On the polytope of non-additive measures

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Abstract

In this paper we deal with the problem of studying the structure of the polytope of non-additive measures for finite referential sets. We give a necessary and sufficient condition for two extreme points of this polytope to be adjacent. We also show that it is possible to find out in polynomial time whether two extremes are adjacent. These results can be extended to the polytope given by the convex hull of monotone boolean functions. We also give some results about the facets and edges of the polytope of non-additive measures; we prove that the diameter of the polytope is 3 for referentials of 3 elements or more. Finally, we show that the polytope is combinatorial and study the corresponding properties; more concretely, we show that the graph of non-additive measures is Hamilton connected if the cardinality of the referential set is not 2.

Keywords: Non-additive measures, monotone boolean functions, adjacency, complexity, diameter, combinatorial polytopes, stack filters.

1 Introduction and basic concepts

Consider a finite referential set $X = \{x_1, ..., x_n\}$ of n elements. The set X is the set of criteria in Decision Making, players in Game Theory, individuals in Welfare Theory, ... Subsets of X are denoted by capital letters $A, B, ...$ and also by $A_1, A_2, ...$ In order to avoid hard notation, for singletons $\{x_i\}$ we will usually omit braces. The set of subsets of X is denoted by $\mathcal{P}(X)$.

A **non-additive measure** [9] or **fuzzy measure** [31] or **capacity** [5] over X is a function $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ satisfying $\mu(\emptyset) = 0$, $\mu(X) = 1$ and $\mu(A) \leq \mu(B)$, $\forall A, B \in \mathcal{P}(X), A \subseteq B$. Note that we need $2^n - 2$ coefficients in order to define a non-additive measure.

From a mathematical point of view, non-additive measures constitute a generalization of probability distributions in which we remove additivity and monotonicity is imposed instead. This extension is perfectly justified in many practical situations, in which additivity is too restrictive. For example, in the field of Decision Making, models based on Probability, as those from von Neumann and Morgenstern [32] or Anscombe and Aumann [3] to cite a few, can lead to inconsistencies due to *risk aversion*, as the well-known paradoxes of Ellsberg [10] or Allais [2]. However, models based on non-additive measures [4, 27] are able to handle and interpret these problems.

Non-additive measures have been successfully applied to model problems in Multicriteria Decision Making and Cooperative Games. In the former case, they allow the decision maker to introduce vetoes and favors in the model [15], as well as interactions among the different criteria [16]. In the theory of Cooperative Games, non-additive measures represent the characteristic function of monotone games, i.e. the pay-off that each coalition can guarantee for itself; they are related to the Shapley value [28]. Other fields related to non-additive measures are combinatorics [26], pseudo-boolean functions [17], etc. This versatility of non-additive measures has led to a huge number of related works, both from a theoretical and from a practical point of view.

We will denote the set of all non-additive measures over X by $\mathcal{FM}(X)$. Notice that $\mathcal{FM}(X)$ is a polytope in $\mathbb{R}^{2^n-2}$ (or $\mathbb{R}^{2^n}$ if we include the coordinates for $\mu(\emptyset)$ and $\mu(X)$).

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An example of non-additive measure is the so-called **unanimity game** over $A \subseteq X$, $A \neq \emptyset$ defined by

$$u_A(B) := \begin{cases} 1 & \text{if A} \subseteq \text{B} \\ 0 & \text{otherwise} \end{cases}$$

Unanimity games are the vertices of a special convex class of non-additive measures called *belief functions*, that appear in the Theory of Evidence [8]. For $\emptyset$, we define the unanimity game $u_\emptyset$ by

$$u_\emptyset(B) := \begin{cases} 1 & \text{if B} \neq \emptyset \\ 0 & \text{if B} = \emptyset \end{cases}$$

Note that $u_\emptyset$ follows a different structure to any other unanimity game (indeed it is not a belief function). We have included this notation because it is the usual notation for this non-additive measure.

On $\mathcal{FM}(X)$ we can define a partial order given by $\mu_1 \leq \mu_2$ if and only if $\mu_1(A) \leq \mu_2(A)$, $\forall A \subseteq X$. If $\mu_1 \leq \mu_2$ or $\mu_2 \leq \mu_1$ we say that μ_1 and μ_2 are **comparable**.

On $\mathcal{FM}(X)$ we can also define the following operations:

- The **supremum** $(\mu_1 \vee \mu_2)(A) := \max(\mu_1(A), \mu_2(A))$, $\forall A \subseteq X$.
- The **infimum** $(\mu_1 \wedge \mu_2)(A) := \min(\mu_1(A), \mu_2(A))$, $\forall A \subseteq X$.
- The **dual** $\overline{\mu}(A) := 1 - \mu(A^c)$, $\forall A \subseteq X$.

A problem arising in practice is the identification of the non-additive measure modelling a certain situation. In [6], we have dealt with the problem of identifying a non-additive measure from sample information through genetic algorithms [14]. The cross-over operator used in the algorithm was the convex combination, possible as $\mathcal{FM}(X)$ is a polytope; this operator allows a reduction in the complexity of the algorithm. However, the use of this operator has the drawback that the search region is reduced in each iteration. Then, in order to ensure that the searched measure is inside the region, we need to consider the extreme points of $\mathcal{FM}(X)$ as the initial population. It has been pointed out in [24] that these extreme points are the set of $\{0, 1\}$-valued measures, i.e. the set of monotone boolean functions of n variables except the constant functions 0 and 1. This extremes are also stack filters [33] and the elements of the free distributive lattice of n generators [29].

Remark that for a $\{0, 1\}$-valued measure μ, there are some subsets A satisfying the following conditions:

$$\begin{array}{ll} \mu(A) = 1, & \\ \mu(B) = 1, & \forall B \supseteq A, \\ \mu(C) = 0, & \forall C \subset A. \end{array}$$

We will call any subset satisfying these conditions a **minimal subset** for μ. Note that a $\{0, 1\}$-valued measure is completely defined by its minimal subsets. To see this, it suffices to remark that $\mu(A) = 1$ if A contains a minimal subset and $\mu(A) = 0$ otherwise. We will denote the families of minimal subsets by $\mathbf{C}, \mathbf{D}$, and so on. The non-additive measure whose minimal subsets are the family $\mathbf{C}$ will denoted by $\mu_{\mathbf{C}}$.

Minimal subsets are also known as *minimal true subsets* or *minimal primes*. Alternatively, one could consider the *maximal false subsets* or *maximal zeroes*, (the subsets whose value is 0 and such that they are maximal in these conditions). If we consider the lattice $(\mathcal{P}(X), \cup, \cap)$, then a minimal subset for a $\{0, 1\}$-valued measure μ can be equivalently defined as a subset of X such that $\mu(A) = 1$ and whose principal filter $\mathcal{F}_A$ and principal ideal $\mathcal{I}_A$ (see [30]) satisfy

$$\mu(B) = 1, \forall B \in \mathcal{F}_A, \ \mu(B) = 0, \forall B \in \mathcal{I}_A \backslash \{A\}.$$

The set of minimal subsets of a $\{0, 1\}$-valued measure determine an **antichain** (collections of sets which are pairwise uncomparable with respect to inclusion, see [1]). Then, the number of vertices of the polytope $\mathcal{FM}(X)$ is the number of different antichains on X. The number of antichains of a set of cardinality n is known as the n-th Dedekind number [7]. The first Dedekind numbers are given in Table 1.

In the values given in this table we have excluded the empty antichain and the antichain which contains only the empty set, as these cases do not lead to a non-additive measure. The form of the general term of this sequence is not known and, in fact, the only Dedekind numbers which have been calculated up today are the first 8. However, their values are bounded:

n	Dedekind numbers
1	1
2	4
3	18
4	166
5	7579
6	7828352
7	2414682040996
8	56130437228687557907786

Table 1: Number of vertices of $\mathcal{FM}(X)$

Theorem 1. *[11] If D_n is the n-th Dedekind number and $E_n := D_n + 2$, then it holds*

$$2^q \leq D_n, E_n \leq 3^q,$$

with $q = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and $\lfloor x \rfloor$ the integer part of x, for all $n \geq 1$.

From the quantities in Table 1, it can be seen that we cannot use the vertices of $\mathcal{FM}(X)$ as initial population when n is big (and $n = 6$ is big!). Another solution consists in generating some non-additive measures through a random algorithm. To determine if a given procedure is indeed random, we have proposed in [20] to compare the number of non-additive measures obtained by the algorithm in two regions with the same hyper-volume. And in order to find such kind of regions we have studied which are the *isometries* (functions maintaining distances) on $\mathcal{FM}(X)$.

Consider $\sigma : X \rightarrow X$ a permutation on X. We define the **symmetry induced by** σ, denoted S_σ, the transformation on $\mathcal{FM}(X)$ such that for any $\mu \in \mathcal{FM}(X)$ the fuzzy measure $S_\sigma(\mu)$ is defined by

$$S_\sigma(\mu)(x_1, ..., x_r) = \mu(x_{\sigma(x_1)}, ..., x_{\sigma(x_r)}), \ \forall \{x_1, ..., x_r\} \subseteq X. \tag{1}$$

We define the **dual transformation**, denoted D, the transformation on $\mathcal{FM}(X)$ given by

$$\begin{array}{rccc} D: & \mathcal{FM}(X) & \rightarrow & \mathcal{FM}(X) \\ & \mu & \mapsto & \overline{\mu} \end{array}$$

Then, the group of isometries on $\mathcal{FM}(X)$ is given by the following theorem

Theorem 2. *[20] If $|X| > 2$, the set of isometries on $\mathcal{FM}(X)$ is given by symmetries and compositions of symmetries with the dual application. In fact, the set of isometries is the semidirect product of the group of symmetries with the cyclic group of order 2 generated by the dual transformation.*

For $|X| = 2$, the group of isometries is isomorphic to the dihedral group D_4 (the group of isometries of the square).

In this paper, we aim to study more properties about the polytope $\mathcal{FM}(X)$. These properties could be interesting in the problem of identification of non-additive measures. Moreover, they might shed light on the structure of $\mathcal{FM}(X)$, and could be useful to look for non-additive measures with additional properties. Besides, many of the results obtained for $\mathcal{FM}(X)$ can be extended to the convex hull of monotone boolean functions.

In next section, we study when two extreme points of $\mathcal{FM}(X)$ are adjacent. We also show that we can find out whether two vertices are adjacent in polynomial time. In Section 4, we study other aspects of $\mathcal{FM}(X)$. First, we study the edges of the polytope; we show that the probability of two measures are adjacent decreases when the cardinality of X grows. We also study the facets of the polytope. We show that the diameter of $\mathcal{FM}(X)$ is 3 for $|X| > 2$. Finally, we show that the graph of $\mathcal{FM}(X)$ is Hamilton connected when $|X| \neq 2$. Corresponding results for monotone boolean functions are also stated. We finish with the conclusions and open problems.

2 Adjacency on $\mathcal{FM}(X)$

Let us address the problem of whether two $\{0,1\}$-valued measures are adjacent. Remark that two vertices of a polytope are adjacent if the middle point of the segment joining them cannot be written as a convex combination of other vertices of the polytope. In this case, the segment is, in fact, an edge of the polytope. Consider a polytope of the form $\{A\vec{x} \leq \vec{b}\}$ where $\vec{x} \in \mathbb{R}^p$ and suppose there are not dummy constraints. Given two vertices $\vec{x}, \vec{y}$, let us denote by B the submatrix of A consisting in the constraints that any point $\vec{z}$ in the segment joining $\vec{x}, \vec{y}$ excluded $\vec{x}$ and $\vec{y}$ satisfies with equality; then, it is well-known that $\vec{x}, \vec{y}$ are adjacent if the range of B is $p-1$.

In the case of $\mathcal{FM}(X)$, these constraints are

$$\mu(\emptyset) = 0, \mu(X) = 1.$$

$$\mu(A) - \mu(A \backslash x_i) \geq 0 \quad \forall A \subseteq X, x_i \in A \tag{2}$$

Then, we have 2 equalities and $n2^{n-1}$ inequalities. Thus, given two vertices μ_1, μ_2, they are adjacent if the range of the constraints in Eq. (2) that $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ satisfies with equality is $2^n - 3$ (or $2^n - 1$ if we include the boundary conditions). Note that the constraints that $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ satisfies with equality are exactly the constraints that both μ_1 and μ_2 satisfy with equality. This provides us with an algorithm to determine whether two vertices are adjacent. However, it is very slow. Indeed, it can be proved that the complexity can grow in a non-polynomial way.

As proved in [19, 23], the problem of determining non-adjacency of vertices of a polytope is, in some cases, NP-complete (see [13] for a definition of NP-complete problems and related notions). However, in this section we give a necessary and sufficient condition of adjacency on $\mathcal{FM}(X)$ which can be used to prove that the problem of determining adjacency can be solved, in this case, in polynomial time in the number of minimal subsets of the vertices (see Theorem 6 below).

Notice that for general boolean functions (not necessarily monotone), the problem of adjacency is easy to solve: It can be proved that two boolean functions are adjacent if and only if their Hamming distance (the number of inputs on which they differ) is 1.

Given a polytope $\mathcal{F}$, the **graph of** $\mathcal{F}$ is defined by the graph whose nodes are the vertices of $\mathcal{F}$ and whose edges join two nodes if and only if they are adjacent.

It can be seen that the Hamming distance between two boolean functions is equal to the length of the shortest path between them in the graph of adjacency. In particular, if the distance of two monotone boolean functions is 1, then they are adjacent. However, the converse is not true; for instance, $u_{x_1} \wedge u_{x_2}$ and u_{x_1} are adjacent as monotone boolean functions although their distance is greater than 1 when $n > 2$ (they differ, at least, in $\{x_1\}$ and in $\{x_1, x_3\}$). In fact, we will show below that $u_{x_1} \vee u_{x_2} \vee \ldots \vee u_{x_n}$ and $u_{x_1} \wedge u_{x_2} \wedge \ldots \wedge u_{x_n}$ are adjacent when $n > 2$ (see Theorem 3), although, as proved in [20], their distance is the maximum possible distance between two extremes of non-additive measures.

The following results are obvious.

Lemma 1. *For $\mu_1, \mu_2 \in \mathcal{FM}(X)$, $\mu_1 \wedge \mu_2$ and $\mu_1 \vee \mu_2$ satisfy*

$$\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 = \frac{1}{2}(\mu_1 \wedge \mu_2) + \frac{1}{2}(\mu_1 \vee \mu_2)$$

and $\mu_1 \wedge \mu_2 \leq \mu_1, \mu_2 \leq \mu_1 \vee \mu_2$.

Corollary 1. *If μ_1 and μ_2 are adjacent extremes of $\mathcal{FM}(X)$, then $\mu_1 \leq \mu_2$ or $\mu_2 \leq \mu_1$.*

Proof: It is enough to notice that if $\mu_1 \not\leq \mu_2$ and $\mu_2 \not\leq \mu_1$, then $\mu_1 \wedge \mu_2$ and $\mu_1 \vee \mu_2$ are both different from μ_1 and μ_2. Since $\mu_1 \vee \mu_2, \mu_1 \wedge \mu_2 \in \mathcal{FM}(X)$ and they are also extremes (they are $\{0,1\}$-valued) we would have, from the previous lemma, that μ_1 and μ_2 are not adjacent, which is a contradiction. ■

However, this is not a sufficient condition. For instance, assume $|X| \geq 2$ and consider $\mu = u_{\{x_i, x_j\}}$ and μ' defined by

$$\mu'(A) := \begin{cases} 1 & \text{if } x_i \in A \text{ or } x_j \in A \\ 0 & \text{otherwise} \end{cases}$$

Then, $\mu < \mu'$ but they are not adjacent as

$$\frac{1}{2}u_{\{x_i,x_j\}} + \frac{1}{2}\mu' = \frac{1}{2}u_{x_i} + \frac{1}{2}u_{x_j}.$$

In fact, if two vertices μ_1 and μ_2 are not comparable, it follows from Lemma 1 that their supremum and infimum are not adjacent, though they are always comparable.

In next definition we introduce the concept of **C**-decomposability, that will be needed in the following results.

Definition 1. *Let* $\mathbf{C}$ *be a collection of subsets of* X *(not necessarily an antichain),* μ *an extreme of* $\mathcal{FM}(X)$ *and let* $\mathbf{D}$ *be the family of all its minimal subsets. We say that* μ *is* $\mathbf{C}-$**decomposable** *if there exists a partition of* $\mathbf{D}$ *in two non-empty subsets* $\mathbf{A}$ *and* $\mathbf{B}$ *such that* $\mathbf{A} \not\subseteq \mathbf{C}$ *and* $\mathbf{B} \not\subseteq \mathbf{C}$, *and if* $A \in \mathbf{A}$ *and* $B \in \mathbf{B}$, *then there exists* $C \in \mathbf{C}$ *such that* $C \subseteq A \cup B$.

We are now in conditions of characterizing the adjacency of two vertices of $\mathcal{FM}(X)$.

Theorem 3. *Let* $\mu, \mu_{\mathbf{C}}$ *be two extremes of* $\mathcal{FM}(X)$ *such that* $\mu > \mu_{\mathbf{C}}$. *Then,* μ *and* $\mu_{\mathbf{C}}$ *are adjacent vertices of* $\mathcal{FM}(X)$ *if and only if* μ *is not* $\mathbf{C}$*-decomposable.*

Proof: $\Rightarrow$) First, let us prove that if μ is **C**-decomposable, then it is not adjacent to $\mu_{\mathbf{C}}$.

Consider the measure $\mu' := \frac{1}{2}\mu_{\mathbf{C}} + \frac{1}{2}\mu$. Let $\{\mathbf{A}, \mathbf{B}\}$ be a partition which makes μ be a **C**-decomposable vertex. Take $C \in \mathbf{C}$. Since $\mu_{\mathbf{C}} \leq \mu$ and $\mu_{\mathbf{C}}(C) = 1$, we have $\mu(C) = 1$, whence there exists a minimal subset F of μ such that $F \subseteq C$. Let us define $\mathbf{C_A} := \{C \in \mathbf{C} : \exists A \in \mathbf{A}, A \subseteq C\}$ and $\mathbf{C_B} := \mathbf{C} \backslash \mathbf{C_A}$. As $\mu > \mu_{\mathbf{C}}$, if $C \in \mathbf{C_B}$, then there exists $B \in \mathbf{B}$ such that $B \subseteq C$. Consider

$$\mu_1(D) := \begin{cases} 1 & \text{if } \exists A \in \mathbf{A} \cup \mathbf{C_B}, A \subseteq D \\ 0 & \text{else} \end{cases} \quad \mu_2(D) := \begin{cases} 1 & \text{if } \exists B \in \mathbf{B} \cup \mathbf{C_A}, B \subseteq D \\ 0 & \text{else} \end{cases}$$

It is easy to check that μ_1 and μ_2 are monotone, whence they are extremes of $\mathcal{FM}(X)$. Let us prove that μ' is equal to $\mu_1' := \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. Let D be a subset of X. We have three different cases:

- Suppose $\mu'(D) = 0$. Then, if $\mu_1'(D) > 0$, this implies that $\mu_1(D) = 1$ or $\mu_2(D) = 1$. As $\mu \geq \mu_{\mathbf{C}}$, we conclude that there must exist a minimal set F of μ such that $F \subseteq D$; but then $\mu'(D) > 0$, a contradiction.
- Suppose $\mu'(D) = \frac{1}{2}$. Then, there exists either $A \in \mathbf{A}$ such that $A \subseteq D$ or $B \in \mathbf{B}$ such that $B \subseteq D$, and for all $C \in \mathbf{C}$, it holds $C \not\subseteq D$. Consequently $\mu_1'(D) \geq \frac{1}{2}$. Suppose $\mu_1'(D) = 1$. Then, there exists $A' \in \mathbf{A} \cup \mathbf{C_B}$ such that $A' \subseteq D$ and $B' \in \mathbf{B} \cup \mathbf{C_A}$ such that $B' \subseteq D$. We know that for every $C \in \mathbf{C}$ it holds $C \not\subseteq D$, whence $A' \in \mathbf{A}$ and $B' \in \mathbf{B}$. But by **C**-decomposability, there exists $C \in \mathbf{C}$ such that $C \subseteq A' \cup B' \subseteq D$, a contradiction.
- Finally, suppose $\mu'(D) = 1$. Thus $\mu_{\mathbf{C}}(D) = 1$, whence there exists $C \in \mathbf{C}$ such that $C \subseteq D$. We can suppose, without loss of generality, that $C \in \mathbf{C_A}$; consequently, $\mu_2(D) = 1$. Since $C \in \mathbf{C_A}$, there must exist $A \in \mathbf{A}$ such that $A \subseteq C$, whence $A \subseteq D$ and $\mu_1(D) = 1$. Therefore, $\mu_1'(D) = 1$.

It only remains to show that $\{\mu_1, \mu_2\} \neq \{\mu, \mu_{\mathbf{C}}\}$. We will prove that $\mu_1 \neq \mu_{\mathbf{C}}$. Since $\mathbf{A} \not\subseteq \mathbf{C}$, there exists $A \in \mathbf{A}$ such that $A \notin \mathbf{C}$. If $\mu_{\mathbf{C}}(A) = 1$, then there exists $C \in \mathbf{C}$ such that $C \subseteq A$ and $C \neq A$. Since $\mu > \mu_{\mathbf{C}}$, it follows that $\mu(C) = 1$, which is a contradiction with the fact that A is a minimal subset of μ. In a similar way we can show that $\mu_2 \neq \mu_{\mathbf{C}}$. Therefore, μ and $\mu_{\mathbf{C}}$ are not adjacent.

$\Leftarrow$) Now, let us show that if $\mu_{\mathbf{C}} < \mu$ and μ is not adjacent to $\mu_{\mathbf{C}}$, then μ is **C**-decomposable. Consider again the measure $\mu' := \frac{1}{2}\mu_{\mathbf{C}} + \frac{1}{2}\mu$. Since $\mu_{\mathbf{C}}$ and μ are not adjacent, there exist $\mu_1, \ldots, \mu_k$ extremes of $\mathcal{FM}(X)$ not all equal to $\mu_{\mathbf{C}}$ or μ and positive numbers $\lambda_1, \ldots, \lambda_k$ such that

$$\mu' = \sum_{i=1}^{k} \lambda_i \mu_i, \ \sum_{i=1}^{k} \lambda_i = 1. \tag{3}$$

We can assume, without loss of generality, that $\mu_1 \neq \mu, \mu_1 \neq \mu_{\mathbf{C}}$. Remark that $\mu \geq \mu_1$; otherwise, there exists $A \subseteq X$ such that $\mu_1(A) = 1$ and $\mu(A) = 0$, whence by Equation (3) it is $\lambda_1 \leq \mu'(A)$, while by definition of μ' it is $\mu'(A) = 0$, a contradiction. Consider $\mathbf{D}$ the collection of minimal subsets of μ and $\mathbf{E}$ the collection of minimal subsets of μ_1. First, let us show that $\mathbf{E} \subseteq \mathbf{C} \cup \mathbf{D}$. Take $A \in \mathbf{E}$; then, $\mu_1(A) = 1$, and from Equation (3) we conclude that $\mu'(A) \geq \lambda_1 > 0$. Since $\mu > \mu_{\mathbf{C}}$, it is $\mu(A) = 1$. There are two cases: $\mu'(A) = 1$ or $\mu'(A) = \frac{1}{2}$.

- If $\mu'(A) = 1$, then $\mu_{\mathbf{C}}(A) = 1$ and there exists $C \in \mathbf{C}$ such that $C \subseteq A$. By definition of μ', we have $\mu'(C) = 1$. If $A \neq C$, then $\mu_1(C) = 0$ (A is minimal for μ_1) and from Equation (3) we have $1 = \mu'(C) < \mu'(A) = 1$, a contradiction.

- If $\mu'(A) = \frac{1}{2}$, since $\mu(A) = 1$, there exists $D \in \mathbf{D}$ such that $D \subseteq A$. If $A \neq D$ then $\mu_1(D) = 0$ and, as before, $\frac{1}{2} = \mu'(D) < \mu'(A) = \frac{1}{2}$, again a contradiction.

Consider then $\mathbf{A} := \mathbf{E} \setminus \mathbf{C}$; it follows that $\mathbf{A} \subseteq \mathbf{D}$; let us also define $\mathbf{B} := \mathbf{D} \setminus \mathbf{A}$. Neither $\mathbf{A}$ nor $\mathbf{B}$ are included in $\mathbf{C}$. By definition, $\mathbf{A} = \mathbf{E}\backslash\mathbf{C} \not\subseteq \mathbf{C}$. If $\mathbf{B} \subseteq \mathbf{C}$, then $\mathbf{A} = \mathbf{D}\backslash\mathbf{C}$, and this would imply $\mathbf{D}\backslash\mathbf{C} = \mathbf{E}\backslash\mathbf{C}$. We know that $\mu \geq \mu_1$; let us prove that $\mu_1 \geq \mu$. Consider a minimal subset H of μ; if $H \in \mathbf{C}$, then $\mu_{\mathbf{C}}(H) = 1$, whence $\mu'(H) = 1$, implying $\mu_1(H) = 1$. Otherwise, $H \in \mathbf{D}\backslash\mathbf{C} = \mathbf{E}\backslash\mathbf{C}$, whence $\mu_1(H) = 1$. Thus, $\mu_1 = \mu$, a contradiction.

Remark that $\mathbf{A}$ and $\mathbf{B}$ are not empty, as they are not included in $\mathbf{C}$. Now take $A \in \mathbf{A}$ and $B \in \mathbf{B}$. We have two cases:

- Suppose $\mu_1(B) = 0$. Since $\mu(B) = 1$, it follows that $\mu'(B) = \frac{1}{2}$. Thus, $\mu'(A \cup B) \geq \mu'(B) + \lambda_1 \mu_1(A \cup B) = \frac{1}{2} + \lambda_1 > \frac{1}{2}$. Then $\mu'(A \cup B) = 1$, which implies $\mu_{\mathbf{C}}(A \cup B) = 1$ and, consequently, there exists $C \in \mathbf{C}$ such that $C \subseteq A \cup B$.

- Otherwise, $\mu_1(B) = 1$. Then, there exists a minimal subset H of μ_1 such that $H \subseteq B$. As $B \in \mathbf{D}$, it is a minimal subset of μ and we conclude that $B = H$ because $\mu > \mu_1$, whence $B \in \mathbf{E}$; on the other hand, $B \notin \mathbf{E}\backslash\mathbf{C}$, so that $B \in \mathbf{C}$, and we have found a subset in $\mathbf{C}$ included in $A \cup B$.

Hence, $\mathbf{A}$ and $\mathbf{B}$ form a $\mathbf{C}$-decomposition of μ and the result holds. ■

Let us give an example of how this result can be applied:

Example 1. *For $\emptyset \neq A, B \subseteq X$, consider the corresponding unanimity games u_A and u_B. As an application of Theorem 3, we conclude that u_A and u_B are adjacent in $\mathcal{FM}(X)$ if and only if $A \subseteq B$ or $B \subseteq A$.*

Moreover, if μ is an extreme of $\mathcal{FM}(X)$ satisfying that $\mu < u_A$, then μ and u_A are adjacent.

Figure 1 (which has been drawn with the help of the Pigale computer program[1]) depicts the adjacency graph for $|X| = 3$. We have used notation borrowed from monotone boolean functions so we use $+$ instead of $\vee$ and concatenation instead of $\wedge$. Also, we use i instead of x_i. Thus, $12 + 3$ stands for $(u_{x_1} \wedge u_{x_2}) \vee u_{x_3}$, and so on. The reason for the particular coloring of the vertices will become apparent in next section.

In we consider maximal false subsets instead of minimal subsets, Theorem 3 can be written as

Corollary 2. *Consider two extreme points μ, μ' of $\mathcal{FM}(X)$ whose maximal zeroes are $\mathbf{D}$ and $\mathbf{C}$, respectively. If $\mu > \mu'$, then μ is adjacent to μ' if and only if there exists a partition $\{\mathbf{A}, \mathbf{B}\}$ in two non-empty subsets of $\mathbf{C}$ such that*

- *Neither $\mathbf{A} \subseteq \mathbf{D}$ nor $\mathbf{B} \subseteq \mathbf{D}$.*

- *For any $A \in \mathbf{A}, B \in \mathbf{B}$, there exists $D \in \mathbf{D}$ such that $A \cap B \subseteq D$.*

Proof: As stated in Theorem 2, the dual transformation is an isometry of $\mathcal{FM}(X)$. Thus, two extremes are adjacent if and only if so are their corresponding duals. On the other hand, a straightforward computation shows that $X\backslash C$ is a maximal zero for μ if and only C is a minimal subset of $\bar{\mu}$. The result follows from these observations and Theorem 3. ■

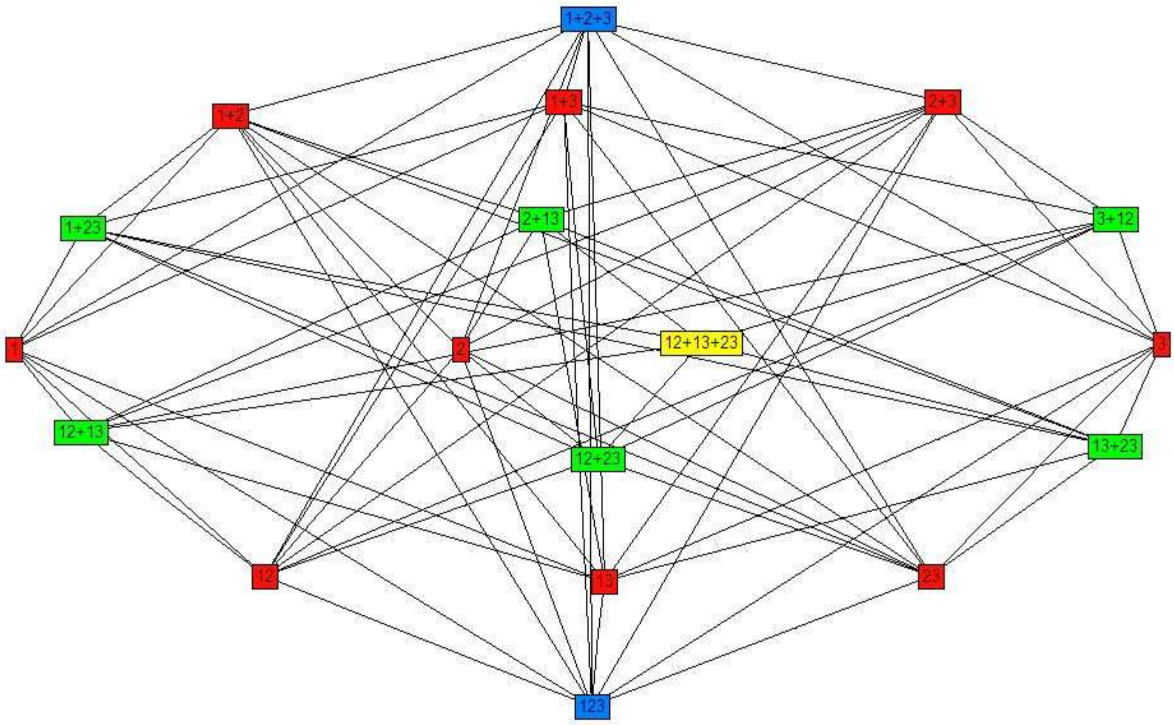
From the proof of Theorem 3, we can also derive the following results:

Corollary 3. *Given two extreme points μ, μ' of $\mathcal{FM}(X)$, if they are not adjacent, then there exist μ_1, μ_2 other extreme points of $\mathcal{FM}(X)$ such that*

$$\mu + \mu' = \mu_1 + \mu_2.$$

[1]PIGALE: Public Implementation of a Graph Algorithm Library and Editor, H. de Fraysseix and P. Ossona de Mendez. http://pigale.sourceforge.net/

Figure 1: Adjacency of vertices of $\mathcal{FM}(X)$ for $|X| = 3$



Corollary 4. *Let μ and μ' be two extremes of $\mathcal{FM}(X)$. Suppose the midpoint between μ and μ' can be written as a convex combination of some vertices of $\mathcal{FM}(X)$, namely,*

$$\frac{1}{2}\mu + \frac{1}{2}\mu' = \sum_{i=1}^{k} \lambda_i \mu_i,\ \lambda_i > 0,\ \forall i = 1, ..., k,\ \sum_{i=1}^{k} \lambda_i = 1.$$

Then, for each $i = 1, \ldots, k$, we have that either $\mu_i = \mu, \mu_i = \mu'$, or there exists μ'_i extreme of $\mathcal{FM}(X)$ such that $\frac{1}{2}\mu + \frac{1}{2}\mu' = \frac{1}{2}\mu_i + \frac{1}{2}\mu'_i$.

This result gives a sufficient and necessary condition for a vertex to appear in a convex combination which is equal to the midpoint of a segment. If μ and μ' are adjacent vertices, then each μ_i is either μ or μ'. The second part of the result only applies when μ and μ' are not adjacent. Notice, however, that in this last case, the measure μ'_i is not necessarily one of the measures $\mu_1, ..., \mu_j$ as next example shows:

Example 2. *For $|X| = 3$, consider $\mu = (u_{x_1} \wedge u_{x_2}) \vee (u_{x_1} \wedge u_{x_3}) \vee (u_{x_2} \wedge u_{x_3})$ and $\mu' = u_{x_1} \wedge u_{x_2} \wedge u_{x_3}$. Then,*

$$\frac{1}{2}\mu + \frac{1}{2}\mu' = \frac{1}{4}[(u_{x_1} \wedge u_{x_2}) \vee (u_{x_1} \wedge u_{x_3})] + \frac{1}{4}[(u_{x_1} \wedge u_{x_2}) \vee (u_{x_2} \wedge u_{x_3})] + \frac{1}{4}[(u_{x_1} \wedge u_{x_3}) \vee (u_{x_2} \wedge u_{x_3})] + \frac{1}{4}(u_{x_1} \wedge u_{x_2} \wedge u_{x_3}).$$

For $(u_{x_1} \wedge u_{x_2}) \vee (u_{x_1} \wedge u_{x_3})$, we obtain

$$\frac{1}{2}\mu + \frac{1}{2}\mu' = \frac{1}{2}[(u_{x_1} \wedge u_{x_2}) \vee (u_{x_1} \wedge u_{x_3})] + \frac{1}{2}(u_{x_2} \wedge u_{x_3}).$$

However, if only two vertices are involved in the convex combination, then we can always recover a whole C-decomposition from them.

Theorem 4. *Suppose $\mu_{\mathbf{D}}$ and $\mu_{\mathbf{C}}$ are two extreme points of $\mathcal{FM}(X)$ such that $\mu_{\mathbf{D}} \geq \mu_{\mathbf{C}}$ and they are not adjacent. Consider $\mu_{\mathbf{E_1}}$ and $\mu_{\mathbf{E_2}}$ two extremes of $\mathcal{FM}(X)$ different to $\mu_{\mathbf{D}}$ and $\mu_{\mathbf{C}}$ such that $\frac{1}{2}\mu_{\mathbf{D}} + \frac{1}{2}\mu_{\mathbf{C}} = \frac{1}{2}\mu_{\mathbf{E_1}} + \frac{1}{2}\mu_{\mathbf{E_2}}$. Then, $\mathbf{A} = \mathbf{E_1} \cap \mathbf{D}$ and $\mathbf{B} = \mathbf{E_2} \cap \mathbf{D}$ form a $\mathbf{C}$-decomposition of $\mu_{\mathbf{D}}$.*

Proof: Let us define $\mu' := \frac{1}{2}\mu + \frac{1}{2}\mu_{\mathbf{C}} = \frac{1}{2}\mu_{\mathbf{E_1}} + \frac{1}{2}\mu_{\mathbf{E_2}}$.

- First, let us prove that $\mathbf{A} \cup \mathbf{B} = \mathbf{D}$. Clearly, $\mathbf{A} \cup \mathbf{B} \subseteq \mathbf{D}$. Consider $D \in \mathbf{D}$. Then $\mu'(D) > 0$, so there exists $E \in \mathbf{E_1} \cup \mathbf{E_2}$ such that $E \subseteq D$. If $E \neq D$ then, since D is minimal and $\mu_{\mathbf{D}} > \mu_{\mathbf{C}}$, it follows that $\mu_{\mathbf{D}}(E) = \mu_{\mathbf{C}}(E) = 0$. But $\frac{1}{2}\mu_{\mathbf{E_1}}(E) + \frac{1}{2}\mu_{\mathbf{E_2}}(E) > 0$, which is a contradiction. Thus, $D = E$ and $D \in (\mathbf{E_1} \cup \mathbf{E_2}) \cap \mathbf{D} = \mathbf{A} \cup \mathbf{B}$.

- Let us now show that $\mathbf{A} \not\subseteq \mathbf{C}$, $\mathbf{B} \not\subseteq \mathbf{C}$. Suppose $\mathbf{A} \subseteq \mathbf{C}$. We will prove that this implies $\mu_{\mathbf{E_1}} \leq \mu_{\mathbf{C}}$. If there exists F such that $\mu_{\mathbf{E_1}}(F) = 1$ and $\mu_{\mathbf{C}}(F) = 0$, then there exists $G \subseteq F$ in $\mathbf{E_1}$ such that $\mu_{\mathbf{E_1}}(G) = 1$ and $\mu_{\mathbf{C}}(G) = 0$. Hence, $\mu'(G) > 0$, whence $\mu_{\mathbf{D}}(G) = 1$. Therefore, there exists $D \in \mathbf{D}$ such that $D \subseteq G$. If $D = G$, then $G \in \mathbf{E_1} \cap \mathbf{D} = \mathbf{A}$ but $G \notin \mathbf{C}$ contrary to the assumption that $\mathbf{A} \subseteq \mathbf{C}$. If $D \subset G$, then $\mu_{\mathbf{E_1}}(D) = 0$ since G is a minimal subset of μ_1. As $\mu'(D) > 0$, we conclude that $\mu_{\mathbf{E_2}}(D) = 1$. Consequently, $\mu_{\mathbf{E_2}}(G) = 1$ since $D \subseteq G$, and then $\mu'(G) = 1$, which implies $\mu_{\mathbf{C}}(G) = 1$, a contradiction. Thus, $\mu_{\mathbf{E_1}} \leq \mu_{\mathbf{C}}$. Also, $\mu_{\mathbf{E_1}} \geq \mu_{\mathbf{C}}$ since $\mu_{\mathbf{C}}(H) = 1$ implies $\mu'(H) = 1$ and, thus, $\mu_{\mathbf{E_1}}(H) = 1$. Then, $\mu_{\mathbf{E_1}} = \mu_{\mathbf{C}}$, which contradicts the hypothesis. Similarly we can prove that $\mathbf{B} \not\subseteq \mathbf{C}$.

- Next, $\mathbf{A}$ and $\mathbf{B}$ are both non empty. This follows from the fact that they are not included in $\mathbf{C}$.

- Finally, let us show that if $A \in \mathbf{A}$ and $B \in \mathbf{B}$, then there exists $C \in \mathbf{C}$ such that $C \subseteq A \cup B$. If $A \in \mathbf{A}$ and $B \in \mathbf{B}$, then $\mu'(A \cup B) = 1$ and thus, $\mu_{\mathbf{C}}(A \cup B) = 1$. Then, there exists $C \in \mathbf{C}$ such that $C \subseteq A \cup B$. ■

Notice that if we apply the construction of the first part of the proof of Theorem 3 to the $\mathbf{C}$-decomposition found in the previous Theorem, then we recover $\mu_{\mathbf{E}_1}$ and $\mu_{\mathbf{E}_2}$.

When dealing with monotone boolean functions, it is usual to include the constant functions 0 and 1, which are not fuzzy measures. For these functions, the following can be shown:

Lemma 2. *The constant functions 0 and 1 are adjacent to any other monotone boolean function.*

Proof: We will prove the result for the constant function 0. By duality the result also holds for function 1. Consider a monotone boolean function μ and take the midpoint of 0 and μ. Then, we obtain a function whose value on X is $\frac{1}{2}$; but this can only be obtained in a convex combination of monotone boolean functions if the constant function 0 is part of the combination and its coefficient is $\frac{1}{2}$. Thus,

$$\frac{1}{2}0 + \frac{1}{2}\mu = \frac{1}{2}0 + \sum_{i=1}^{k} \lambda_i \mu_i, \; \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = \frac{1}{2}.$$

But this means that

$$\mu = \sum_{i=1}^{k} 2\lambda_i \mu_i,$$

whence $\mu = \mu_i, \forall i = 1, ..., k$ because μ is an extreme. ■

Remark that the collections of minimal subsets for 0 and 1 are respectively $\emptyset$ and $\{\emptyset\}$. This allows us to extend Theorem 3 for any monotone boolean function.

Corollary 5. *Let $\mu, \mu_{\mathbf{C}}$ be two extremes of the set of the convex hull of monotone boolean functions such that $\mu > \mu_{\mathbf{C}}$. Then, μ and $\mu_{\mathbf{C}}$ are adjacent vertices of this polytope if and only if μ is not $\mathbf{C}$-decomposable.*

Similarly, Corollaries 2, 3 and 4 also apply to the convex hull of monotone boolean functions.

Let us now study the complexity of determining whether two $\{0, 1\}$-valued measures are adjacent. Consider two antichains $\mathbf{C}$ and $\mathbf{D}$ in X corresponding to the minimal subsets of two extreme points of $\mathcal{FM}(X)$. If these measures are not comparable, then we already know that they are not adjacent by Corollary 1. Let us then assume that $\mu_{\mathbf{C}} < \mu_{\mathbf{D}}$.

Definition 2. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two collections of subsets of* X*. We associate to them the graph* $G_{\mathbf{C},\mathbf{D}}$ *whose nodes are the elements of* $\mathbf{D} \setminus \mathbf{C}$ *and such that there is an edge between* A *and* B *if there is no* $C \in \mathbf{C}$ *such that* $C \subseteq A \cup B$*.*

Theorem 5. *Let* $\mu_{\mathbf{C}}$ *and* $\mu_{\mathbf{D}}$ *be two extremes of* $\mathcal{FM}(X)$ *such that* $\mu_{\mathbf{C}} < \mu_{\mathbf{D}}$*. Then* $\mu_{\mathbf{C}}$ *and* $\mu_{\mathbf{D}}$ *are adjacent if and only if the graph* $G_{\mathbf{C},\mathbf{D}}$ *is connected.*

Proof: From Theorem 3, we know that $\mu_{\mathbf{C}}$ and $\mu_{\mathbf{D}}$ are adjacent if and only $\mu_{\mathbf{D}}$ is not $\mathbf{C}$-decomposable.

$\Rightarrow$) Suppose that $G_{\mathbf{C},\mathbf{D}}$ is not connected. Then, we have at least two connected components. Take one connected component, namely $\mathbf{A}$, and consider

$$\mathbf{D}_1 := \mathbf{A} \cup (\mathbf{D} \cap \mathbf{C}),\ \mathbf{D}_2 := \mathbf{D} \backslash \mathbf{D}_1.$$

By construction, $\mathbf{D}_1$ and $\mathbf{D}_2$ are not empty nor included in $\mathbf{C}$. And by the construction of $G_{\mathbf{C},\mathbf{D}}$, they determine a $\mathbf{C}$-decomposition.

$\Leftarrow$) Suppose on the other hand that $G_{\mathbf{C},\mathbf{D}}$ is connected. If $\mu_{\mathbf{C}}$ and $\mu_{\mathbf{D}}$ are not adjacent, then there exists $\{\mathbf{D}_1, \mathbf{D}_2\}$ a $\mathbf{C}-$decomposition of $\mu_{\mathbf{D}}$. But then, we can consider $\mathbf{A} := \mathbf{D}_1 \backslash \mathbf{C}$ and $\mathbf{B} := \mathbf{D}_2 \backslash \mathbf{C}$; as $\{\mathbf{D}_1, \mathbf{D}_2\}$ form $\mathbf{C}-$decomposition of $\mu_{\mathbf{D}}$, it follows that $\mathbf{A}$ and $\mathbf{B}$ are not empty. All nodes of $\mathbf{D} \backslash \mathbf{C}$ are either in $\mathbf{A}$ or in $\mathbf{B}$, and by construction of $G_{\mathbf{C},\mathbf{D}}$, no nodes $A \in \mathbf{A}, B \in \mathbf{B}$ are joined by an edge. Thus, $G_{\mathbf{C},\mathbf{D}}$ is not connected, a contradiction. ■

Corollary 6. *Let* $\mu_{\mathbf{C}}$ *and* $\mu_{\mathbf{D}}$ *be two extremes of* $\mathcal{FM}(X)$ *such that* $\mu_{\mathbf{C}} < \mu_{\mathbf{D}}$ *and they are not adjacent. If* c *is the number of connected components of* $G_{\mathbf{C},\mathbf{D}}$ *and* $d = |\mathbf{D} \cap \mathbf{C}|$*, then the number of different* $\mathbf{C}$*-decompositions of* $\mu_{\mathbf{D}}$ *is* $(2^c - 2)2^{d-1}$*.*

Proof: If $\mathbf{H}_1, \ldots, \mathbf{H}_c$ are the connected components of $G_{\mathbf{C},\mathbf{D}}$, then all the subsets of each $\mathbf{H}_i$ must be together either in $\mathbf{A}$ or in $\mathbf{B}$ in the decomposition; then, for $\mathbf{H}_j$ with $j = 1, \ldots, c$ we can choose whether $\mathbf{H}_j \subseteq \mathbf{A}$ or $\mathbf{H}_j \subseteq \mathbf{B}$. We only need to exclude the case in which no $\mathbf{H}_j$ goes to $\mathbf{A}$ or $\mathbf{B}$. On the other hand, given a decomposition, for each D in $\mathbf{D} \cap \mathbf{C}$, we can choose whether D is included in $\mathbf{A}$ or in $\mathbf{B}$. This gives the value $(2^c - 2)2^d$. Finally, notice that we can interchange the roles of $\mathbf{A}$ and $\mathbf{B}$, whence the number of possible $\mathbf{C}$-decompositions halves. ■

Theorem 6. *Let* μ *and* μ' *two extremes of* $\mathcal{FM}(X)$*. We can check whether they are adjacent in polynomial time in the number of minimal subsets of* μ *plus the number of minimal subsets of* μ'*.*

Proof: We can check whether $\mu < \mu'$ (or $\mu > \mu'$) in polynomial time in the number of minimal subsets of the extremes. If they are not comparable, then they are not adjacent. Otherwise, we can construct their associated graph (Definition 2) in polynomial time and check whether it is connected or not (for instance by breadth-first search, see [25]) also in polynomial time. The result follows then from Theorem 5. ■

As the monotone boolean functions 0 and 1 are adjacent to any other monotone boolean function, the following can be deduced:

Corollary 7. *Let* μ *and* μ' *two extremes of the convex hull of monotone boolean functions. We can check whether they are adjacent in polynomial time in the number of minimal subsets of* μ *plus the number of minimal subsets of* μ'*.*

3 The polytope $\mathcal{FM}(X)$ and its adjacency graph

In this section, we study other properties of $\mathcal{FM}(X)$. First, we show that given two monotone boolean functions, it is quite uncommon that they are adjacent. To simplify the proof, in this case we include the constant functions 0 and 1.

Lemma 3. *The probability of two monotone boolean functions of* n *variables taken at random being comparable (with the order relation) tends to zero when* n *tends to infinity.*

Proof: It is a well-known fact [12] that the number of pairs (f, g) of monotone boolean functions on n variables such that $f \leq g$ is equal to the number of monotone boolean functions on $n+1$ variables, that is, E_{n+1}.

Then, the proportion of pairs (f, g) of monotone boolean functions on n variables such that $f \leq g$ is $\frac{E_{n+1}}{E_n^2}$.

On the other hand, we know from Theorem 1 that

$$2^{h(n)} < E_n < 3^{h(n)}$$

with $h(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Then,

$$\frac{E_{n+1}}{E_n^2} < \frac{3^{h(n+1)}}{(2^{h(n)})^2} = \frac{3^{h(n+1)}}{4^{h(n)}} = 3^{\frac{h(n+1)}{h(n)}} \left(\frac{3}{4}\right)^{h(n)}.$$

But $\frac{h(n+1)}{h(n)} \leq 2$ for all n, whence

$$\frac{E_{n+1}}{E_n^2} < 9\left(\frac{3}{4}\right)^{h(n)},$$

so that this quotient tends to 0 when n tends to infinity. The number of comparable pairs (first component less than or equal to the second or vice versa) is, at most, twice this number. ■

As an immediate consequence this lemma and Corollary 1, we have the following result.

Corollary 8. *The probability of two monotone boolean functions of n variables taken at random being adjacent tends to zero when n tends to infinity.*

This result also applies to the extremes of $\mathcal{FM}(X)$, since the monotone boolean functions 0 and 1 are adjacent to any other monotone boolean function (Lemma 2).

Some values of the probability of two vertices of $\mathcal{FM}(X)$ being adjacent are given in the following table.

n	2	3	4	5
Probability	0.5	0.45062	0.23015	0.07189

Table 2: Probabilities of two vertices being adjacent for different cardinalities

Consider the graph of adjacency $\mathcal{FM}(X)$. Let us define the distance $d(\mu_1, \mu_2)$ between two extremes μ_1, μ_2 of $\mathcal{FM}(X)$ as the number of edges of the shortest path between them in this graph. We have shown in the previous corollary that the distance between two vertices is greater than 1 with probability tending to 1 when $|X|$ tends to infinity. Now, we will study the *diameter* of the graph, i.e. the maximum distance between two extremes.

Lemma 4. *If $|X| > 2$ and μ_1, μ_2 are two extremes of $\mathcal{FM}(X)$ such that both μ_1 and μ_2 are either adjacent to u_X or to $u_\emptyset$ (not necessarily both adjacent to the same) then $d(\mu_1, \mu_2) \leq 3$.*

Proof: It is obvious because u_X and $u_\emptyset$ are adjacent when $|X| > 2$. ■

Note that for $|X| = 2$, u_X and $u_\emptyset$ are not adjacent as $\{\{x_1\}, \{x_2\}\}$ determine a $\{X\}-$decomposition. For $|X| = 1$, $u_X = u_\emptyset$.

Let us now study the distance when we consider extremes that are not adjacent to u_X nor $u_\emptyset$. We start characterizing these extremes.

Lemma 5. *If $|X| > 3$ and μ is an extreme of $\mathcal{FM}(X)$ which is neither adjacent to u_X nor to $u_\emptyset$, then there exists $x_i \in X$ such that μ can be written as*

$$\mu = \left(\bigvee_{j \neq i} u_{\{x_i, x_j\}}\right) \vee u_{X \setminus \{x_i\}}.$$

Proof: Let $\mathbf{C}$ be the collection of minimal subsets of μ. Since μ is not adjacent to $u_\emptyset$ and $u_\emptyset > \mu$, there exists a $\mathbf{C}$-decomposition $\{\mathbf{A}, \mathbf{B}\}$ of $\{\{x_1\}, \{x_2\}, \ldots, \{x_n\}\}$, the minimal subsets of $u_\emptyset$. This implies that in $\mathbf{C}$ there exists either a singleton or a subset of two elements; otherwise, if we take a subset in $\mathbf{A}$ and another subset in $\mathbf{B}$, there would be no element of $\mathbf{C}$ contained in the corresponding union and $\{\mathbf{A}, \mathbf{B}\}$ would not be a $\mathbf{C}-$decomposition.

Let us prove that there is no singleton in $\mathbf{C}$. If there is a singleton, say $\{x_i\}$, then as μ is not adjacent to u_X, there exist a $\{X\}-$decomposition of μ, whence the set $X \setminus \{x_i\}$ must be also a minimal subset of μ. But then $\mathbf{C}$ is exactly $\{\{x_i\}, X \setminus \{x_i\}\}$. As $n > 3$, it is $\mu \neq u_\emptyset$. Consider the $\mathbf{C}-$decomposition $\{\mathbf{A}, \mathbf{B}\}$ and assume, without loss of generality, that $\{x_i\} \in \mathbf{A}$; note that $\mathbf{A} \neq \{\{x_i\}\}$, as otherwise $\mathbf{A} \subseteq \mathbf{C}$. Consider $\{x_j\} \in \mathbf{A}\backslash\{\{x_i\}\}$ and $\{x_k\} \in \mathbf{B}$. Then, $\{x_j, x_k\}$ is not contained in any minimal subset of $\mathbf{C}$, a contradiction.

Let us now show that either $\mathbf{A}$ or $\mathbf{B}$ has only one singleton. Suppose that both $\mathbf{A}$ and $\mathbf{B}$ have more than one element, say $\{x_1\}, \{x_2\} \in \mathbf{A}$ and $\{x_3\}, \{x_4\} \in \mathbf{B}$. As $\mathbf{C}$ has no singletons, this implies that the subsets $\{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}$ are in $\mathbf{C}$. Since μ is not adjacent to u_X, there must exist $\{\mathbf{D}, \mathbf{E}\}$ a $\{X\}$-decomposition of μ. Suppose that $\{x_1, x_3\}$ is in $\mathbf{D}$. Then, $\{x_1, x_4\}$ and $\{x_2, x_3\}$ must be also in $\mathbf{D}$, since their union with $\{x_1, x_3\}$ is not the total set X and then the condition of $\{X\}$-decomposability would fail. Again by $\{X\}$-decomposability, there exists a minimal subset E of μ in $\mathbf{E}$ such that $E \cup \{x_1, x_3\} = X$. Then, $x_2, x_4 \in E$ and E is exactly $\{x_2, x_4\}$ because $\{x_2, x_4\}$ is a minimal subset of μ. But $\{x_2, x_3\} \cup \{x_2, x_4\} \neq X$, and $\{\mathbf{D}, \mathbf{E}\}$ cannot be a $\{X\}$-decomposition of μ. Hence, we can assume that $\mathbf{A}$ has only one element, say $\{x_i\}$ and, consequently, $\mathbf{B} = \{\{x_1\}, \{x_2\}, \ldots, \{x_n\}\} \setminus \{\{x_i\}\}$.

Then, $\mathbf{C}$ contains all the subsets of the form $C_j = \{x_i, x_j\}$ with $j \neq i$. Consider $\{\mathbf{D}, \mathbf{E}\}$ a $\{X\}$-decomposition of μ. Note that all the minimal subsets C_j must be in either $\mathbf{D}$ or $\mathbf{E}$, since $C_j \cup C_k \neq X$, $\forall j, k$. Suppose w.l.g. that they are in $\mathbf{D}$; then, for $E \in \mathbf{E}$, we necessarily have $E \cup C_j = X$ for every $j \neq i$. The only set satisfying this condition and candidate for being a minimal subset is $X \setminus \{x_i\}$, thus obtaining the expression for μ in the Lemma. ■

Lemma 6. *Suppose* $|X| > 3$. *If* $\mu_i := \left(\bigvee_{j \neq i} u_{\{x_i, x_j\}}\right) \vee u_{X \setminus \{x_i\}}$, $\forall x_i \in X$, *then* $d(\mu_i, \mu) \leq 3$ *for any* μ *extreme point of* $\mathcal{FM}(X)$.

Proof: It is easy to check that, for all i and j, the extremes μ_i and μ_j are both adjacent to the extreme $u_{X\setminus\{x_i\}} \vee u_{X\setminus\{x_j\}}$. Thus $d(\mu_i, \mu_j) = 2$ since μ_i and μ_j are not adjacent (neither $\mu_i > \mu_j$ nor $\mu_j > \mu_i$). Now, remark that μ_i is also adjacent to $u_{X\setminus\{x_i\}}$ which, in turn, is adjacent to u_X and to $u_\emptyset$. Hence $d(\mu_i, u_X) = d(\mu_i, u_\emptyset) = 2$ and the result follows from the previous lemma. ■

Corollary 9. *The diameter of* $\mathcal{FM}(X)$ *when* $|X| > 3$ *is at most 3.*

In next result, we will prove that it is indeed 3.

Lemma 7. *Assume* $|X| > 3$. *There exist vertices in* $\mathcal{FM}(X)$ *whose distance is at least 3.*

Proof: Consider μ the extreme point of $\mathcal{FM}(X)$ whose minimal subsets are $\{x_1, x_2\}$ and $X\backslash\{x_1, x_2\}$. Let μ' be the extreme point whose minimal subsets are $\mathbf{C} := \{A_i, B_i\}_{i=3}^n$ with $A_i := \{x_1, x_i\}$, $B_i := \{x_2, x_i\}$. Note that they are not comparable, whence $d(\mu, \mu') > 1$. We will prove that $d(\mu, \mu') \geq 3$.

To show this, we need to prove that it is not possible to find another extreme point μ'' being adjacent to both μ and μ'. We will show that any extreme point of $\mathcal{FM}(X)$ adjacent to μ is not adjacent to μ'.

Suppose μ'' is adjacent to μ. Then, either $\mu'' > \mu$ or $\mu > \mu''$.

First, let us assume that $\mu'' > \mu$. If μ'' cannot be compared with μ', then μ'' and μ' are not adjacent and we are done. Moreover, it cannot be $\mu' > \mu''$, as there exists a minimal subset A for μ'' such that $A \subseteq \{x_1, x_2\}$. Then, it suffices to study the case when $\mu'' > \mu'$. Let us define

$$\mathbf{A} := \{A \,|\, A \text{ minimal subset for } \mu'', A \subseteq \{x_1, x_2\}\}, \, \mathbf{B} := \{A \,|\, A \text{ minimal subset for } \mu'', A \notin \mathbf{A}\}.$$

As $\mu'' > \mu$, we conclude that both $\mathbf{A} \neq \emptyset, \mathbf{B} \neq \emptyset$. By construction, $\mathbf{A} \not\subseteq \mathbf{C}$. On the other hand, as $\mu'' > \mu$, there exists B a minimal subset of μ'' such that $B \subseteq X\backslash\{x_1, x_2\}$. By definition of $\mathbf{B}, B \in \mathbf{B}$, whence $\mathbf{B} \not\subseteq \mathbf{C}$.

Let us show that $\mathbf{A}, \mathbf{B}$ are a $\mathbf{C}$-decomposition for the minimal subsets of μ'. For $A \in \mathbf{A}$, assume w.l.g. that $x_1 \in A$. Given $B \in \mathbf{B}$, there exists $x_i \in B \backslash \{x_1, x_2\}$. Therefore, $\{x_1, x_i\} \subseteq A \cup B$. We conclude that μ'' is not adjacent to μ'.

Let us now study the case when $\mu > \mu''$ and let us denote by $\mathbf{D}$ the set of minimal subsets of μ''. As μ'' and μ are adjacent, then μ is not $\mathbf{D}$-decomposable. Consider the partition $\mathbf{A} := \{\{x_1, x_2\}\}$, $\mathbf{B} := \{X \backslash \{x_1, x_2\}\}$ of the minimal subsets of μ. We have four different cases:

- If $\mathbf{A} \subseteq \mathbf{D}, \mathbf{B} \subseteq \mathbf{D}$, then $\mu = \mu''$, a contradiction.
- If $\mathbf{A} \subseteq \mathbf{D}$ and $\mathbf{B} \not\subseteq \mathbf{D}$, then $\mu''(x_1, x_2) = 1$; on the other hand, $\mu'(x_1, x_2) = 0$, whence $\mu' \not> \mu''$. It is not possible that $\mu'' > \mu'$, because this would imply $\mu > \mu'' > \mu'$ and μ and μ' are not comparable. Therefore, μ' and μ'' are not comparable and consequently, they are not adjacent.
- If $\mathbf{A} \not\subseteq \mathbf{D}$ and $\mathbf{B} \subseteq \mathbf{D}$, then $\mu''(X \backslash \{x_1, x_2\}) = 1$; on the other hand $\mu'(X \backslash \{x_1, x_2\}) = 0$, whence $\mu' \not> \mu''$. Again it is not possible that $\mu'' > \mu'$, because this would imply $\mu > \mu'' > \mu'$. Therefore, μ' and μ'' are not adjacent.
- Otherwise, $\mathbf{A} \not\subseteq \mathbf{D}, \mathbf{B} \not\subseteq \mathbf{D}$, and $\mathbf{A}, \mathbf{B}$ is a $\mathbf{D}$-decomposition, no matter which is $\mathbf{D}$. Then μ and μ'' are not adjacent, a contradiction.

This finishes the proof. ∎

For $|X| = 3$, we can study the distance of two extreme points of $\mathcal{FM}(X)$ in Figure 1. We have used red color for those vertices which are simultaneously adjacent to 123 and $1 + 2 + 3$ (which are, in turn, colored blue), and green color for those vertices which are adjacent to $12 + 13 + 23$ (colored yellow). Thus, all green vertices are at distance 2 or less among them, and all red vertices are at distance 2 or less among them. Also, any red vertex is adjacent to at least one green vertex (and vice versa), so green and red vertices are at distance 3 at most. Since $12 + 13 + 23$ is adjacent to all green vertices, it is at distance at most 2 of all red vertices. Similarly, 123 and $1 + 2 + 3$ are at distance at most 2 of all green vertices. Finally, the distance between $12 + 13 + 23$ and 123 is 3, the distance between $12 + 13 + 23$ and $1 + 2 + 3$ is again 3, and the distance between 123 and $1 + 2 + 3$ is 1.

Joining all these results, we can state the following Theorem.

Theorem 7. *If $|X| \geq 3$, then the diameter of the graph of adjacency of the extremes of $\mathcal{FM}(X)$ is exactly 3.*

For $|X| = 1$, the diameter of $\mathcal{FM}(X)$ is 0, and for $|X| = 2$, it is 2. Notice that in the convex hull of all monotone boolean functions, Lemma 7 does not hold, as any measure is adjacent to the constant functions 0 and 1. In this case, the diameter is always 2, except for $|X| = 1$, whose diameter is 1 (note that there are three measures in this case: the constant 0, the constant 1 and the function whose value is 0 for $\emptyset$ and 1 for X; all are adjacent to each other).

Moreover, the following holds:

Theorem 8. *The probability that an extreme point of $\mathcal{FM}(X)$ chosen at random is adjacent to $u_\emptyset$ tends to 1 when $n = |X|$ tends to infinity.*

Proof: It is a known fact (see e.g. [18]) that if μ is an extreme point of $\mathcal{FM}(X)$, then the cardinality of the smallest minimal subset of μ is at least $\frac{n-3}{2}$ with probability tending to 1 when n tends to infinity.

As $\mu < u_\emptyset$, if they are not adjacent, we have a decomposition of $\{\{x_1\}, ..., \{x_n\}\}$, the minimal subsets of $u_\emptyset$. But with probability tending to 1, this is not possible. ∎

From this theorem, we can derive the following results:

Corollary 10. *The probability that an extreme point of $\mathcal{FM}(X)$ chosen at random is adjacent to u_X tends to 1 when $|X|$ tends to infinity.*

Proof: It suffices to apply duality and note that if μ is adjacent to $u_\emptyset$, then its dual is adjacent to u_X. As the dual application is bijective (Theorem 2), the result follows from the previous Theorem. ∎

Corollary 11. *The probability that two extreme points μ, μ' of $\mathcal{FM}(X)$ chosen at random satisfy $d(\mu, \mu') = 2$ tends to 1 when $n = |X|$ tends to infinity.*

Proof: By Theorem 8, the probability of both μ, μ' being adjacent to $u_\emptyset$ tends to 1. Thus, $d(\mu, \mu') \leq 2$ with probability tending to 1 when n tends to infinity. On the other hand, we know from Corollary 8, that they are not adjacent with probability tending to 1. Joining both results, the corollary holds. ■

As a consequence, the probability that two extreme points μ, μ' of $\mathcal{FM}(X)$ chosen at random satisfy $d(\mu, \mu') = 3$, i.e. they are at maximal distance, tends to 0 when $|X|$ tends to infinity. Next table shows the probability of the different distances for different values of n. Distance 0 appears when $\mu = \mu'$.

n	$P(d(\mu, \mu') = 0)$	$P(d(\mu, \mu') = 1)$	$P(d(\mu, \mu') = 2)$	$P(d(\mu, \mu') = 3)$	D_n	Pairs at distance 1
2	0.25	0.5	0.25	0	4	8
3	0.05555	0.45062	0.44444	0.04938	18	146
4	0.00602	0.23015	0.75091	0.01292	166	6342
5	0.00013	0.07189	0.92797	$2.78 * 10^{-6}$	7579	4129670

Table 3: Probabilities of different distances for $\mathcal{FM}(X)$

From the last two columns of this table we can see that the number of adjacent vertices for each extreme point is not the same except for the case $|X| = 2$. Otherwise, the number pairs that are at distance one should be divisible by the number of vertices D_n, and this does not hold for $n \geq 3$.

Notice that if we embed $\mathcal{FM}(X)$ in $\mathcal{FM}(X \cup \{x_{n+1}\})$, the distance between two measures $\mu_1, \mu_2 \in \mathcal{FM}(X)$ and the distance of the corresponding embedded measures $\mu'_1, \mu'_2 \in \mathcal{FM}(X \cup \{x_{n+1}\})$, where μ'_1 and μ'_2 are defined by

$$\mu'_i(A) := \begin{cases} \mu_i(A) & \text{if } A \subseteq X \\ \mu_i(A \backslash \{x_{n+1}\}) & \text{if } x_{n+1} \in A \end{cases}$$

can be different.

Note that from the definition of μ'_i, it can be seen that the minimal subsets for μ_i and μ'_i are the same. Now, the following can be shown:

Lemma 8. *Consider μ_1 and μ_2 two monotone boolean functions such that there exists $x_i \in X$ satisfying that it does not belong to any minimal subset of μ_1 and μ_2. Then, $d(\mu_1, \mu_2) \leq 2$.*

Proof: If μ_1 and μ_2 are adjacent, then $d(\mu_1, \mu_2) = 1$ and we are done. Suppose then that they are not adjacent; then, they are adjacent to u_X as the families of minimal subsets cannot be $\{X\}$-decomposed. This finishes the proof. ■

As a corollary we obtain the following result:

Corollary 12. *If we consider the polytope $\mathcal{FM}(X)$ as a sub-polytope of $\mathcal{FM}(X \cup \{x_{n+1}\})$, then the measures in $\mathcal{FM}(X)$ whose distance is 3 are at distance 2 in $\mathcal{FM}(X \cup \{x_{n+1}\})$.*

Proof: Notice that the distance is not 1, as they are not adjacent and the minimal subsets do not vary when embedding. Applying the previous lemma, the result holds. ■

Let us now study the facets of $\mathcal{FM}(X)$. More concretly, we address now the problem of obtaining the number of vertices in a facet of the polytope. The facets of a polytope are given for the points satisfying with equality a non-dummy constraint.

Lemma 9. *There are not dummy constraints in $\mathcal{FM}(X)$.*

Proof: Given a constraint of Equation (2) $\mu(A) - \mu(A \backslash x_i) \geq 0, x_i \in A$, it suffices to find a measure satisfying any constraint of $\mathcal{FM}(X)$ except this one.

Consider the set function given by $\mu(A) = 0, \mu(B) = 1, \forall B \not\subset A$, $\mu(B) = 0, \forall B \subseteq A, B \neq A\backslash x_i, \mu(A\backslash x_i) = 1$. Then, it can be easily checked that this set function satisfies any constraint defining $\mathcal{FM}(X)$ except the constraint $\mu(A) - \mu(A\backslash x_i) \geq 0$. ■

Then, given the facet defined by $\mu(A) = \mu(A\backslash x_i)$, for some $\emptyset \neq A \subseteq X$, the vertices in it are those satisfying either $\mu(A) = \mu(A\backslash x_i) = 0$ or $\mu(A) = \mu(A\backslash x_i) = 1$.

Lemma 10. *Consider $A, B \subset X$ such that $|A| = |B|$. Then, the number of vertices in any facet defined by $\mu(A) = \mu(A\backslash x_i), x_i \in A$ is the same as in any facet defined by $\mu(B) = \mu(B\backslash x_j), x_j \in B$.*

Proof: Consider a permutation $\sigma : X \to X$ such that $\sigma(x_i) = x_j$ and $\sigma(A) = B$. As $|A| = |B|$, it is always possible to define such permutation.

Let us denote by $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) the set of vertices in the facet defined by $\mu(A) = \mu(A\backslash x_i), x_i \in A$ (resp. $\mu(B) = \mu(B\backslash x_j), x_j \in B$). Consider the mapping S_σ defined by Equation (1). By Theorem 2, S_σ is an isometry and it maps $\mathcal{C}_1$ in $\mathcal{C}_2$, whence the result holds. ■

Lemma 11. *Consider $A, B \subset X$ such that $|B| = |X| - |A| + 1$. Then, the number of vertices in any facet defined by $\mu(A) = \mu(A\backslash x_i), x_i \in A$ is the same as in any facet defined by $\mu(B) = \mu(B\backslash x_j), x_j \in B$.*

Proof: By the previous lemma, it suffices to prove the result for $B = X\backslash(A\backslash x_i)$ and $j = i$. Let μ be a vertex such that $\mu(A) = \mu(A\backslash x_i) = 0$ and consider the dual measure $\bar{\mu}$ of μ. Then,

$$\bar{\mu}(X\backslash A) = 1 - \mu(A) = 1.$$

As $\bar{\mu} \in \mathcal{FM}(X)$, it follows by monotonicity that $\bar{\mu}(X\backslash(A\backslash x_i)) = 1$.

The same can be done to show that for any μ satisfying $\mu(A) = \mu(A\backslash x_i) = 1$, we have $\bar{\mu}(X\backslash(A\backslash x_i)) = \bar{\mu}(X\backslash A) = 0$. As the dual application is a bijective mapping in $\mathcal{FM}(X)$, (it is an isometry as noted in Theorem 2), the result holds. ■

For $|A| = 1$, the following can be stated:

Lemma 12. *Suppose $A = \{x_i\}$. The number of vertices in the corresponding facet of $\mathcal{FM}(X)$ is $D_n - D_{n-1} - 1$.*

Proof: As $\mu(\emptyset) = 0$, the only vertices in the facet are those such that $\mu(x_i) = 0$. This means that $\{x_i\}$ is not a minimal subset of μ. Let us denote by $\mathcal{C}_1$ the set of antichains satisfying that $\{x_i\}$ is a minimal subset and by $\mathcal{C}_2$ the set of antichains in $X\backslash x_i$ and the empty antichain. Consider the mapping

$$\begin{array}{cccc} f: & \mathcal{C}_1 & \to & \mathcal{C}_2 \\ & \{A_1, ..., A_r, \{x_i\}\} & \mapsto & \{A_1, ..., A_r\} \end{array}$$

Then, f is well-defined because $x_i \notin A_j, \forall j = 1, ..., r$; moreover, f is bijective. As $|\mathcal{C}_2| = D_{n-1} + 1$, the number of vertices in the facet is $D_n - D_{n-1} - 1$. ■

Joining Lemma 11 and Lemma 12, we conclude that the number of vertices in the facet defined by $\mu(X\backslash x_i) = 1$ is $D_n - D_{n-1} - 1$.

From Lemma 10, we know that the number of vertices in a facet only depends on the cardinality of the set defining it. Let us denote $A_i := \{x_1, \ldots, x_i\}$, $i = 1, \ldots, n$ and $A_0 := \emptyset$. We will call the facet defined by $\mu(A_i) = \mu(A_{i-1})$, $i = 1, \ldots, n$, the **A_i-facet**. The number of vertices in the A_i-facet will be denoted by F_i. From Lemmas 11 and 12, we already know that $F_1 = F_n = D_n - D_{n-1} - 1$ and that $F_i = F_{n-i+1}$ for $i = 1, \ldots, n$. The following result shows further relationship between the F_i numbers and the Dedekind numbers.

Theorem 9. *For every n it holds*

$$F_1 + F_2 + \ldots + F_n = (n-1)D_n.$$

Proof: Consider μ, any extreme of $\mathcal{F}M(X)$ with $|X| = n$. We know that $\mu(A_0) = 0$ and $\mu(A_n) = 1$. Hence, there exists $k \geq 1$ such that $\mu(A_0) = \mu(A_1) = \ldots = \mu(A_{k-1}) = 0$ and $\mu(A_k) = \mu(A_{k+1}) = \ldots = \mu(A_n) = 1$. Thus, μ is in every A_i-facet with the only exception of the A_k-facet. That is, every extreme of $\mathcal{FM}(X)$ is in exactly $n-1$ A_i-facets. The number of extremes of $\mathcal{FM}(X)$ is D_n, hence the result. ■

In Table 4 we give the exact values of the F_i numbers for some values of $|X|$. The values for $|X| = 2, 3, 4$ can be deduced from the previous results. A computer program was implemented to explicitly count the vertices in each facet and supply the values for $|X| = 5, 6$.

	F_1	F_2	F_3	F_4	F_5	F_6
$\|X\| = 2$	2	2	-	-	-	-
$\|X\| = 3$	13	10	13	-	-	-
$\|X\| = 4$	147	102	102	147	-	-
$\|X\| = 5$	7412	5739	4014	5739	7412	-
$\|X\| = 6$	7820772	7240284	4509824	4509824	7240284	7820772

Table 4: Values of F_i

Finally, let us study other properties of the graph of $\mathcal{FM}(X)$.

Definition 3. *[22] A convex polytope is* **combinatorial** *if it satisfies the following conditions:*

- *All its vertices are $\{0, 1\}$-valued.*
- *If vertices x and y are not adjacent, then there exist two other vertices u and v such that $x + y = u + v$.*

Then, as the extreme points of $\mathcal{FM}(X)$ are the $\{0, 1\}$-valued measures, applying Corollary 3, we have:

Corollary 13. *$\mathcal{FM}(X)$ is a combinatorial polytope.*

By Lemma 2, the result also holds for the convex hull of all monotone boolean functions.

A graph is **Hamilton connected** if every pair of distinct nodes is joined by a Hamilton path. For combinatorial polyhedra, the following can be shown:

Theorem 10. *[22] Let G be the graph of a combinatorial polytope. Then G is either a hypercube or else is Hamilton connected.*

Then, the following holds:

Corollary 14. *The graph of $\mathcal{FM}(X)$ is Hamilton connected for $|X| \neq 2$.*

Proof: Remark that $\mathcal{FM}(X)$ is a hypercube for $n = 1$ and $n = 2$. For $n = 1$, the graph of $\mathcal{FM}(X)$ is trivially Hamilton connected. Moreover, it is easy to see that the graph is not Hamilton connected for $n = 2$.

If $n > 2$, it suffices from Theorem 10 to show that the graph of $\mathcal{FM}(X)$ is not an hypercube. But this holds, as the hypercube of dimension n has diameter n; from Theorem 8, the diameter of $\mathcal{FM}(X)$ is 3 if $|X| \geq 3$. Thus, if $\mathcal{FM}(X)$ is a hypercube, it must be the 3-dimensional one. On the other hand, the hypercube of dimension 3 has 8 vertices, and this is not a Dedekind number. ■

For the convex hull of monotone boolean functions, we can adapt the previous proof to conclude that the corresponding graph is Hamilton connected for any cardinality. For $|X| = 2$, note that it is not an hypercube, as there are two more measures than in the case of fuzzy measures.

4 Conclusions and open problems

In this paper we have studied some properties of the polytpe $\mathcal{FM}(X)$. Many of the results that we have obtained apply also to the convex hull of monotone boolean functions.

We have characterized from a mathematical point of view whether two vertices of $\mathcal{FM}(X)$ are adjacent. Moreover, we have proved that given two vertices, it can be computed in polynomial time if they are adjacent. We have also found all the possibilities of expressing the sum of two vertices of $\mathcal{FM}(X)$ that are not adjacent as sum of two other vertices.

Next, we have studied the edges and the facets of $\mathcal{FM}(X)$. We have proved that the probability of two vertices chosen at random are adjacent tends to 0 when the cardinality of X grows. For the facets, we have proved that the number of vertices in a facet depends on the cardinality of the subset defining the facet. We have also shown that there seems to be a duality relationship for the facets.

We have proved that $\mathcal{FM}(X)$ has diameter 3 when $|X| \geq 3$ and that two vertices chosen at random are at distance 2 with probability tending to 1 when $|X|$ tends to infinity.

Finally, we have shown that $\mathcal{FM}(X)$ is combinatorial, whence we have concluded that the graph of this polytope is Hamilton connected for $|X| \neq 2$.

We think that these results can shed light on the structure of $\mathcal{FM}(X)$ and the convex hull of monotone boolean functions. Moreover, these results could be interesting in the problem of identifying a non-additive measure. For example, if we consider the identification through genetic algorithms, we know that the set of vertices cannot be used as the initial population [6]. However, it could be interesting to study the performance of the algorithm if we consider a subset of vertices such that two vertices in this initial population are not adjacent to each other.

When dealing with non-additive measures, we are usually restricted to a subset of $\mathcal{FM}(X)$ satisfying an additional property. An interesting subfamily of non-additive measures are the so-called k-additive measures, where k varies in $\{1, ..., n-1\}$ [16]. These measures constitute a gap between probabilities and general non-additive measures. Let us denote by $\mathcal{FM}^k(X)$ the set of all non-additive measures being at most k-additive. We have proved in [21] that, unexpectedly, the set of vertices of this polytope is not the set of boolean functions in $\mathcal{FM}^k(X)$. The results obtained in this paper could help us to study the structure of these vertices.

Finally, there are some problems that could be interesting to study related to $\mathcal{FM}(X)$. In this sense, it might be interesting to study more deeply the number of vertices in a facet. Another problem is to determine the number of adjacent vertices to a given extreme point of $\mathcal{FM}(X)$. For this, more research is needed.

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