ_2 are 2-symmetric measures w.r.t. the corresponding partitions. Let us now define $m = 0.75m_1 + 0.25m_2$. For this new measure, we have

$$m(x_1) = 0.5,\ m(x_2) = 0.75 * 0.5,\ m(x_3) = 0.25 * 0.5,\ m(A) = 0, \ \ \textit{otherwise,}$$

and consequently μ is a 3-symmetric measure as $m(x_1), m(x_2)$ and $m(x_3)$ attain different values, whence so do $\mu(x_1), \mu(x_2)$ and $\mu(x_3)$.

However, the result holds when we maintain the same partition:

Proposition 12. *Let $\mu_1, \mu_2 \in \mathcal{FM}(A_1, ..., A_p)$. Then, $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2 \in \mathcal{FM}(A_1, ..., A_p), \forall \lambda \in [0, 1]$.*

Proof: The proof is straightforward. It suffices to show that A_i is a subset of indifference for any $i = 1, ..., p$. Consider $B, C \subseteq A_i$ such that $|B| = |C|$. Then, for any $D \subseteq X\backslash A_i$,

$$\mu(B \cup D) = \lambda\mu_1(B \cup D) + (1-\lambda)\mu_2(B \cup D) = \lambda\mu_1(C \cup D) + (1-\lambda)\mu_2(C \cup D) = \mu(C \cup D),$$

as $\mu_1, \mu_2 \in \mathcal{FM}(A_1, ..., A_p)$. This finishes the proof. ■

Contrary to the k-additive case, for p-symmetric measures the following can be shown:

Theorem 2. *The set of extreme points of $\mathcal{FM}(A_1, ..., A_p)$ is the set of $\{0,1\}$-valued measures that are also in $\mathcal{FM}(A_1, ..., A_p)$.*

Proof: We will make the proof using properties of *unimodular matrices* (see [3]). An alternative proof can be done adapting the one of Proposition 1.

From System (9), the convex polyhedron $\mathcal{FM}(A_1, ..., A_p)$ can be written in terms of μ as

$$A\vec{u} \leq \vec{b},$$

where $\vec{u}$ is a vector representing the values of $\mu(A)$ and this system represents constraints in the form

$$-\mu(A) + \mu(A\backslash x_i) \leq 0,$$

$$-\mu(x_i) \leq 0,$$

and

$$\mu(X\backslash x_i) \leq 1.$$

Then, $\vec{b}$ has integer coefficients. This implies that if we prove that A is unimodular, all the extreme points have integer coefficients. Of course, if A is unimodular, so it is A^t, the transpose of matrix A.

Remark that in A^t, we have three different types of columns: some of then have two non-zero values (one value 1 and another one -1); some others have only one non-zero value (a -1 or a 1). Then, A^t can be written as $(B, I, -I)$ where B stands for columns with two non-zero values, I for columns with one 1 and rest of 0's and $-I$ for columns with one -1 and rest of 0's.

As columns in B have only one value 1 and only one value -1, this implies that B is unimodular. But then, so it is the matrix (B, Id) where Id represents the identity matrix and analogously, $(-B, -Id, Id)$ is unimodular. Then, $(B, Id, -Id)$ is unimodular and consequently, $(B, I, -I)$ is unimodular, as it is a submatrix of a unimodular matrix.

Consequently, the extreme points of $\mathcal{FM}(A_1, ..., A_p)$ have integer coefficients, and then they are $\{0,1\}$- valued measures contained in $\mathcal{FM}(A_1, ..., A_p)$. As any $\{0,1\}$-valued measure in $\mathcal{FM}(A_1, ..., A_p)$ is an extreme point of the polyhedron, this finishes the proof. ■

We have proved that the vertices of p-symmetric measures are indeed the set of $\{0,1\}$-valued measures for different values of p and the partition of X in subsets of indifference. However, as for $\mathcal{FM}^k(X)$, this provides us with a huge number of measures. For example, if we consider the 2-symmetric case, the following can be shown:

Proposition 13. *The number of $\{0,1\}$-valued measures in $\mathcal{FM}(A_1, A_2)$ with $|A_1| \geq |A_2|$ is*

$$\frac{(|A_1|+2)^{|A_2|+1}}{(|A_2|+1)!} - 2.$$

Proof: Using the matrix representation (Lemma 1), the measures can be represented in a $(|A_1|+1) \times (|A_2|+1)$ matrix M. Moreover, as μ is a $\{0,1\}$-valued measure, it follows that values in M are 0's and 1's. We look for the number of possible measures in these conditions.

Due to monotonicity conditions (Definition 1), this problem is equivalent to determining, for each row in M, the first position with value 1. For any row, there are $|A_1|+2$ possible positions (the last value representing a row of 0's). Then, we have

$$(|A_1|+2)^{|A_2|+1}$$

possibilities. However, not all of them are non-additive measures because of the monotonicity conditions. Given a $(|A_2|+1)$-vector of the positions of the first 1 in each row, $(i_0, ..., i_{|A_2|})$, the only possibility satisfying the monotonicity conditions is the one for which $i_0 \leq i_1 \leq ... \leq i_{|A_2|}$. Then, only one possibility over $(|A_2|+1)!$ satisfies the monotonicity conditions.

Finally, two possibilities, representing a matrix of only 0's or only 1's must be removed because of the boundary conditions. ■

6 Conclusions and future research

In this paper we have studied the vertices of some subfamilies of non-additive measures. These extreme points are necessary in order to ensure that the solution of the problem of identification can be reached when using our algorithm.

We have proved that the vertices of p-symmetric measures are indeed the set of $\{0,1\}$-valued measures for different values of p and the partition of X in subsets of indifference. This fact, together with the matrix representation of p-symmetric measures, allows a simple way to compute the vertices.

On the other hand, we have shown that there exist vertices of the set of k-additive measures that are not $\{0,1\}$-valued measures. However, the $\{0,1\}$-valued measures are the vertices of $\mathcal{FM}^k(X)$ for the special cases of $k=1$ and $k=2$.

As future research, we have, first of all, the problem of obtaining explicitly the set of vertices of k-additive measures. However, this is purely a theoretical problem; from a practical point of view, this is not useful, as the number of vertices is too high, even if we only consider the $\{0,1\}$-valued measures, and should be reduced before applying the algorithm; otherwise the complexity grows tremendously.

This is indeed our main open problem: how can the number of vertices be reduced in the initial population? In this case, several options must be studied. For example, are there vertices "more important" than others in a certain way? Another solution is to consider as initial population the vertices that are "near" the available data following some criterion of proximity. In both cases, further research is needed.

An interesting problem comes from the fact that both $\mathcal{FM}^k(X)$ and $\mathcal{FM}(A_1, ..., A_p)$ are not dense in $\mathcal{FM}(X)$. Then, using a subfamily of $\mathcal{FM}(X)$ dense and easy to compute would be an interesting problem. However, to our knowledge, no such family has been defined yet.

Finally, we have the problem of adding in our algorithm other subfamilies of non-additive measures, as belief functions, λ-measures, k-intolerant measures, etc.

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References

[1] M. Allais. Le comportement de l'homme rationnel devant le risque: critique des postulats de l'école américaine. *Econometrica*, (21):503–546, 1953. In French.

[2] F. J. Anscombe and R. J. Aumann. A definition of subjective probability. *The Annals of Mathematical Statistics*, (34):199–205, 1963.

[3] M. Bazaraa, J. Jarvis, and H. Sherali. *Linear programming and network flows.* John Wiley and Sons, 1990.

[4] C. Berge. *Principles of combinatorics*, volume 72 of *Mathematics in Science and Engineering.* Academic Press, 1971.

[5] A. Chateauneuf. Modelling attitudes towards uncertainty and risk through the use of Choquet integral. *Annals of Operations Research*, (52):3–20, 1994.

[6] A. Chateauneuf and J.-Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, (17):263–283, 1989.

[7] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, (5):131–295, 1953.

[8] E. F. Combarro and P. Miranda, Identification of fuzzy measures from sample data with genetic algorithms. *Computers and Operations Research*, to appear.

[9] D. Denneberg. *Non-additive measures and integral.* Kluwer Academic, 1994.

[10] D. Dubois and H. Prade. A class of fuzzy measures based on triangular norms. *Int. J. General Systems*, 8:43–61, 1982.

[11] D. Dubois, H. Prade, and R. Sabbadin. Qualitative decision theory with Sugeno integrals. In M. Grabisch, T. Murofushi, and M. Sugeno, editors, *Fuzzy measures and integrals*, number 40 in Studies in Fuzziness and Soft Computing, pages 314–331. Physica-Verlag, 2000.

[12] C. A. Bana e Costa and J.-C. Vansnick. MACBETH- An interative path towards the construction of cardinal value function. *International Transactions in Operational Research*, 1(4):489–500, 1994.

[13] D. Ellsberg. Risk, ambiguity, and the Savage axioms. *Quart. J. Econom.*, (75):643–669, 1961.

[14] D. E. Goldberg. *Genetic algorithms in search, optimization and machine learning.* Addison-Wesley, 1989.

[15] M. Grabisch. Pattern classification and feature extraction by fuzzy integral. In *3rd European Congress on Intelligent Techniques and Soft Computing (EUFIT)*, pages 1465–1469, Aachen (Germany), August 1995.

[16] M. Grabisch. k-order additive discrete fuzzy measures. In *Proceedings of 6th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU)*, pages 1345–1350, Granada (Spain), 1996.

[17] M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 5:587–607, 1997.

[18] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, (92):167–189, 1997.

[19] M. Grabisch. The interaction and Möbius representations of fuzzy measures on finite spaces, k-additive measures: a survey. In M. Grabisch, T. Murofushi, and M. Sugeno, editors, *Fuzzy Measures and Integrals-Theory and Applications*, number 40 in Studies in Fuzziness and Soft Computing, pages 70–93. Physica–Verlag, 2000.

[20] M. Grabisch and J.-M. Nicolas. Classification by fuzzy integral-performance and tests. *Fuzzy Sets and Systems, Special Issue on Pattern Recognition*, (65):255–271, 1994.

[21] P. L. Hammer and R. Holzman. On approximations of pseudo-boolean functions. *Zeitschrift für Operations Research. Mathematical Methods of Operations Research*, (36):3–21, 1992.

[22] J. H. Holland. *Adaptation in natural and artificial systems.* Ann Arbor: The University of Michigan Press, 1975.

[23] J.-L. Marichal. k-intolerant capacities and Choquet integrals. In *Proceedings of Tenth International Conference of Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU)*, pages 601–608, Perugia (Italy), July 2004.

[24] J.-L. Marichal. Tolerant or intolerant character of interacting criteria in aggregation by the Choquet integral. *European Journal of operational Research*, 155(3):771–791, June 2004.

[25] J.-L. Marichal, P. Meyer, and M. Roubens. Sorting multiattribute alternatives : the TOMASO method. *Computer and Operations Resarch*, to appear.

[26] P. Miranda, E.F. Combarro, and P. Gil. Extreme points of some families of fuzzy measures. In *Proceedings of the Second International Summer School on Aggregation Operators and their applications AGOP*, pages 129–135, Alcalá de Henares (Spain), July 2003.

[27] P. Miranda and M. Grabisch. Optimization issues for fuzzy measures. In *Proceedings of Seventh International Conference of Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU)*, pages 1204–1212, Paris (France), 1998.

[28] P. Miranda and M. Grabisch. Characterizing k-additive fuzzy measures. In *Proceedings of Eighth International Conference of Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU)*, pages 1063–1070, Madrid (Spain), July 2000.

[29] P. Miranda and M. Grabisch. p-symmetric fuzzy measures. In *Proceedings of Ninth International Conference of Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU)*, pages 545–552, Annecy (France), July 2002.

[30] P. Miranda and M. Grabisch. p-symmetric bi-capacities. *Kybernetica*, 40(2):421–440, 2004.

[31] P. Miranda, M. Grabisch, and P. Gil. p-symmetric fuzzy measures. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 10 (Suppl.):105–123, 2002.

[32] P. Miranda, M. Grabisch, and P. Gil. Identification of non-additive measures from sample data. In R. Kruse G. Della Riccia, D. Dubois and H.-J. Lenz, editors, *Planning based on Decision Theory*, volume 472 of *CISM Courses and Lectures*, pages 43–63. Springer Verlag, 2003.

[33] D. Radojevic. The logical representation of the discrete Choquet integral. *Belgian Journal of Operations Research, Statistics and Computer Science*, 38 (2-3):67-89, 1998.

[34] G. C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, (2):340–368, 1964.

[35] D. Schmeidler. Integral representation without additivity. *Proceedings of the American Mathematical Society*, (97(2)):255–261, 1986.

[36] G. Shafer. *A Mathematical Theory of Evidence.* Princeton University Press, Princeton, New Jersey, (USA), 1976.

[37] L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the theory of Games*, volume II of *Annals of Mathematics Studies*, pages 307–317. Princeton University Press, 1953.

[38] L. A. Skornjakov. *Elements of lattice theory.* Adam Hilger Ltd., 1977.

[39] M. Sugeno. *Theory of fuzzy integrals and its applications.* PhD thesis, Tokyo Institute of Technology, 1974.

[40] M. Sugeno and T. Terano. A model of learning based on fuzzy information. *Kybernetes*, (6):157–166, 1977.

[41] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behaviour.* Princeton University Press, New Jersey (USA), 1944.

[42] Z. Wang and G. Klir. *Fuzzy measure theory.* Plenum, New York, 1992.

[43] J. Wang, Z. Wang, K. Xu and G. Klir. Using genetic algorithms to determine nonnegative monotone set functions for information fusion in environments with random perturbation. *International Journal of Intelligent Systems*, 14:949–962, 1999.

[44] R. R. Yager. On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Transactions on Systems, Man and Cybernetics*, (18):183–190, 1988.