On the structure of some families of fuzzy measures*

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Abstract

The generation of fuzzy measures is an important question arising in the practical use of these operators. In this paper we deal with the problem of developing a random generator of fuzzy measures. More concretely, we study some of the properties that any random generator should satisfy. These properties lead to some theoretical problems concerning the group of isometries that we tackle in the paper for some subfamilies of fuzzy measures.

Keywords: Fuzzy measures, random generation, isometric transformations.

1 Introduction

Fuzzy measures [28] are generalizations of probability distributions in which we remove additivity and monotonicity is imposed instead. Fuzzy measures, together with Choquet integral [4] have been successfully applied in many fields, as Decision Under Uncertainty [2, 27], Multicriteria Decision Making [14, 15], Cooperative Game Theory [13], Combinatorics [26], pseudo-Boolean functions [17], etc.

Consider a situation that can be modelled through the Choquet integral w.r.t. a fuzzy measure (possibly, restricted to a subfamily). Next step is to obtain such a measure. When dealing with the practical identification of fuzzy measures, we usually have information about some prototypical examples and the problem consists in finding the fuzzy measure that best fits the data.

If the considered proximity criterion is the squared error, different techniques exist to solve the problem. For example, Grabisch and Nicolas [16] have developed a method based on solving a quadratic problem, in [1] Beliakov *et al.* acknowledge the case when the measures are symmetric or 2 or 3-additive, and in [12], Grabisch proposes an *ad hoc* algorithm for the problem; on the other hand, Wang *et al.* [30] have developed a method based on genetic algorithms [11]. Although in the two last cases a suboptimal solution is found, the computational cost is, sometimes, greatly reduced.

In a previous paper [5], we have proposed a method based on genetic algorithms to deal with the problem of identification of some convex families of fuzzy measures; in that paper, the cross-over operator considered in the genetic algorithm was the convex combination of fuzzy measures. This operator has the advantage that the convex combination of fuzzy measures is a fuzzy measure and this also holds for some subfamilies of fuzzy measures, as k-additive measures [15] and p-symmetric measures [22, 24] when the partition of indifference is fixed. Moreover, the simulations carried out with this algorithm suggest that the method is stable with respect to the presence of noise in the sample data [5]. This property is specially appealing, as exact values seldom appear in practical situations.

However, the convex combination reduces the search region in each iteration. This implies that the initial population must be carefully chosen so that the fuzzy measure fitting the data is curbed inside the region. To bear on this problem, the only option is to use as initial population the set of vertices of the corresponding subfamily (when this is possible, i.e. the subfamily determines a convex polyhedron).

On the other hand, as pointed out in [5], the number of extreme points for the general case coincides with the n-th Dedekind number [7], and similar results can be found for other subfamilies. The value of the n-th

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Dedekind number is very large (see Table 1 below) and this makes the use of the set of vertices unfeasible for large values of n (and $n = 8$ is already large!). Consequently, we are dragged to seek another initial population.

Another possibility is to initialize the population with a random generator of fuzzy measures. In this case, it is important that any measure can be obtained using the algorithm and that the method avoids any trend in the generated measures.

In this case, we are forced to consider some algorithms "seeming to be" random in order to generate the corresponding fuzzy measure. Additionally, we can deal with several algorithms, all of them seeming random at a first glance. How to choose the best one?

In this paper, we propose some properties that any random generator should satisfy. Then, to any procedure intending to be random, we should check if it satisfies these properties and we choose the procedure leading to best results. In this paper, we stress on two of these properties, namely the group of isometric transformations and the set of measures that remain invariant for these transformations. We treat the problem for three different cases: the general case, the p-symmetric case and the k-additive case.

The paper is organized as follows: We give the basic concepts on fuzzy measures, k-additive measures and p-symmetric measures in Section 2; in this section we also explain with more detail our algorithm of identification of fuzzy measures and the necessity of developing random procedures. Section 3 deals with the problem of obtaining the group of isometric transformations for each of these subfamilies of fuzzy measures. Section 4 treats the problem of finding the set of all invariant measures w.r.t. any isometric transformation for each subfamily. We finish with the conclusions and open problems.

2 Basic background

In order to be self-contained and to fix notation, we introduce in this section the basic results that will be needed throughout the paper.

Let X be a finite referential set of n elements, $X = \{x_1, ..., x_n\}$. Subsets of X are denoted by capital letters A, B, and so on, and also by $A_1, A_2, ...$ Sometimes we use the notation $\{x_{i_1}, ..., x_{i_r}\}$, specially for singletons and pairs; in order to avoid hard notation, we usually remove braces in this case. Matrices are denoted by $\mathbf{B}, \mathbf{M}, \mathbf{B}', ...$, and vectors are denoted by $\vec{v}, \vec{w}$, and so on.

Definition 1 *A* **fuzzy measure** *[28] (also called* **capacity** *[4] or* **non-additive measure** *[8]) on X is a function μ that assigns to each subset of X a real value between 0 and 1 satisfying*

1. *$\mu(\emptyset) = 0$ and $\mu(X) = 1$.*
2. *If $A \subseteq B$ then $\mu(A) \leq \mu(B)$.*

We will denote by $\mathcal{FM}(X)$ the set of fuzzy measures on X. Remark that $\mathcal{FM}(X)$ is a convex polyhedron.

Definition 2 *Let μ be a fuzzy measure over X; we define the* **dual measure of** *μ as the fuzzy measure $\bar{\mu}$ given by $\bar{\mu}(A) = 1 - \mu(A^c)$.*

Definition 3 *A* **unanimity game** *over $A \subseteq X$, $A \neq \emptyset$ is a fuzzy measure defined by*

$$u_A(B) = \begin{cases} 1 & \text{if A} \subseteq \text{B} \\ 0 & \text{otherwise} \end{cases}$$

For $\emptyset$, we define the unanimity game by

$$u_\emptyset(B) = \begin{cases} 1 & \text{if B} \neq \emptyset \\ 0 & \text{if B} = \emptyset \end{cases}$$

Consider a function $f : X \to \mathbb{R}^+$, whose corresponding scores on each element x_i are $f(x_1), \ldots, f(x_n)$. From the point of view of Multicriteria Decision Making, in order to compare different objects, we need to obtain an overall score from $f(x_1), \ldots, f(x_n)$. This is done through an aggregation operator [18], the Choquet integral [4] being among the most popular.

Definition 4 *The* **Choquet integral** *of a function*

$$f : X \to [0, 1]$$

with respect to a fuzzy measure μ on X is defined by

$$\mathcal{C}_\mu(f) := \sum_{i=1}^{n} (f(x_{(i)}) - f(x_{(i-1)}))\mu(B_i),$$

where $\{x_{(1)}, \ldots, x_{(n)}\}$ *is a permutation of the set* $\{x_1, \ldots, x_n\}$ *satisfying*

$$0 = f(x_{(0)}) \leq f(x_{(1)}) \leq \ldots \leq f(x_{(n)}),$$

and

$$B_i = \{x_{(i)}, \ldots, x_{(n)}\}.$$

The Choquet integral is a generalization of the concept of expected value, and models based on it are generalizations of the expected utility model.

Notice that $2^n - 2$ coefficients are needed in order to determine a fuzzy measure on a set of n elements. This fact makes the use of these measures unfeasible in practice for large values of n. In an attempt to reduce complexity, some subfamilies of non-additive measures have been defined, e.g. k-additive measures [13], p-symmetric measures [22], k-intolerant measures [19], λ-measures [29], or more generally, decomposable measures [9]. In this paper, we will deal with k-additive measures and p-symmetric measures.

The concept of k-additivity is based on the Möbius transform.

Definition 5 *[26] Let* μ *be a set function (not necessarily a fuzzy measure) on* X*. The* **Möbius transform (or inverse)** *of* μ *is another set function on* X *defined by*

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B), \forall A \subseteq X. \tag{1}$$

The Möbius transform given, the original set function can be recovered through the *Zeta transform* [3]:

$$\mu(A) = \sum_{B \subseteq A} m(B). \tag{2}$$

The Möbius transform represents the importance that a subset can attain on its own, without considering its different parts.

Definition 6 *[13] A fuzzy measure* μ *is said to be* k**-additive** *if its Möbius transform vanishes for any* $A \subseteq X$ *such that* $|A| > k$ *and there exists at least one subset* A *of exactly* k *elements such that* $m(A) \neq 0$.

In this sense, a probability measure is just a 1-additive measure [13] and k-additive measures constitute a middle term between probabilities and general fuzzy measures. For a k-additive measure, the number of coefficients is reduced to

$$\sum_{i=1}^{k} \binom{n}{i}.$$

More about k-additive measures can be found e.g. in [15]. We will denote the set of fuzzy measures in $\mathcal{FM}(X)$ being at most k-additive by $\mathcal{FM}^k(X)$; remark that $\mathcal{FM}^k(X)$ is a convex polyhedron.

Let us now turn to the concept of p-symmetry. This concept appears as a middle term between symmetric measures and general fuzzy measures.

Definition 7 *A fuzzy measure* μ *is said to be* **symmetric** *if it satisfies for any* $A, B \in \mathcal{P}(X)$,

$$|A| = |B| \Rightarrow \mu(A) = \mu(B).$$

The definition of p-symmetric measure is based on the concept of indifferent elements and subsets of indifference.

Definition 8 *[24] Given two elements* x_i, x_j *of the universal set* X *and* $\mu \in \mathcal{FM}(X)$*, we say that* x_i *and* x_j *are* **indifferent elements** *for* μ *if and only if*

$$\forall A \subseteq X \setminus \{x_i, x_j\}, \ \mu(A \cup \{x_i\}) = \mu(A \cup \{x_j\}).$$

If x_i and x_j are indifferent, they have exactly the same behavior.

Definition 9 *[24] Given a subset* A *of* X *and* $\mu \in \mathcal{FM}(X)$*, we say that* A *is a* **subset of indifference** *for* μ *if and only if* $\forall B_1, B_2 \subseteq A$, $|B_1| = |B_2|$ *and* $\forall C \subseteq X \setminus A$*, it is*

$$\mu(B_1 \cup C) = \mu(B_2 \cup C).$$

Two elements in a subset of indifference A are indifferent elements in the sense of Definition 8. Thus, all the elements in A have the same behavior.

Let Π_1, Π_2 be two partitions of X. We say that Π_2 is **coarser** than Π_1, denoted $\Pi_1 \sqsubseteq \Pi_2$ if

$$\forall A_1 \in \Pi_1, \exists A_2 \in \Pi_2, \text{ s.t. } A_1 \subseteq A_2.$$

We are now in a position to define p-symmetric measures.

Definition 10 *[24] Given a fuzzy measure μ, we say that μ is a* p-**symmetric measure** *if and only if the coarsest partition of the universal set in subsets of indifference is $\{A_1, ..., A_p\}, A_i \neq \emptyset, \forall i \in \{1, ..., p\}$.*

The existence and unicity of this partition has been proved in [23]. We will denote by $\mathcal{FM}(A_1, ..., A_p)$ the set of fuzzy measures for which A_i, $i = 1, ..., p$, is a subset of indifference (but not necessarily p-symmetric! Indeed, any symmetric measure belongs to $\mathcal{FM}(A_1, ..., A_p)$). It can be easily seen that $\mathcal{FM}(A_1, ..., A_p)$ is a convex polyhedron for a fixed partition $\{A_1, ..., A_p\}$.

As all the elements in the same subset of indifference have the same behavior, when dealing with a fuzzy measure in $\mathcal{FM}(A_1, ..., A_p)$, we only need to know the number of elements of each A_i that belong to a given subset B of the universal set X. Therefore, the following result holds:

Lemma 1 *[24] If $\{A_1, ..., A_p\}$ is a partition of X, then in order to define a measure in $\mathcal{FM}(A_1, ..., A_p)$, any $C \subseteq X$ can be identified with a p-dimensional vector $(c_1, ..., c_p)$ with $c_i := |C \cap A_i|$.*

This property allows a reduction in the complexity of the measure:

Lemma 2 *[24] Let μ be a p-symmetric measure w.r.t. the indifference partition $\{A_1, ..., A_p\}$. Then, it can be represented in a $(|A_1| + 1) \times \cdots \times (|A_p| + 1)$ matrix whose coefficients are defined by*

$$\mathbf{M}(i_1, ..., i_p) := \mu(i_1, ..., i_p), i_j \in \{0, ..., |A_j|\}.$$

More properties about p-symmetric measures and their behavior for Choquet integral can be found in [24, 23].

Let us now turn to the problem of identification. Assume a situation that can be modelled through the Choquet integral w.r.t. a fuzzy measure restricted to a subfamily $\mathcal{F}$; next step is to obtain such a measure. For this, suppose we have some prototypical examples for which we know the score on each $x_i \in X$ and the corresponding overall score, this overall score being the Choquet integral. We assume that all these scores are numerical; if this is not the case, they should be transformed into numerical data through a tool dealing with ordinal values, as MACBETH [10] or TOMASO [20]. Thus, the sample information can be written as m functions $f_1, \ldots, f_m$ representing the scores on n criteria and another m values $y_1, \ldots, y_m$ denoting their corresponding Choquet values. Consider a subfamily of fuzzy measures $\mathcal{F}$. We look for a fuzzy measure $\mu \in \mathcal{F}$ minimizing

$$\sum_{i=1}^{m} (\mathcal{C}_\mu(f_i) - y_i)^2.$$

In a previous paper [5], we have proposed a method based on genetic algorithms to deal with this problem. The cross-over operator considered in such algorithm is the convex combination

$$\lambda\mu_1 + (1 - \lambda)\mu_2,$$

with $\lambda \in [0, 1]$ chosen at random. When $\mathcal{F}$ is convex, the convex combination is a suitable operator, as it is not necessary to check on each step if the resulting measure is in $\mathcal{F}$.

However, this operator has the drawback that the search region is reduced in each iteration (Figure 1), and thus, if the initial population is not chosen carefully, then it might be the case that only bad approximations could be found among the convex combinations of those initial individuals.

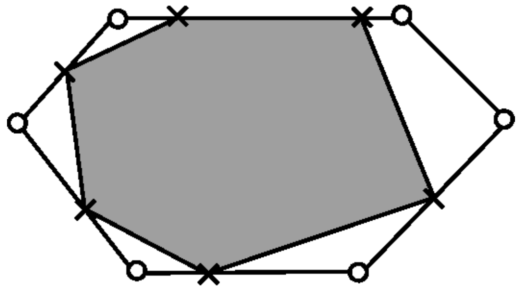
To solve this problem, the best option is to use as initial population the set of vertices of the corresponding subfamily. In this sense, the following results for $\mathcal{FM}(X), \mathcal{FM}^k(X)$ and $\mathcal{FM}(A_1, ..., A_p)$ can be proved:

Theorem 1 *[25] The set of $\{0, 1\}$-valued measures constitutes the set of vertices of $\mathcal{FM}(X)$.*

Theorem 2 *[21] There are vertices of the set $\mathcal{FM}^k(X), k > 2$, that are not $\{0, 1\}$-valued measures.*

Proposition 1 *[21] The set of extreme points of $\mathcal{FM}^1(X)$ (resp. $\mathcal{FM}^2(X)$) are the $\{0, 1\}$-valued measures that are in $\mathcal{FM}^1(X)$ (resp. $\mathcal{FM}^2(X)$).*

Figure 1: Reduction of search space due to convex combination



Theorem 3 *[21] The set of extreme points of $\mathcal{FM}(A_1, ..., A_p)$ is the set of $\{0,1\}$-valued measures that are also in $\mathcal{FM}(A_1, ..., A_p)$.*

Let us focus on the general case. As pointed out in [5], the number of $\{0,1\}$-valued measures coincides with the n-th Dedekind number [7], and similar results can be found for k-additive measures and p-symmetric measures. The first Dedekind numbers are given in Table 1.

n	Dedekind numbers
1	1
2	4
3	18
4	166
5	7579
6	7828352
7	2414682040996
8	56130437228687557907786

Table 1: Number of vertices of $\mathcal{FM}(X)$

From this table, it can be seen that the use of the set of vertices is unfeasible for large values of n. Consequently, we have to look for another initial population.

Another possibility is to initialize the population with a random generator of fuzzy measures, although in this case we cannot ensure that the measure fitting the data is inside the search region. In this case, we have the problem of determining whether a given procedure is indeed random. To face this problem, we propose some properties that any random generator should satisfy. Then, to any procedure intending to be random, we should check if it satisfies these properties. In this paper, we deal with two of these properties: the group of isometric transformations and the set of measures remaining invariant by these transformations. We treat the problem for three different convex subfamilies of fuzzy measures: $\mathcal{FM}(X), \mathcal{FM}^k(X)$ and $\mathcal{FM}(A_1, ..., A_p)$.

3 The group of isometric transformations

If a procedure generating fuzzy measures in $\mathcal{F}$ is random, then it has no trend to obtain measures in a subregion of $\mathcal{F}$. Therefore, we can expect subregions with the same hyper-volume to have a similar number of generated measures. To check this property, we have to pin out subregions with the same hyper-volume, and a compelling way to obtain such subregions is considering the image of a subregion through an isometric invariant transformation, i.e. a transformation leaving invariant the search space and keeping distances (and thus, hyper-volumes).

More concretely, we identify a fuzzy measure μ with a vector in $\mathbb{R}^r$ for a suitable choice of r depending on the representation ($r = 2^n - 2$ for $\mathcal{FM}(X)$ and $\mathcal{FM}^k(X)$, $r = (|A_1| + 1)...(|A_p| + 1)$ for $\mathcal{FM}(A_1, ..., A_p)$)

where each coordinate is given by $\mu(B), B \subseteq X, B \neq \emptyset, X$ (resp. $\mu(b_1, ..., b_p)$ for $\mathcal{FM}(A_1, ..., A_p)$), for an order (for example the binary order), and we consider the Euclidean distance in $\mathbb{R}^r$, so that

$$d(\mu_1, \mu_2) = \sqrt{\sum_{B \subseteq X} (\mu_1(B) - \mu_2(B))^2}.$$

Definition 11 *Let $\mathcal{F}$ be a family of fuzzy measures. A surjective function $f : \mathcal{F} \to \mathcal{F}$ is an* **isometry** *if*

$$d(\mu_1, \mu_2) = d(f(\mu_1), f(\mu_2)), \forall \mu_1, \mu_2 \in \mathcal{F}.$$

Remark that an isometry is a bijective mapping on $\mathcal{F}$. Let us denote by $\mathcal{G}(\mathcal{F})$ the set of isometries of $\mathcal{F}$.

For $\mathcal{G}(\mathcal{F})$, it is easy to see that the identity function is an isometry and that the composition of functions in $\mathcal{G}(\mathcal{F})$ is an isometry. Moreover, as an isometry is a bijective mapping it has an inverse function, and this inverse is an isometry, too. Therefore, $\mathcal{G}(\mathcal{F})$ forms a group under usual composition of functions.

We start our study with some previous results:

Lemma 3 *Suppose $\mathcal{F}$ is a convex subfamily of fuzzy measures on X. If $f \in \mathcal{G}(\mathcal{F})$, then*

$$f(\lambda\mu_1 + (1-\lambda)\mu_2) = \lambda f(\mu_1) + (1-\lambda) f(\mu_2), \forall \mu_1, \mu_2 \in \mathcal{F}, \lambda \in [0, 1].$$

Proof: If $\mu_1 = \mu_2$ the result is obvious. Suppose then that $\mu_1 \neq \mu_2$ and consider $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ with $\mu_1, \mu_2 \in \mathcal{FM}(X)$ and λ in $[0, 1]$. Let us denote $d := d(\mu_1, \mu_2)$. Clearly,

$$d(\mu_1, \mu) = (1-\lambda)d, \; d(\mu_2, \mu) = \lambda d. \tag{3}$$

Then, as $f \in \mathcal{G}(\mathcal{F})$,

$$d(f(\mu_1), f(\mu)) = d(\mu_1, \mu) = (1-\lambda)d, \; d(f(\mu_2), f(\mu)) = d(\mu_2, \mu) = \lambda d.$$

Applying the triangular inequality,

$$d = d(f(\mu_1), f(\mu_2)) \leq d(f(\mu_1), f(\mu)) + d(f(\mu), f(\mu_2)) = (1-\lambda)d + \lambda d = d,$$

whence the equality holds, and we deduce that $f(\mu)$ is in the segment joining $f(\mu_1)$ and $f(\mu_2)$. But then, by Equation (3),

$$f(\lambda\mu_1 + (1-\lambda)\mu_2) = \lambda f(\mu_1) + (1-\lambda) f(\mu_2), \forall \mu_1, \mu_2 \in \mathcal{F}, \lambda \in [0, 1].$$

This finishes the proof. ■

As a consequence, we have

Corollary 1 *If $f \in \mathcal{G}(\mathcal{F})$ and $\mathcal{F}$ is a convex polyhedron, then f maps vertices in vertices.*

Let us denote by $\mathcal{G}(X)$ the group of isometries for $\mathcal{FM}(X)$, and $\mathcal{G}(A_1, ..., A_p)$ (respectively $\mathcal{G}^k(X)$) the corresponding group for $\mathcal{FM}(A_1, ..., A_p)$ (resp. $\mathcal{FM}^k(X)$). The goal of this section is to determine these groups.

3.1 The general case

In this case, we identify $\mu \in \mathcal{FM}(X)$ with a $(2^n - 2)$-vector whose components are $\mu(A), A \subseteq X, A \neq X, \emptyset$ for a given order. Therefore, $\mathcal{FM}(X)$ is a convex polyhedron in $\mathbb{R}^{2^n - 2}$.

Let us start with some definitions.

Definition 12 *Consider $\sigma : X \to X$ a permutation on X. We define the* **symmetry induced by** σ*, denoted S_σ, the transformation on $\mathcal{FM}(X)$ such that for any $\mu \in \mathcal{FM}(X)$, the fuzzy measure $S_\sigma(\mu)$ is defined by*

$$S_\sigma(\mu)(x_1, ..., x_r) = \mu(x_{\sigma(x_1)}, ..., x_{\sigma(x_r)}), \forall \{x_1, ..., x_r\} \subseteq X.$$

Definition 13 *We define the* **dual transformation***, denoted D, the transformation on $\mathcal{FM}(X)$ given by*

$$D : \begin{array}{ccc} \mathcal{FM} & \to & \mathcal{FM} \\ \mu & \hookrightarrow & \overline{\mu} \end{array}.$$

Lemma 4 *Let $\mathcal{F}$ be a subfamily of fuzzy measures.*

1. *If S_σ is an internal operation on $\mathcal{F}$, then $S_\sigma \in \mathcal{G}(\mathcal{F})$.*
2. *If D is an internal operation on $\mathcal{F}$, then $D \in \mathcal{G}(\mathcal{F})$.*

Proof: It suffices to compute the distance:

1. Remark that S_σ applied on μ just produces a reordenation of vector μ. Then, for any $\mu_1, \mu_2 \in \mathcal{F}$,

$$d(S_\sigma(\mu_1), S_\sigma(\mu_2)) = \sqrt{\sum_{B\subseteq X} (S_\sigma(\mu_1)(B) - S_\sigma(\mu_2)(B))^2} = \sqrt{\sum_{B\subseteq X} (\mu_1(B) - \mu_2(B))^2} = d(\mu_1, \mu_2).$$

2. For the dual application,

$$d(D(\mu_1), D(\mu_2)) = \sqrt{\sum_{B\subseteq X} (D(\mu_1)(B) - D(\mu_2)(B))^2} = \sqrt{\sum_{B\subseteq X} (1 - \mu_1(B^c) - (1 - \mu_2(B^c)))^2}$$

$$= \sqrt{\sum_{B\subseteq X} (\mu_1(B) - \mu_2(B))^2} = d(\mu_1, \mu_2).$$

This finishes the proof. ■

For the general case, the following can be proved:

Theorem 4 *If $|X| > 2$, the set $\mathcal{G}(X)$ is given by symmetries and compositions of symmetries with the dual application. In fact, G(X) is the semidirect product of the group of symmetries with the cyclic group or order 2 generated by the dual transformation.*

Proof: By Lemma 4, we already know that S_σ, $\forall \sigma$ permutation on X and D are transformations in $\mathcal{G}(X)$. We prove that the reverse also holds in several lemmas:

Lemma 5 *Let f be a transformation in $\mathcal{G}(X)$. Then, f can be written as*

$$f(\mu) = \mathbf{B}\mu + \vec{b}, \forall \mu \in \mathcal{FM}(X),$$

where $\mathbf{B}$ is a square matrix and $\vec{b} := f(u_X)$.

Proof: Consider $f : \mathcal{FM}(X) \to \mathcal{FM}(X)$ in $\mathcal{G}(X)$. Define $\vec{b} := f(u_X)$. Note that $u_X = \vec{0}$, as we have removed $u_X(\emptyset)$ and $u_X(X)$. Let us define

$$\begin{array}{cccc} f_1 : & \mathcal{FM}(X) & \to & \mathbb{R}^{2^n-2} \\ & \mu & \hookrightarrow & f(\mu) - \vec{b} \end{array}$$

As $\mathcal{FM}(X)$ is convex, for $\mu_1, \mu_2 \in \mathcal{FM}(X)$, $\lambda \in [0, 1]$, it follows that $\lambda\mu_1 + (1-\lambda)\mu_2 \in \mathcal{FM}(X)$. From Lemma 3, we know that f keeps convex combinations in $\mathcal{FM}(X)$. Then,

$$f_1(\lambda\mu_1 + (1-\lambda)\mu_2) = f(\lambda\mu_1 + (1-\lambda)\mu_2) - \vec{b} = \lambda f(\mu_1) + (1-\lambda)f(\mu_2) - \vec{b} = \lambda f_1(\mu_1) + (1-\lambda)f_1(\mu_2).$$

Therefore, f_1 keeps convex combinations, too. Moreover, $f_1(u_X) = \vec{0}$.

Consider the family of unanimity games $\{u_A\}_{A\subseteq X, A\neq\emptyset, X}$. Let us see that this family is a basis for the vector space $\mathbb{R}^{2^n-2}$. As there are $2^n - 2$ vectors in the family, it suffices to show that they are linearly independent. Suppose that $\exists \lambda_1, ..., \lambda_r, u_{A_1}, ..., u_{A_r}$, such that

$$\sum_{i=1}^{r} \lambda_i u_{A_i} = \vec{0}.$$

As the Möbius transform is a linear transformation on $\mathcal{FM}(X)$, the Möbius transform m of a fuzzy measure μ can be written as

$$m := \mathbf{M}\mu,$$

where $\mathbf{M}$ is a square matrix. Then, as the Möbius transform is a bijective transformation,

$$\vec{0} = \mathbf{M}(\sum_{i=1}^{r} \lambda_i u_{A_i}) = \sum_{i=1}^{r} \lambda_i \mathbf{M} u_{A_i}. \quad (4)$$

On the other hand, it can be easily seen that $\mathbf{M}u_{A_i}$ is a vector with value 1 in the coordinate corresponding to A_i and 0 otherwise. This implies that $\lambda_i = 0, \forall i$ in Equation (4). Thus, $\{u_A\}_{A\subseteq X, A\neq\emptyset, X}$ is a basis of $\mathbb{R}^{2^n-2}$.

We extend f_1 to all $\mathbb{R}^{2^n-2}$ by

$$\begin{array}{cccc} \bar{f}_1: & \mathbb{R}^{2^n-2} & \rightarrow & \mathbb{R}^{2^n-2} \\ & \vec{v} = \sum_{i=1}^{2^n-2} \alpha_i u_{A_i} & \hookrightarrow & \sum_{i=1}^{2^n-2} \alpha_i f_1(u_{A_i}) \end{array}$$

As $\{u_A\}_{A\subseteq X, A\neq\emptyset, X}$ is a basis of $\mathbb{R}^{2^n-2}$ and $f_1(u_X) = \vec{0}$, it follows that $\bar{f}_1$ is well-defined and $\bar{f}_1(\mu) = f_1(\mu), \forall \mu \in \mathcal{FM}(X)$ as f_1 is linear on this set. Moreover, $\bar{f}_1$ is a linear application on $\mathbb{R}^{2^n-2}$ and thus, it can be written as

$$\bar{f}_1(\vec{v}) = \mathbf{B}\vec{v},$$

where $\mathbf{B}$ is a square matrix. Let us now define

$$\begin{array}{cccc} \bar{f}: & \mathbb{R}^{2^n-2} & \rightarrow & \mathbb{R}^{2^n-2} \\ & \vec{v} & \hookrightarrow & \bar{f}_1(\vec{v}) + \vec{b} \end{array}$$

For $\mu \in \mathcal{FM}(X)$, it is

$$\mathbf{B}\mu + \vec{b} = \bar{f}(\mu) = \bar{f}_1(\mu) + \vec{b} = f_1(\mu) + \vec{b} = f(\mu),$$

whence the result. ■

Lemma 6 *If $n > 2$, given $f \in \mathcal{G}(X)$ such that $f(\mu) = \mathbf{B}\mu + \vec{b}$. Then, necessarily $\vec{b} \in \{\vec{0}, \vec{1}\}$.*

Proof: It is clear that

$$d(u_\emptyset, u_X) = \max_{\mu_1,\mu_2\in\mathcal{FM}(X)} d(\mu_1, \mu_2).$$

Moreover, this is the only maximum when $n > 2$ (if $n = 2$, we have $d(u_{x_1}, u_{x_2}) = d(u_X, u_\emptyset)$).

On the other hand, as f maps $\mathcal{FM}(X)$ in $\mathcal{FM}(X)$, we have that $f(u_X), f(u_\emptyset) \in \mathcal{FM}(X)$. And, as f keeps distances, we conclude that $\{f(u_X), f(u_\emptyset)\} = \{u_X, u_\emptyset\}$.

Now, as $u_X = \vec{0}, u_\emptyset = \vec{1}$, it is $f(u_X) = \mathbf{B}u_X + \vec{b} = \vec{b} \in \{\vec{0}, \vec{1}\}$. ■

Lemma 7 *Suppose $n > 2$. Let $f \in \mathcal{G}(X)$ such that $f(\mu) = \mathbf{B}\mu$. Then, if μ is a symmetric measure, $f(\mu) = \mu$.*

Proof: As the set of symmetric fuzzy measures is a convex polyhedron of $\mathcal{FM}(X)$, it suffices to show the result for the extreme points of the polyhedron. It is immediate to see that these vertices are given by

$$\mu_k(A) := \left\{ \begin{array}{ll} 1 & \text{if } |A| \geq k \\ 0 & \text{otherwise} \end{array} \right., \; k = 1, ..., n.$$

In particular, $u_X = \mu_n$, $u_\emptyset = \mu_1$.

Consider $x_i \in X$ and the unanimity game $u_{X\backslash x_i}$. As f keeps distances,

$$1 = d(u_X, u_{X\backslash x_i})^2 = d(\mathbf{B}u_X, \mathbf{B}u_{X\backslash x_i})^2 = d(u_X, \mathbf{B}u_{X\backslash x_i})^2.$$

This implies that $\mathbf{B}u_{X\backslash x_i}$ has exactly one value 1 and $2^n - 3$ values 0. Moreover, as $\mathbf{B}u_{X\backslash x_i} \in \mathcal{FM}(X)$ and maps vertices in vertices (Corollary 1), we conclude that $\mathbf{B}u_{X\backslash x_i} = u_{X\backslash x_j}$ by monotonicity.

On the other hand, $d(\mu_{n-1}, u_{X\backslash x_i})^2 = n - 1$. Moreover, $d(\mu_{n-1}, u_X)^2 = n \Rightarrow \mathbf{B}\mu_{n-1}$ is a vertex with exactly n values 1.

Joining both results, we conclude that $\mathbf{B}\mu_{n-1}(X\backslash x_j) = 1$. As this can be done for any $x_i \in X$ and f is bijective, we conclude that $\mathbf{B}\mu_{n-1} = \mu_{n-1}$.

Consider now $\mu \in \mathcal{FM}(X)$ given by

$$\mu(X\backslash x_i) = 1, \forall x_i \in X, \exists!\, x_j, x_k \in X \,|\, \mu(X\backslash\{x_j, x_k\}) = 1, \; \mu(A) = 0 \text{ otherwise.}$$

As f keeps distances, and $d(\mu_{n-1}, \mu)^2 = 1, d(u_X, \mu)^2 = n + 1$, we deduce $d(\mathbf{B}\mu_{n-1}, \mathbf{B}\mu)^2 = d(\mu_{n-1}, \mathbf{B}\mu)^2 = 1$ and $d(\mathbf{B}u_X, \mathbf{B}\mu)^2 = d(u_X, \mathbf{B}\mu)^2 = n + 1$. Thus, by monotonicity, $\mu' := \mathbf{B}\mu$ can be written as

$$\mu'(X\backslash x_i) = 1, \forall x_i \in X, \exists!\, x_{j'}, x_{k'} \in X \,|\, \mu'(X\backslash\{x_{j'}, x_{k'}\}) = 1, \; \mu'(A) = 0 \text{ otherwise.}$$

For μ_{n-2} we have $d(\mu_{n-1}, \mathbf{B}\mu_{n-2})^2 = \binom{n}{2}, d(u_X, M\mu_{n-2})^2 = \binom{n}{2} + \binom{n}{1}$. Therefore, $\mathbf{B}\mu_{n-2}$ has $\binom{n}{2} + \binom{n}{1}$ values 1 and $\binom{n}{1}$ correspond to subsets $X\backslash\{x_i, x_i\} \in X$.

On the other hand, $d(\mathbf{B}\mu_{n-2}, \mathbf{B}\mu)^2 = \binom{n}{2} - 1$, whence $\mathbf{B}\mu_{n-2}(X \backslash \{x_{j'}, x_{k'}\}) = 1$. As this can be done for any $x_j, x_k \in X$, we conclude that $\mathbf{B}\mu_{n-2} = \mu_{n-2}$.

Following this process, we prove that $\mathbf{B}\mu_i = \mu_i$, $i = 1, ..., n$, whence the result. ■

Lemma 8 *Consider $f \in \mathcal{G}(X)$ such that $f\mu = \mathbf{B}\mu$. Let μ be a $\{0,1\}$-valued measure. Then, the number of subsets A of cardinality i such that $\mu(A) = 1$ coincides with the number of subsets C of cardinality i such that $\mathbf{B}(\mu)(C) = 1$.*

Proof: As the transformation keeps distances, for any $\{0,1\}$-valued measure μ,

$$d(u_X, \mu) = d(\mathbf{B}u_X, \mathbf{B}\mu) = d(u_X, \mathbf{B}\mu).$$

Therefore, $\mathbf{B}\mu$ has exactly the same number of values 1 as μ.

We will prove the by induction on i that the result holds for cardinality $n - i$.

For $i = 1$, applying Lemma 7,

$$d(\mu_{n-1}, \mu) = d(\mathbf{B}\mu_{n-1}, \mathbf{B}\mu) = d(\mu_{n-1}, \mathbf{B}\mu),$$

whence we conclude that $\mathbf{B}\mu$ has exactly the same number of values 1 for subsets of type $X \backslash x_i$ as μ.

Consider $i > 1$ and suppose the result holds until $i - 1$. Applying Lemma 7,

$$d(\mu_{n-i}, \mu) = d(\mathbf{B}\mu_{n-i}, \mathbf{B}\mu) = d(\mu_{n-i}, \mathbf{B}\mu).$$

This means that the number of subsets A such that $|A| \geq n - i, \mu(A) = 1$ is the same as the number of subsets C such that $|C| \geq n - i, \mathbf{B}\mu(C) = 1$. Applying now the induction hypothesis, the result holds. ■

Lemma 9 *Let $f \in \mathcal{G}(X)$ such that $f(\mu) = \mathbf{B}\mu$. Then f is necessarily a symmetry.*

Proof: As f keeps convex combinations (Lemma 3), it suffices to show the result for the extreme points of $\mathcal{FM}(X)$, i.e. the $\{0,1\}$-valued measures (Theorem 1).

If $\mu = u_{X \backslash x_i}$ for some $x_i \in X$, we have already shown in the proof of Lemma 7 that $\mathbf{B}u_{X \backslash x_i} = u_{X \backslash x_j}$ for some $x_j \in X$. As f is bijective, the mapping

$$\begin{array}{cccc} \sigma : & X & \rightarrow & X \\ & x_i & \hookrightarrow & x_j \end{array}$$

is a permutation on X. We will prove that $f = S_\sigma$.

We prove the result applying induction on $r \equiv$ maximal value such that $\exists A \subseteq X, |A| = n - r$, $\mu(A) = 1$.

- For $r = 1$, we show the result again by induction on $s \equiv$ number of subsets of cardinality $n - r$ whose value is 1.

  If $s = 1$, then $\mu = u_{X \backslash x_i}$ and the result holds.

  Assume $s > 1$ and suppose the result holds until $s - 1$. Consider $X \backslash x_i$ such that $\mu(X \backslash x_i) = 1$ and define μ' by

  $$\mu'(A) := \left\{ \begin{array}{cl} \mu(A) & \text{if } A \neq X \backslash x_i \\ 0 & \text{otherwise} \end{array} \right. \tag{5}$$

  Then, $d(\mu', \mu) = 1$, whence $d(f(\mu'), f(\mu)) = 1$. By Lemma 8, this implies that $f(\mu)(A) = 1$ when $f(\mu')(A) = 1$ and that there exists a subset $X \backslash x_j$ such that $f(\mu)(X \backslash x_j) = 1, f(\mu')(X \backslash x_j) = 0$. Moreover, this holds for any μ' defined by Equation (5) and any $X \backslash x_i$ such that $\mu(X \backslash x_i) = 1$. Therefore, the result holds applying the induction hypothesis on s.

- Assume $r > 1$ and that the result holds until $r - 1$. We apply again induction on $s \equiv$ number of subsets of cardinality $n - r$ whose value is 1.

  If $s = 1$, then $\exists! A, |A| = n - r$ such that $\mu(A) = 1$. We have two different cases:

  - If $\mu = u_A$, with $|A| = n - r$, we consider

    $$\mu'(C) := \left\{ \begin{array}{cl} \mu(C) & \text{if } C \neq A \\ 0 & \text{if } C = A \end{array} \right.$$

    Then, $d(\mu', \mu) = 1$, whence $d(f(\mu'), f(\mu)) = 1$. This means that $f(\mu)(C) = 1$ whenever $f(\mu')(C) = 1$ and that there exists a subset D such that $f(\mu)(D) = 1, f(\mu')(D) = 0$. Moreover, by Lemma 8, it follows that $|D| = |A|$. And $f(\mu') = S_\sigma(\mu')$ by induction. This implies that D is the image of A through σ; otherwise, as there exists at least two elements in $X \backslash A$, monotonicity would fail. Therefore, the result holds.

– Consider $x_i \in X\backslash A$ and suppose that μ is defined by

$$\mu(C) := \begin{cases} 1 & \text{if } A \subseteq C \text{ or } C = X\backslash x_i \\ 0 & \text{otherwise} \end{cases}$$

Then, considering μ' as defined before, we have $d(\mu', \mu) = 1$, whence $d(f(\mu'), f(\mu)) = 1$. Thus $f(\mu)(C) = 1$ whenever $f(\mu')(C) = 1$ and there exists a subset D such that $f(\mu)(D) = 1, f(\mu')(D) = 0$. Moreover, by Lemma 8, it follows that $|D| = |A|$. By induction, we know that $f(\mu') = S_\sigma(\mu')$. On the other hand, $f(u_A) = S_\sigma(u_A)$ and $f(u_A, \mu) = 1$. Joining both results, we conclude that $f(\mu) = S_\sigma(\mu)$.

Following this process for measures whose distance from u_A is 2, 3, and so on, and applying in each case that $f = S_\sigma$ for previous steps, we can prove the result for any measure for which $s = 1$.

Assume $s > 1$ and suppose the result holds until $s - 1$. Take A such that $\mu(A) = 1$ and $|A| = n - r$. Let μ'_A be defined by

$$\mu'_A(B) := \begin{cases} \mu(B) & \text{if } B \neq A \\ 0 & \text{if } C = A \end{cases}$$

Then, $d(\mu', \mu) = 1$, whence $d(f(\mu'), f(\mu)) = 1$. By induction, $f(\mu'_A) = S_\sigma(\mu'_A)$ and this can be done for any A such that $\mu(A) = 1$ and $|A| = n - r$. Therefore, $f(\mu) = S_\sigma(\mu)$.

This finishes the proof. ■

Lemma 10 *Let us define*

$$\mathcal{G}_0(X) := \{\mathbf{B}\mu \in \mathcal{G}(X)\},\ \mathcal{G}_1(X) := \{\mathbf{B}\mu + \vec{1} \in \mathcal{G}(X)\}.$$

Then, $|\mathcal{G}_0(X)| = |\mathcal{G}_1(X)|$.

Proof: Consider the binary order. Then, if the position of $A \subseteq X$ is i, then the corresponding position for A^c in the binary order is $2^n - 2 - i$.

Consider a transformation $f \in \mathcal{G}_1(X)$ with matrix $\mathbf{B}$. Define $\mathbf{B}'$ by $\mathbf{B}'(i, j) = -\mathbf{B}(2^n - 2 - i, j)$. Let us show that $f'(\mu) := \mathbf{B}'\mu$ is a transformation in $\mathcal{G}_0(X)$.

- First, let us show that $\mathbf{B}'\mu \in \mathcal{FM}(X), \forall \mu \in \mathcal{FM}(X)$. Consider $A \subseteq B$ and $A, B \neq X, \emptyset$. Then,

$$(\mathbf{B}'\mu)(A) = r \Leftrightarrow (-\mathbf{B}\mu)(A^c) = r \Leftrightarrow (\mathbf{B}\mu)(A^c) = -r \Leftrightarrow (\mathbf{B}\mu)(A^c) + 1 = 1 - r.$$

  As $f \in \mathcal{G}_1(X)$, it follows that $\mathbf{B}\mu + \vec{1} \in \mathcal{FM}(X)$. On the other hand, as $B^c \subseteq A^c$, by monotonicity,

$$(\mathbf{B}\mu)(B^c) + 1 \leq 1 - r \Leftrightarrow (\mathbf{B}\mu)(B^c) \leq -r \Leftrightarrow (-\mathbf{B}\mu)(B^c) \geq r \Leftrightarrow (\mathbf{B}'\mu)(B) \geq r.$$

  Therefore, monotonicity holds. Moreover, $\mathbf{B}'\mu(C) \in [0, 1], \forall C \subseteq X, C \neq X, \emptyset$ as $\mathbf{B}\mu + \vec{1} \in \mathcal{FM}(X)$ and consequently $\mathbf{B}\mu(C^c) \in [-1, 0]$.

- Let us prove that f' is an injective mapping on $\mathcal{FM}(X)$.
  Consider $\mu_1, \mu_2 \in \mathcal{FM}(X)$ such that $\mathbf{B}'\mu_1 = \mathbf{B}'\mu_2$. Then, for any $A \subseteq X, A \neq \emptyset, X$, we have

$$(\mathbf{B}'\mu_1)(A) = (\mathbf{B}'\mu_2)(A) \Leftrightarrow (-\mathbf{B}\mu_1)(A^c) = (-\mathbf{B}\mu_2)(A^c) \Leftrightarrow (\mathbf{B}\mu_1)(A^c) + 1 = (\mathbf{B}\mu_2)(A^c) + 1.$$

  But this means that $\mu_1 = \mu_2$ because $f \in \mathcal{G}(X)$ and consequently, bijective.

- Let us prove that $f'\mu$ is a surjective mapping on $\mathcal{FM}(X)$. Consider $\mu \in \mathcal{FM}(X)$ and let us show that we can obtain another fuzzy measure μ_1 such that $B'\mu_1 = \mu$.
  We denote by $\bar{\mu}$ the dual measure of μ. As $f \in \mathcal{G}(X)$, there exists μ_1 such that $\mathbf{B}\mu_1 + \vec{1} = \bar{\mu}$. Now, for any $A \subseteq X, A \neq \emptyset, X$, we have

$$(\mathbf{B}\mu_1)(A) + 1 = \bar{\mu}(A) \Leftrightarrow (\mathbf{B}\mu_1)(A) = \bar{\mu}(A) - 1 = -\mu(A^c) \Leftrightarrow (-\mathbf{B}'\mu_1)(A^c) = -\mu(A^c) \Leftrightarrow \mathbf{B}'\mu_1 = \mu$$

Therefore, $f'(\mu) = \mathbf{B}'\mu$ is a transformation in $\mathcal{G}_0(X)$.
On the other hand, it is clear that the mapping

$$\begin{array}{rccc} F: & \mathcal{G}_1(X) & \rightarrow & \mathcal{G}_0(X) \\ & \mathbf{B}\mu + \vec{1} & \hookrightarrow & \mathbf{B}'\mu \end{array}$$

is a bijective application, whence the result. ∎

As shown in Lemma 9, the set $\mathcal{G}_0(X)$ is the set of symmetric transformations following a permutation on X. Consider $f \in \mathcal{G}_0(X)$, and define

$$f' : \begin{array}{ccc} \mathcal{FM} & \rightarrow & \mathcal{FM} \\ \mu & \hookrightarrow & \overline{f(\mu)} \end{array}$$

As $f(u_X) = u_X$ and $\bar{u}_X = u_\emptyset = \vec{1}$, it follows that $f' \in \mathcal{G}_1(X)$. Moreover, different transformations in $\mathcal{G}_0(X)$ lead to different transformations in $\mathcal{G}_1(X)$, whence applying Lemma 10 we conclude that transformations in $\mathcal{G}_1(X)$ are symmetries composed with the dual application.

Finally, we show that $\mathcal{G}(X)$ is the semidirect product of symmetries with the dual application:

Lemma 11 *The group $\mathcal{G}_0(X)$ is a normal subgroup of $\mathcal{G}(X)$.*

Proof: Take g in $\mathcal{G}(X)$ and h in $\mathcal{G}_0(X)$. We must show that $g^{-1}hg$ belongs to $\mathcal{G}_0(X)$. If $g(u_X) = u_X$, then $g^{-1}(u_X) = u_X$ and the result clearly holds. If not, we know that $g(u_X) = u_\emptyset$ and $h(u_\emptyset) = u_\emptyset$. Thus,

$$(g^{-1}hg)(u_X) = g^{-1}(h(g(u_X))) = g^{-1}(h(u_\emptyset)) = g^{-1}(u_\emptyset) = u_X,$$

and consequently $g^{-1}hg \in \mathcal{G}_0(X)$. ∎

This finishes the proof of the theorem. ∎

Corollary 2 *If $|X| = n$, ($n > 2$), then the cardinality of $\mathcal{G}(X)$ is*

$$2(n!).$$

Theorem 5 *If $n = 2$, the group $\mathcal{G}(X)$ is isomorphic to the dihedral group D_4 (the group of isometries of the square).*

Proof: Each measure in $\mathcal{FM}(X)$ is determined by its value on x_1 and x_2. Thus, the function

$$C : \mathcal{FM}(X) \rightarrow [0,1] \times [0,1]$$

defined by

$$C(\mu) = (\mu(x_1), \mu(x_2))$$

is a bijection between $\mathcal{FM}(X)$ and the unit square. Also, C keeps distances (i.e. $d(C(\mu_1), C(\mu_2)) = d(\mu_1, \mu_2)$) so both sets have the same group of isometries. ∎

3.2 The group $\mathcal{G}(A_1, ..., A_p)$

For this subsection, we consider the vector expression of p-symmetric measures stated in Lemma 1. Thus, $\mathcal{FM}(A_1, ..., A_p)$ can be seen as a convex polyhedron in $\mathbb{R}^{(|A_1|+1)\cdots(|A_p|+1)}$. We start again with some definitions.

Definition 14 *Consider $\mathcal{FM}(A_1, ..., A_p)$ and suppose there exist j, k ($j < k$) such that $|A_j| = |A_k|$. We define the* **transposition between** A_j **and** A_k*, denoted $p_{j,k}$, as the function $p_{j,k} : \mathcal{FM}(A_1, ..., A_p) \rightarrow \mathcal{FM}(A_1, ..., A_p)$ such that for any $\mu \in \mathcal{FM}(A_1, ..., A_p)$, $p_{j,k}(\mu)$ is defined for any subset $(i_1, ..., i_p)$ by*

$$p_{j,k}(\mu)(i_1, ..., i_p) := \mu(i_1, ..., i_k, ..., i_j, ..., i_p).$$

For $\mathcal{FM}(A_1, ..., A_p)$, the corresponding result is:

Theorem 6 *If $|X| > 2$ or $|X| = 2, p = 1$, the set of isometric transformations in $\mathcal{G}(A_1, ..., A_p)$ are the compositions of transpositions between subsets of indifference with the same cardinality, and compositions of these transformations with the dual application. In fact, $\mathcal{G}(A_1, ..., A_p)$ is the semidirect product of the group generated by the transpositions between subsets of indifference with the same cardinality with the cyclic group or order 2 generated by the dual transformation.*

Proof: The theorem can be shown translating the corresponding lemmas of Theorem 4. However, we propose here another alternative proof.

Lemma 12 *Consider $\mathcal{FM}(A_1, ..., A_p)$ and suppose there exist j, k $(j < k)$ such that $|A_j| = |A_k|$. Then, $p_{j,k} \in \mathcal{G}(A_1, ..., A_p)$.*

Proof: Trivial. ∎

Lemma 13 *If $\mu \in \mathcal{FM}(A_1, ..., A_p)$, then $\bar{\mu} \in \mathcal{FM}(A_1, ..., A_p)$.*

Proof: Consider $B, C \subseteq X$ with the same vector representation for the partition $\{A_1, ..., A_p\}$ (Lemma 1).

$$\bar{\mu}(B) = 1 - \mu(B^c) = 1 - \mu(C^c) = \bar{\mu}(C),$$

as B^c and C^c have the same vector representation. Therefore, the result holds. ∎

Then, translating the proof of Lemma 4, we have

Corollary 3 *Consider $\mathcal{FM}(A_1, ..., A_p)$. Then, $D \in \mathcal{G}(A_1, ..., A_p)$.*

As $\mathcal{G}(A_1, ..., A_p)$ is a group, we conclude that compositions of transpositions between subsets of indifference with the same cardinality and these functions composed with the dual transformation are in $\mathcal{G}(A_1, ..., A_p)$.

Let us prove that these are the only transformations in the group.

Remark that $u_X, u_\emptyset \in \mathcal{FM}(A_1, ..., A_p)$ for any partition $\{A_1, ..., A_p\}$ of X. Then, just translating the proof of Lemma 6, we have

Lemma 14 *If $n > 2$, given $f \in \mathcal{G}(A_1, ..., A_p)$, we necessarily have $\{f(u_X), f(u_\emptyset)\} = \{u_X, u_\emptyset\}$.*

Let us define

$$\mathcal{G}_0(A_1, ..., A_p) := \{f \in \mathcal{G}(A_1, ..., A_p) : f(u_X) = u_X\}.$$

It can be seen that $\mathcal{G}_0(A_1, ..., A_p)$ is a subgroup of $\mathcal{G}(A_1, ..., A_p)$:

- The identity function is in $\mathcal{G}_0(A_1, ..., A_p)$.
- If $f, g \in \mathcal{G}_0(A_1, ..., A_p)$, then $f \circ g(u_X) = u_X$, whence $f \circ g \in \mathcal{G}_0(A_1, ..., A_p)$.
- If $f \in \mathcal{G}_0(A_1, ..., A_p)$, then $f^{-1}(u_X) = u_X$, whence $f^{-1} \in \mathcal{G}_0(A_1, ..., A_p)$.

Definition 15 *Consider $\mu \in \mathcal{FM}(X)$. We say that $\mu^* \in \mathcal{FM}(X)$* **dominates** *$\mu$, and we denote it $\mu^* \geq \mu$, if and only if*

$$\mu^*(A) \geq \mu(A), \forall A \subseteq X.$$

Lemma 15 *If $f \in \mathcal{G}_0(A_1, ..., A_p)$, then $\mu_1 \leq \mu_2$ implies $f(\mu_1) \leq f(\mu_2)$ for all extremes μ_1, μ_2 of $\mathcal{FM}(A_1, ..., A_p)$.*

Proof: As $u_X \in \mathcal{FM}(A_1, ..., A_p)$, the number of subsets for which an extreme $\mu \in \mathcal{FM}(A_1, ..., A_p)$ attains value 1 is invariant under f, since $d(f(\mu), f(u_X)) = d(\mu, u_X)$ and $d(f(\mu), f(u_X)) = d(f(\mu), u_X)$ as $f \in \mathcal{G}_0(A_1, ..., A_p)$. Let us denote

$$n_1 := d(\mu_1, u_X), n_2 := d(\mu_2, u_X).$$

Since $\mu_1 \leq \mu_2$ we have

$$d(f(\mu_1), f(\mu_2)) = d(\mu_1, \mu_2) = n_2 - n_1.$$

On the other hand, consider

$$\begin{aligned} p &:= |\{A \subseteq X : f(\mu_1)(A) = f(\mu_2)(A) = 1\}|, \\ p_1 &:= |\{A \subseteq X : f(\mu_1)(A) = 1, f(\mu_2)(A) = 0\}|, \\ p_2 &:= |\{A \subseteq X : f(\mu_2)(A) = 1, f(\mu_1)(A) = 0\}|. \end{aligned}$$

Then, $p_1 + p_2 = d(f(\mu_1), f(\mu_2)) = n_2 - n_1$. Also,

$$p + p_1 = |\{A \subseteq X : f(\mu_1)(A) = 1\}| = d(f(\mu_1), u_X) = d(\mu_1, u_X) = n_1.$$

Analogously, $p + p_2 = n_2$ and, substracting, we obtain $p_2 - p_1 = n_2 - n_1$, which we know is equal to $p_1 + p_2$. Thus, $p_2 - p_1 = p_2 + p_1$ and $p_1 = 0$. Hence, $f(\mu_1) \leq f(\mu_2)$. ∎

As a corollary, we have:

Corollary 4 *If $f \in \mathcal{G}_0(A_1, ..., A_p)$, then $\mu_1 \leq \mu_2$ implies $f(\mu_1) \leq f(\mu_2)$ for all $\mu_1, \mu_2 \in \mathcal{FM}(A_1, ..., A_p)$.*

Proof: The result follows directly from the previous lemma and Lemma 3. ■

Now, we consider the supremum $\vee$ and infimum $\wedge$ operations, given for any $C \subseteq X$ by

$$(\mu_1 \vee \mu_2)(C) := \max(\mu_1(C), \mu_2(C)),\ (\mu_1 \vee \mu_2)(C) := \min(\mu_1(C), \mu_2(C)),\ \forall \mu_1, \mu_2 \in \mathcal{FM}(A_1, ..., A_p).$$

Lemma 16 *The operations $\vee$ and $\wedge$ are internal operations on $\mathcal{FM}(A_1, ..., A_p)$.*

Proof: Take $\mu_1, \mu_2 \in \mathcal{FM}(A_1, ..., A_p)$. Let B, C be two subsets with the same vector representation for $\{A_1, ..., A_p\}$. Then, $\mu_1(B) = \mu_1(C), \mu_2(B) = \mu_2(C)$, whence $(\mu_1 \vee \mu_2)(B) = (\mu_1 \vee \mu_2)(C)$. The same can be done for $\wedge$. ■

Moreover, if μ_1, μ_2 are extreme points, so they are $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$.
The following proposition holds.

Lemma 17 *Given $f \in \mathcal{G}_0(A_1, ..., A_p)$, and $\mu_1, \mu_2 \in \mathcal{FM}(A_1, ..., A_p)$, we have*

$$f(\mu_1 \vee \mu_2) = f(\mu_1) \vee f(\mu_2),\ f(\mu_1 \wedge \mu_2) = f(\mu_1) \wedge f(\mu_2).$$

Proof: It suffices to show that

- $f(\mu_1 \vee \mu_2) \geq f(\mu_1)$.
- $f(\mu_1 \vee \mu_2) \geq f(\mu_2)$.
- For any $\mu \in \mathcal{FM}(A_1, ..., A_p)$ such that $\mu \geq f(\mu_1), \mu \geq f(\mu_2)$, it is $\mu \geq f(\mu_1 \vee \mu_2)$.

The first two conditions hold by Corollary 4. For the last condition, consider f^{-1} the inverse transformation of f. Such transformation exists as $f \in \mathcal{G}(A_1, ..., A_p)$. Then, by Corollary 4:

$$f(\mu_1) \leq \mu \Rightarrow \mu_1 \leq f^{-1}(\mu),\ \ f(\mu_2) \leq \mu \Rightarrow \mu_2 \leq f^{-1}(\mu).$$

Thus, $\mu_1 \vee \mu_2 \leq f^{-1}(\mu)$, whence, again by Corollary 4, $f(\mu_1 \vee \mu_2) \leq \mu$.
The same proof can be done for the infimum. ■

Definition 16 *Given a partition of indifference $\{A_1, ..., A_p\}$, we define the p-**symmetric unanimity game** over $(b_1, ..., b_p)$ and we will denote it by $u_{(b_1,...,b_p)}$, the measure in $\mathcal{FM}(A_1, ..., A_p)$ given by*

$$u_{(b_1,...,b_p)}(i_1, ..., i_p) = \begin{cases} 1 & \text{if } \mathrm{i_j \geq b_j,\ \forall j = 1, ..., p} \\ 0 & \text{otherwise} \end{cases}$$

Remark that for any measure $\mu \in \mathcal{FM}(A_1, ..., A_p)$, there are some subsets $(b_1, ..., b_p)$ satisfying the following conditions:

$$\begin{array}{l} \mu(b_1, ..., b_p) > 0, \\ \mu(c_1, ..., c_p) = 0, \quad \text{if } c_i \leq b_i, i = 1, ..., p \text{ and } \exists i \,|\, c_i < b_i. \end{array} \tag{6}$$

This leads us to introduce the following concept:

Definition 17 *Consider $\mu \in \mathcal{FM}(A_1, ..., A_p)$. We will say that a subset $(b_1, ..., b_p)$ is a **minimal subset** for μ if it satisfies condition (6).*

Definition 18 *We will call the unanimity games of type $u_{(0,...,r_j,...,0)}$, with $r_j \in \{1, ..., |A_j|\}$ **atomic p-symmetric unanimity games**.*

Remark that atomic p-symmetric unanimity games are the only $\{0, 1\}$-valued measures $\mu \in \mathcal{FM}(A_1, ..., A_p)$ satisfying

$$\begin{array}{l} \mu = \mu_1 \wedge \mu_2 \Rightarrow \mu = \mu_1 \text{ or } \mu = \mu_2, \\ \mu = \mu_1 \vee \mu_2 \Rightarrow \mu = \mu_1 \text{ or } \mu = \mu_2. \end{array} \tag{7}$$

Lemma 18 *If μ is an extreme point of $\mathcal{FM}(A_1, ..., A_p)$, then it can be written as suprema of infima of the atomic p-symmetric unanimity games.*

Proof: Let us denote the minimal subsets of μ by $(b_1^1, ..., b_p^1), ..., (b_1^s, ..., b_p^s)$. Then,

$$\mu = \bigvee_{i=1}^{s} u_{(b_1^i,...,b_p^i)}.$$

On the other hand,

$$u_{(b_1^i,...,b_p^i)} = \bigwedge_{j=1}^{p} u_{(0,...,0,b_j^i,0,...,0)}.$$

Joining both results, we obtain the result. ■

Lemma 19 *Let f be a transformation in $\mathcal{G}_0(A_1, ..., A_p)$ Then, f is defined by the images of the atomic p-symmetric unanimity games.*

Proof: From Corollary 1 f is determined by its image on the extremes. Thus, the result follows from Lemmas 17 and 18. ■

Therefore, it suffices to obtain the image of these measures.

Lemma 20 *Consider $f \in \mathcal{G}_0(A_1, ..., A_p)$. Then f maps atomic p-symmetric unanimity games on atomic p-symmetric unanimity games.*

Proof: Let us consider $u_{(0,...,0,r_j,0,...,0)}$ and assume $f(u_{(0,...,0,r_j,0,...,0)})$ is not an atomic p-symmetric unanimity game. Then, suppose w.l.g. that there exist μ_1, μ_2 two $\{0, 1\}$-valued measures in $\mathcal{FM}(A_1, ..., A_p)$ such that

$$f(u_{(0,...,0,r_j,0,...,0)}) = \mu_1 \wedge \mu_2.$$

Then,

$$u_{(0,...,0,r_j,0,...,0)} = f^{-1}(\mu_1) \wedge f^{-1}(\mu_2).$$

On the other hand, $f^{-1}(\mu_1)$ and $f^{-1}(\mu_2)$ are $\{0, 1\}$-valued measures in $\mathcal{FM}(A_1, ..., A_p)$ by Corollary 1. And they are different because f is one-to-one. But this contradicts Equation (7). The same can be done for the supremum. This finishes the proof. ■

Lemma 21 *Consider $f \in \mathcal{G}_0(A_1, ..., A_p)$. Then, f can be written as a composition of transpositions between subsets of indifference of the same cardinality.*

Proof: By Lemma 19, it suffices to study the image of the atomic p-symmetric unanimity games. Moreover, by Lemma 20, we already know that f maps atomic p-symmetric unanimity games on atomic p-symmetric unanimity games. Therefore, $f(u_{(0,...,0,r_j,0,...,0)}) = u_{(0,...,0,r_i,0,...,0)}$.

On the other hand, f keeps distances and $u_X \in \mathcal{FM}(A_1, ..., A_p)$, so that f maintains the number of subsets with measure 0. The number of subsets with value 0 for $(0, ..., 0, r_j, 0, ..., 0)$ are

$$(|A_1| + 1) \cdots (|A_{j-1}| + 1)r_j(|A_{j+1}| + 1) \cdots (|A_p| + 1). \tag{8}$$

For $f(u_{(0,...,0,r_j,0,...,0)})$, the number of values 0 are

$$(|A_1| + 1) \cdots (|A_{i-1}| + 1)r_i(|A_{i+1}| + 1) \cdots (|A_p| + 1). \tag{9}$$

Let us suppose w.l.g. that $|A_1| \geq |A_2| \geq ... \geq |A_p|$. Then, for $u_{(|A_1|,0,...,0)}$, applying Equations (8) and (9), it is

$$|A_1|(|A_i| + 1) = (|A_1| + 1)r_i \Rightarrow r_i = \frac{(|A_i| + 1)|A_1|}{|A_1| + 1}.$$

Moreover, as $r_i \in \mathbb{Z}$, we necessarily have that $|A_1| + 1$ divides $|A_i| + 1$, whence $|A_1| = |A_i|$. Therefore, $u_{(|A_1|,0,...,0)}$ can only be mapped on $u_{(0,...,0,|A_i|,0,...,0)}$ with $|A_i| = |A_1|$.

Consider now $u_{(i,0,...,0)}$. By Lemma 20, $f(u_{(i,0,...,0)}) = u_{(0,...,0,j,0,...,0)}$. But by monotonicity, $f(u_{(i,0,...,0)}) \geq f(u_{(|A_1|,0,...,0)})$. This implies that the image of any $u_{(i,0,...,0)}$ is an atomic p-symmetric unanimity game on the same subset of indifference as $f(u_{(|A_1|,0,...,0)})$. Finally, as the number of values 0 remains constant by f, the value i remains the same, too.

This can be done for any A_i such that $|A_i| = |A_1|$. Therefore, f acts as a composition of transpositions between these subsets of indifference.

Once the image of these p-symmetric unanimity games is fixed, we take A_j such that $|A_j| < |A_1|$ and $|A_j|$ maximal in these conditions. We can apply the same proof to conclude that $u_{(0,\ldots,r_j,\ldots,0)}$ is mapped to $u_{(0,\ldots,r_k,\ldots,0)}$, with $r_k = r_j$ and $|A_k| = |A_j|$.

Following this process, we obtain the result. ■

Lemma 22 *Suppose $n > 2$ or $n = 2, p = 1$. Consider $f \in \mathcal{G}(A_1, ..., A_p)$ such that $f \notin \mathcal{G}_0(A_1, ..., A_p)$. Then, f is necessarily the composition of a transformation in $\mathcal{G}_0(A_1, ..., A_p)$ with the dual transformation.*

Proof: If $f \in \mathcal{G}(A_1, ..., A_p)$ but $f \notin \mathcal{G}_0(A_1, ..., A_p)$, this implies (Lemma 14) that $f(u_X) = u_\emptyset$. Consider $\bar{f}$ given by

$$\begin{array}{cccc} \bar{f}: & \mathcal{FM}(A_1, ..., A_p) & \rightarrow & \mathcal{FM}(A_1, ..., A_p) \\ & \mu & \hookrightarrow & \overline{f(\mu)} \end{array}.$$

Then, $\bar{f}$ is the composition of f with the dual application. On the other hand, $\bar{f}(u_X) = \bar{u}_\emptyset = u_X$, whence $\bar{f} \in \mathcal{G}_0(A_1, ..., A_p)$. As the dual application is its own inverse, we conclude that f is the composition of a transformation in $\mathcal{G}_0(A_1, ..., A_p)$ with the dual application. ■

Finally, we show that $\mathcal{G}(A_1, ..., A_p)$ is the semidirect product of symmetries between subsets of the same cardinality with the dual application:

Lemma 23 *The group $\mathcal{G}_0(A_1, ..., A_p)$ is a normal subgroup of $\mathcal{G}(A_1, ..., A_p)$.*

Proof: It is just a translation of the one for Lemma 11. ■

This finishes the proof of the Theorem. ■

Definition 19 *Let $\{A_1, ..., A_p\}$ be a partition of indifference on X. We will denote by $\mathcal{C}_i$ the subfamily of all subsets of indifference of cardinality i.*

Corollary 5 *If $|X| > 2$ or $|X| = 2, p = 1$, the cardinality of $\mathcal{G}(A_1, ..., A_p)$ is*

$$2(|\mathcal{C}_1|! \cdots |\mathcal{C}_k|!),$$

where k is the maximal cardinality of the subsets of indifference.

The case $|X| = p = 2$ is solved in Theorem 5.

3.3 The k-additive case

The k-additive case is more complicated. We consider again that $\mu \in \mathcal{FM}(A_1, ..., A_p)$ can be identified with a $(2^n - 2)$-vector whose components are $\mu(A), A \subseteq X, A \neq X, \emptyset$ for a given order. Therefore, $\mathcal{FM}(X)$ is a convex polyhedron in $\mathbb{R}^{2^n-2}$.

At this point, it must be noted that we could have considered the representation given by the Möbius inverse. In this case, the number of coefficients is more reduced and it is the natural representation of k-additive measures. However, if the Möbius transform is used, it follows that D is no more an isometry:

Example 1 *Consider $|X| = 4$ and the fuzzy measures μ_1, μ_2 whose corresponding Möbius inverse are given by*

$$m_1(x_1) = 1, m_1(x_2) = 1, m_1(x_1, x_2) = -1, m_1(A) = 0, \text{ otherwise.}$$
$$m_2(x_3) = 1, m_2(x_4) = 1, m_1(x_3, x_4) = -1, m_1(A) = 0, \text{ otherwise.}$$

Then, $d(m_1, m_2)^2 = 6$. On the other hand, the dual measures $\overline{\mu}_1, \overline{\mu}_2$ has as Möbius inverse:

$$\bar{m}_1(x_1, x_2) = 1, \bar{m}_1(A) = 0, \text{ otherwise,}$$

$$\bar{m}_2(x_3, x_4) = 1, \bar{m}_2(A) = 0, \text{ otherwise,}$$

whence $d(\bar{m}_1, \bar{m}_2)^2 = 2$.

We will come back to this point below.

The following can be shown:

Proposition 2 *Let $k \in \{1, ..., n\}$. Then, symmetries and compositions of symmetries with the dual application are transformations in $\mathcal{G}^k$.*

Proof: It is clear that symmetries are transformations in $\mathcal{G}^k(X)$. Then, it suffices to show that the dual transformation of a measure in $\mathcal{FM}^k(X)$ is in $\mathcal{FM}^k(X)$. Consider $\mu \in \mathcal{FM}^k(X)$ with Möbius transform m and let us denote by $\bar{\mu}$ its dual measure, with Möbius transform $\bar{m}$. For $A \subseteq X$, applying the Zeta transform (Eq. (2)) and Eq. (1),

$$\bar{m}(A) = \sum_{B\subseteq A} \bar{\mu}(B)(-1)^{|A\setminus B|} = \sum_{B\subseteq A} (1-\mu(B^c))(-1)^{|A\setminus B|} = \sum_{B\subseteq A} (-1)^{|A\setminus B|} - \sum_{B\subseteq A} \mu(B^c)(-1)^{|A\setminus B|}$$

$$= -\sum_{B\subseteq A} \mu(B^c)(-1)^{|A\setminus B|} = -\sum_{B\subseteq A} (-1)^{|A\setminus B|} \sum_{C\subseteq B^c} m(C).$$

Consider $C \subseteq X$ and let us see how many times $m(C)$ appears in the last expression. We have three different cases:

- If $C \cap A = \emptyset$, then $C \subseteq A^c \subseteq B^c$. Therefore, $m(C)$ appears in all $\mu(B^c)$ and thus, $m(C)$ appears multiplied by
$$-\sum_{B\subseteq A} (-1)^{|A\setminus B|} = -(1-1)^{|A|} = 0.$$
- If $C \cap A = D \neq A$, as C appears in any $\mu(B^c)$ such that $C \subseteq B^c \Leftrightarrow B \cap C = \emptyset$. Thus, $m(C)$ appears multiplied by
$$-\sum_{B\subseteq A, B\cap C=\emptyset} (-1)^{|A\setminus B|} = -(1-1)^{|A\setminus D|} = 0.$$
- If $C \cap A = A$, then C only appears when $B = \emptyset$. Therefore, $m(C)$ appears once and multiplied by $(-1)^{|A|+1}$.

Consequently, $\bar{m}(A) = \sum_{C\supseteq A} m(C)(-1)^{|A|+1}$. Thus, if $\mu \in \mathcal{FM}^k(X)$ and $|A| > k$, it follows that $\bar{m}(A) = 0$, whence $\bar{\mu} \in \mathcal{FM}^k(X)$.

Now, applying Lemma 4, the result holds. ■

For the special case of probabilities, the following can be shown:

Proposition 3 *The set $\mathcal{G}^1(X)$ is given by symmetries.*

Proof: First, remark that for probabilities, the dual transformation coincides with the identity. On the other hand, any transformation in $\mathcal{G}^1(X)$ must map vertices into vertices (Corollary 1). As the vertices of $\mathcal{FM}^1(X)$ are u_{x_i}, $x_i \in X$ (Proposition 1), the result holds. ■

For the 2-additive case, the following can be shown:

Proposition 4 *If $n > 2$, the set $\mathcal{G}^2(X)$ consists in symmetries and compositions of symmetries with the dual application.*

Proof: First, we will show that if f is in $\mathcal{G}^2(X)$ then for every i it holds $f(u_i) = u_j$ for some j. We will denote $s(\mu_1, \mu_2) = d(\mu_1, \mu_2)^2$. It is routine to check that for i, j and k, all different, it holds:

- $s(u_i, u_j) = s(u_i, u_j \vee u_k) = s(u_i, u_j \wedge u_k) = 2^{n-1}$
- $s(u_i, u_i \vee u_j) = s(u_i, u_i \wedge u_j) = 2^{n-2}$
- $s(u_i \vee u_j, u_i \vee u_k) = s(u_i \wedge u_j, u_i \wedge u_k) = 2^{n-2}$
- $s(u_i \vee u_j, u_i \wedge u_j) = s(u_i \vee u_j, u_i \wedge u_k) = 2^{n-1}$

Also, if $n > 3$ we can consider i, j, k and l, all different, and then:

- $s(u_i \vee u_j, u_k \vee u_l) = s(u_i \wedge u_j, u_k \wedge u_l) = 3 \cdot 2^{n-3}$
- $s(u_i \vee u_j, u_k \wedge u_l) = 9 \cdot 2^{n-4}$

We know that f takes vertices into vertices, and in the case of 2-additive measures the vertices are exactly $\{u_i : i = 1, \ldots, n\} \cup \{u_i \vee u_j : 1 \leq i < j \leq n\} \cup \{u_i \wedge u_j : 1 \leq i < j \leq n\}$ (Proposition 1). Thus, if $n > 3$ it is clear that for every i it holds $f(u_i) = u_j$ for some j, since f keeps distances and also s, and $s(u_i, \mu)$ with μ another extreme is always even while for the other type of vertices this value is sometimes odd.

Let us focus on the case $n = 3$. Suppose that, for some i, $f(u_i)$ is not u_j for some j. Then, we can assume, without loss of generality, that $f(u_1) = u_1 \vee u_2$ (if not, we compose f with symmetries and/or the dual transformation). We know that $s(u_1, u_2) = s(u_1, u_3) = s(u_1, u_2 \vee u_3) = s(u_1, u_2 \wedge u_3) = 4$, and for any other extreme μ it holds $s(u_1, \mu) = 2$. Consequently

$$\{f(u_2), f(u_3), f(u_2 \vee u_3), f(u_2 \wedge u_3)\} = \{u_3, u_1 \wedge u_2, u_1 \wedge u_3, u_2 \wedge u_3\},$$

since the left set includes the image of all extremes which are at squared distance 4 from u_1, and the right set includes all those wich are at squared distance 4 from $f(u_1)$.

We have to consider several different cases:

- $f(u_2) = u_3$. Then, it must hold $s(u_3, f(u_3)) = s(f(u_2), f(u_3)) = s(u_2, u_3) = 4$. But $s(u_3, u_1 \wedge u_3) = s(u_3, u_2 \wedge u_3) = 2$, and then $f(u_3) = u_1 \wedge u_2$. Thus,

$$\{f(u_2 \vee u_3), f(u_2 \wedge u_3)\} = \{u_1 \wedge u_3, u_2 \wedge u_3\},$$

  but $s(u_2 \vee u_3, u_2 \wedge u_3) = 4$ and $s(u_1 \wedge u_3, u_2 \wedge u_3) = 2$, which is a contradiction.

- $f(u_2) = u_1 \wedge u_2$. Since $s(u_1 \wedge u_2, u_1 \wedge u_3) = s(u_1 \wedge u_2, u_2 \wedge u_3) = 2$, it must hold $f(u_3) = u_3$. Thus,

$$\{f(u_2), f(u_3)\} = \{u_3, u_1 \wedge u_2\},$$

  and, hence, again

$$\{f(u_2 \vee u_3), f(u_2 \wedge u_3)\} = \{u_1 \wedge u_3, u_2 \wedge u_3\},$$

  which we already know is a contradiction.

- $f(u_2) = u_1 \wedge u_3$. In this case $s(u_1 \wedge u_3, u_3) = s(u_1 \wedge u_3, u_1 \wedge u_2) = s(u_1 \wedge u_3, u_2 \wedge u_3) = 2$, which contradicts the fact that $s(f(u_2), f(u_3)) = s(u_2, u_3) = 4$.

- $f(u_2) = u_2 \wedge u_3$. Same as the previous case.

Then, we know that for each i, $f(u_i) = u_j$ for some j. We can suppose, without loss of generality, that $f(u_i) = u_i$ for every i (if not, compose with the adequate symmetry). We must show that f is either the identity of the dual application. Consider $f(u_i \vee u_j)$ with $i \neq j$. Since f takes extremes into extremes, we have two possibilities:

- $f(u_i \vee u_j) = u_k \vee u_l$. Since $f(u_i) = u_i$ and $f(u_j) = u_j$, we know that $s(u_k \vee u_l, u_i) = s(u_i \vee u_j, u_i) = 2^{n-2}$, and $s(u_k \vee u_l, u_j) = s(u_i \vee u_j, u_j) = 2^{n-2}$, and then $\{i, j\} = \{k, l\}$.

- $f(u_i \vee u_j) = u_k \wedge u_l$. After composition with the dual transformation we are in the previous case and we can deduce $\{i, j\} = \{k, l\}$.

This finishes the proof. ■

The general k-additive case is much more complicated. The reason is that the set of vertices of $\mathcal{FM}^k(X)$ is no longer the set of $\{0, 1\}$-valued measures in $\mathcal{FM}^k(X)$ (Theorem 2). However, as we will see below, Proposition 2 suffices to obtain surprising results.

4 The set of invariant measures

Let us now consider another point of view. Assume $\mathcal{F}$ is a convex family of fuzzy measures and let μ_g be the center of gravity of $\mathcal{F}$. Let f be an isometric transformation on $\mathcal{F}$. Then, μ_g should remain invariant by f. On the other hand, if a procedure is random, it has not any trend and thus, the center of gravity of the generated measures should be near μ_g. Therefore, it makes sense to study the set of invariant measures for any $f \in \mathcal{G}(\mathcal{F})$. If there is only one invariant measure, this is the center of gravity; if this is not the case (the arithmetic mean of the vertices is invariant and it does not necessarily coincide with μ_g), at least we will obtain a subset of measures where μ_g is.

Let us denote by $\lfloor x \rfloor$ the integer part of x.

4.1 The general case

For the general case, the following can be shown:

Proposition 5 *The set of measures in $\mathcal{FM}(X)$ that are invariant for any transformation in $\mathcal{G}(X)$ is a convex polyhedron whose vertices are the symmetric measures $\mu_1, ..., \mu_{\lfloor \frac{n}{2} \rfloor+1}$ given by*

$$\mu_i(B_j) := \begin{cases} 0 & \text{if j < i} \\ \frac{1}{2} & \text{if i} \leq \text{j} \leq \lfloor \frac{\text{n}}{2} \rfloor \\ 1-\mu(B_{n-j}) & \text{if j} > \lfloor \frac{\text{n}}{2} \rfloor \end{cases} \quad (10)$$

where B_j represents any subset of X such that $|B_j| = j$.

Proof: From Theorem 4, we already know that $\mathcal{G}(X)$ consists in symmetries and compositions between symmetries and the dual transformation. Therefore, it suffices to study which are the measures in $\mathcal{FM}(X)$ that are both self-dual and symmetric.

If $\mu \in \mathcal{FM}(X)$ is symmetric, then $\mu(A)$ only depends on $|A|, \forall A \subseteq X$. Let us denote with some abuse of notation $\mu(i) = \mu(A)$, $|A| = i$, $i = 1, ..., n-1$.

On the other hand, as μ is also self-dual, it satisfies $\mu(A) = 1 - \mu(A^c)$, $\forall A \subseteq X$.

Joining both conditions, we obtain the following set of equations:

$$\mu(i) = 1 - \mu(n-i),\ i = 1, ..., n-1.$$

Thus, it suffices to determine the values of $\mu(i)$ for $i = 1, ..., \lfloor \frac{n}{2} \rfloor$.

Let us check that any measure in these conditions can be put as a convex combination of the measures of Equation (10). First, remark that $\mu_i, i = 1, ..., \lfloor \frac{n}{2} \rfloor + 1$ are symmetric and self-dual. Let μ be any symmetric self-dual measure and consider

$$\mu' := 2\mu(1)\mu_1 + 2(\mu(2) - \mu(1))\mu_2 + ... + 2(\frac{1}{2} - \mu(\lfloor \frac{n}{2} \rfloor))\mu_{\lfloor \frac{n}{2} \rfloor+1}.$$

The measure μ' thus defined is symmetric and self-dual. Then, μ' is a convex combination of $\mu_1, ..., \mu_{\lfloor \frac{n}{2} \rfloor+1}$, and it suffices to prove that $\mu' = \mu$. Consider $i \in \{1, ..., \lfloor \frac{n}{2} \rfloor\}$.

$$\mu'(i) = \frac{1}{2}[2\mu(1) + 2(\mu(2) - \mu(1)) + ... + 2(\mu(i) - \mu(i-1))] = \mu(i).$$

Finally, it is clear that $\mu_i, i = 1, ..., \lfloor \frac{n}{2} \rfloor + 1$ are extreme points. ■

4.2 Invariant measures by transformations in $\mathcal{G}(A_1, ..., A_p)$

Let us turn to $\mathcal{FM}(A_1, ..., A_p)$. We need some previous definitions.

Definition 20 *Let $\{A_1, ..., A_p\}$ be a partition of indifference on X. Consider $B \equiv (b_1, ..., b_i, ..., b_j, ..., b_p)$. If $|A_i| = |A_j|$, we define the set $B^{i,j}$ by*

$$B^{i,j} \equiv (b_1, ..., b_j, ..., b_i, ..., b_p).$$

Definition 21 *Let $\{A_1, ..., A_p\}$ be a partition of indifference on X. Assume $|A_{i_1}| = ... = |A_{i_r}|$. We say that a family of subsets $\mathcal{D}$ is* **symmetric with respect to** *$\{A_{i_1}, ..., A_{i_r}\}$ if*

$$\forall B \in \mathcal{D}, B^{i_j, i_l} \in \mathcal{D},\ \forall j, l \in \{1, ..., r\}.$$

Definition 22 *Let $\{A_1, ..., A_p\}$ be a partition of indifference on X. Assume $|A_{i_1}| = ... = |A_{i_r}|$. We say that a measure $\mu \in \mathcal{FM}(A_1, ..., A_p)$ is* **symmetric with respect to** *$\{A_{i_1}, ..., A_{i_r}\}$ if*

$$\mu(B) = \mu(B^{i_j, i_l}),\ \forall j, l \in \{1, ..., r\}.$$

Proposition 6 *The set of measures in $\mathcal{FM}(A_1, ..., A_p)$ that are invariant for any transformation in $\mathcal{G}(A_1, ..., A_p)$ is a convex polyhedron whose vertices are the measures in $\mathcal{FM}(A_1, ..., A_p)$ satisfying:*

1. *They are self-dual measures.*
2. *They are symmetric with respect to $\mathcal{C}_i, i = 1, ..., k$, where k is the maximal cardinality of the subsets of indifference (Definition 19).*
3. *If $B \subseteq X, |B| \leq \lfloor \frac{n}{2} \rfloor$ and $\mu(B) \neq 0$, then $\mu(B) = \frac{1}{2}$.*

Remark that the first condition implies that any minimal subset of μ has cardinality at most $\lfloor \frac{n}{2} \rfloor + 1$.

Conditions 1 and 2 characterize the set of all measures in $\mathcal{FM}(A_1, ..., A_p)$ invariant by $\mathcal{G}(A_1, ..., A_p)$. The last condition characterizes the vertices of the convex polyhedron.

Proof: Let us denote the set of all measures in $\mathcal{FM}(A_1, ..., A_p)$ being symmetric with respect to $\mathcal{C}_i, i = 1, ..., k$, by $\mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$. It is clear that this set is a convex polyhedron.

Lemma 24 *The vertices of* $\mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$ *are the* $\{0, 1\}$*-valued measures in* $\mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$.

Proof: Consider $\mu \in \mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$. Let us define the families of subsets $\mathcal{D}_1, ..., \mathcal{D}_r$ satisfying the following conditions:

- If $B \in \mathcal{D}_i, C \in \mathcal{D}_j, i < j \Rightarrow \mu(B) < \mu(C)$.
- If $B, B' \in \mathcal{D}_i \Rightarrow \mu(B) = \mu(B')$.
- $\bigcup_{i=1}^r \mathcal{D}_i = \{B \in \mathcal{P}(X), \mu(B) > 0\}$, $\mathcal{D}_i \neq \emptyset$.

Indeed, $\mathcal{D}_i, i = 1, ..., r$ form a partition of subsets in X with positive measure grouping the subsets with the same measure. Let us define:

$$\mu_i(B) := \begin{cases} 1 & \text{if } B \in \mathcal{D}_j,\ j \geq i \\ 0 & \text{otherwise} \end{cases} \quad i = 0, ..., r-1.$$

As $\mu \in \mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$, it follows that $\mu_i \in \mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$, $i = 1, ..., r$.

Consider $B_i \in \mathcal{D}_i, i = 0, ..., r$, and define

$$\mu' := \mu(B_1)\mu_1 + (\mu(B_2) - \mu(B_1))\mu_2 + ... + (\mu(B_r) - \mu(B_{r-1}))\mu_r.$$

Let us check that $\mu' = \mu$. For any $B \subseteq X$, we have two different cases:

- If $\mu(B) = 0$, then $\mu'(B) = 0$.
- Otherwise, $B \in \mathcal{D}_i$ for some $i = 1, ..., r$. Therefore,

$$\mu'(B) = 2[\mu(B_1) + (\mu(B_2) - \mu(B_1)) + ... + (\mu(B_i) - \mu(B_{i-1}))] = \mu(B_i) = \mu(B).$$

On the other hand, any $\{0, 1\}$-valued measure in $\mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$ is an extreme point because it is an extreme point of $\mathcal{FM}(A_1, ..., A_p)$. This finishes the proof. ■

¿From Theorem 6, we know that $\mathcal{G}(A_1, ..., A_p)$ consists in the set of bijections between subsets of indifference in $\mathcal{C}_i$ and these transformations composed with the dual application. This implies that for any $\mu \in \mathcal{FM}(A_1, ..., A_p)$ remaining invariant for any transformation in $\mathcal{G}(A_1, ..., A_p)$, $\mu \in \mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$ and it is self-dual. As μ is self-dual, it suffices to define it for subsets B such that $|B| \leq \lfloor \frac{n}{2} \rfloor$. On the other hand, by monotonicity and the dual condition, $\mu(B) \leq \frac{1}{2}$, $\forall B \subseteq X$, $|B| \leq \lfloor \frac{n}{2} \rfloor$. Moreover, these two conditions characterize the set of invariant measures.

Let us define

$$\mu'(B) := \begin{cases} 2\mu(B) & \text{if } |B| \leq \lfloor \frac{n}{2} \rfloor \\ 1 & \text{otherwise} \end{cases}$$

As $\mu \in \mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$, it follows that $\mu' \in \mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$. Then, by Lemma 24, there exist measures $\mu'_1, ..., \mu'_r \in \mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$ being $\{0, 1\}$-valued such that

$$\mu' = \sum_{i=1}^r \alpha_i \mu'_i, \ \sum_{i=1}^r \alpha_i = 1, \alpha_i \geq 0, \ \forall i = 1, ..., r.$$

Let us define

$$\mu_i(B) := \begin{cases} \frac{\mu'_i(B)}{2} & \text{if } |B| \leq \lfloor \frac{n}{2} \rfloor \\ 1 - \frac{\mu'_i(B^c)}{2} & \text{otherwise} \end{cases}$$

Note that any $\mu_i, i = 1, ..., r$, are in the conditions stated in the proposition. As $\mu'_i \in \mathcal{FM}(A_1, ..., A_p)_{\mathcal{C}_1,...,\mathcal{C}_k}$, so are $\mu_i, i = 1, ..., r$. Then, $\forall B \subseteq X, |B| \leq \lfloor \frac{n}{2} \rfloor$,

$$\mu(B) = \sum_{i=1}^r \alpha_i \mu_i(B), \ \sum_{i=1}^r \alpha_i = 1, \alpha_i \geq 0, \ \forall i = 1, ..., r.$$

By self-duality, we conclude that $\mu = \sum_{i=1}^r \alpha_i \mu_i$. On the other hand, it is straightforward to prove that μ_i is an extreme point of the set of invariant measures. This finishes the proof. ■

4.3 The k-additive case

Finally, let us study the k-additive case. We already know that the set of transformations in $\mathcal{G}^k(X)$ is at least the set of symmetries and compositions of symmetries with the dual measure (Proposition 2). This implies that the set of invariant measures in $\mathcal{FM}^k(X)$ is included in the set of all symmetric self-dual measures in $\mathcal{FM}^k(X)$. The structure of these measures depends on the values of k and n. Theorem 5 shows the form of the n-additive case. However, an important situation arises when k is small and n is large; this is the usual case in practice, as the decision maker finds difficult dealing with interactions of more than two or three elements. For this case, the following can be shown:

Theorem 7 *Suppose $|X| = n$ and consider $k \leq \lfloor \frac{n}{2} \rfloor$. The only measure in $\mathcal{FM}^k(X)$ that is invariant for any transformation in $\mathcal{G}^k(X)$ is the arithmetic mean of the vertices, given by*

$$\mu(B_i) = \frac{i}{n}, \, \forall B_i \subseteq X, \, |B_i| = i.$$

Proof: Let $\mu \in \mathcal{FM}^k(X)$ be an invariant measure for any transformation in $\mathcal{G}^k(X)$. By symmetry, $\mu(A)$ only depends on $|A|, \forall A \subseteq X$. Let us denote $\mu(i) := \mu(A)$, $|A| = i$, $i = 1, ..., n-1$. As μ is also self-dual, it satisfies $\mu(A) = 1 - \mu(A^c)$, $\forall A \subseteq X$, whence

$$\mu(i) = 1 - \mu(n-i), \, i = 1, ..., n-1.$$

If we write these conditions for $i = 1, ..., k-1$ in terms of the Möbius transform of μ, that we will denote by m, and we add the condition $\sum_{A \subseteq X} m(A) = 1$, we obtain:

$$\begin{array}{ccc} \sum_{i=1}^k \binom{n}{i} m_i & = & 1 \\ m_1 & = & 1 - [\binom{n-1}{1} m_1 + ... + \binom{n-1}{k} m_k] \\ ... & ... & ... \\ \binom{k-1}{1} m_1 + ... + \binom{k-1}{k-1} m_{k-1} & = & 1 - [\binom{n-k+1}{1} m_1 + ... + \binom{n-k+1}{k} m_k] \end{array}$$

This system is equivalent to:

$$\begin{array}{ccc} \binom{n}{1} m_1 + ... + \binom{n}{k} m_k & = & 1 \\ \binom{1}{1} m_1 + \binom{n-1}{1} m_1 + ... + \binom{n-1}{k} m_k & = & 1 \\ ... & ... & ... \\ \binom{k-1}{1} m_1 + ... + \binom{k-1}{k-1} m_{k-1} + \binom{n-k+1}{1} m_1 + ... + \binom{n-k+1}{k} m_k & = & 1 \end{array}$$

The matrix of the system is given by

$$\mathbf{M}_1 = \begin{pmatrix} n & \binom{n}{2} & ... & \binom{n}{k-1} & \binom{n}{k} \\ n & \binom{n-1}{2} & ... & \binom{n-1}{k-1} & \binom{n-1}{k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{k-1}{1} + \binom{n-k+1}{1} & +\binom{k-1}{2} + \binom{n-k+1}{2} & ... & \binom{k-1}{k-1} + \binom{n-k+1}{k-1} & \binom{n-k+1}{k} \end{pmatrix}$$

Let us now see that the determinant of the system is different of 0. Substracting the r-th row to the $(r+1)$-th row, for r=k-1, ..., 1, we obtain that the coefficient $(r+1, i)$ is given by

$$-\binom{n-r}{i} + \binom{n-r-1}{i} - \binom{r}{i} + \binom{r+1}{i} = -\binom{n-r-1}{i-1} + \binom{r}{i-1}, r, i = 1, ..., k.$$

The system turns into

$$\mathbf{M}_2 = \begin{pmatrix} n & \binom{n}{2} & ... & \binom{n}{k-1} & \binom{n}{k} \\ 0 & -\binom{n-1}{1} & ... & -\binom{n-1}{k-2} & -\binom{n-1}{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & +\binom{k-2}{1} - \binom{n-k}{1} & ... & +\binom{k-2}{k-2} - \binom{n-k}{k-2} & -\binom{n-k}{k-1} \end{pmatrix}$$

But now, we can repeat the process for $r = k-1, ..., 2$ on $\mathbf{M}_2$ to obtain a new matrix $\mathbf{M}_3$, and so on until we arrive to $\mathbf{M}_k$. This matrix is an upper triangular matrix whose elements in the main diagonal are non-null. Therefore, our system has only one solution.

On the other hand, it can be easily checked that the arithmetic mean is a solution of the system.

Finally, as any transformation in $\mathcal{G}^k(X)$ maps vertices into vertices, the arithmetic mean of the vertices must remain invariant.

Joining both results, we obtain that the arithmetic mean of the vertices is given by

$$\mu(B_i) = \frac{i}{n}, \forall B_i \subseteq X, |B_i| = i.$$

This finishes the proof. ■

It is interesting to remark the differences between the general case (Proposition 5) and the k-additive case for small values of k (Theorem 7).

Theorem 7 is specially interesting, as it provides the center of gravity, while in other cases, we only have a region containing this measure.

5 Conclusions and open problems

In this paper we have studied some of the properties that any random generator of fuzzy measures should satisfy. We think this could be useful in the comparison of different procedures.

The development of these properties leads to some problems about the structure of fuzzy measures.

We have studied the group of isometric transformations for $\mathcal{FM}(X), \mathcal{FM}(A_1, ..., A_p)$ and $\mathcal{FM}^k(X)$. This group is completely determined for $\mathcal{FM}(X)$ and $\mathcal{FM}(A_1, ..., A_p)$, being an open problem for $\mathcal{FM}^k(X)$. The reason lies in the fact that the set of vertices of $\mathcal{FM}^k(X)$ has not been determined yet.

Next, we have studied the set of invariant measures for any isometric transformation in each family. Although the set of isometric transformations in $\mathcal{FM}^k(X)$ has not been completely defined, the results obtained suffice to completely determine the set of invariant measures when $k \leq \frac{n}{2}$.

As explained when dealing with the k-additive case, $\mathcal{G}(\mathcal{F})$ can vary if we change the representation. Indeed, if we consider the Möbius transform, the dual transformation is not an isometric transformation. This implies that Theorem 7 (and consequently, the center of gravity) is no longer valid if we use the Möbius transform. On the other hand, the Möbius transform is the natural representation of k-additive measures; then, it could be interesting to study $\mathcal{G}^k(X)$ with the representation in terms of Möbius transform.

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