Adjacency on the order polytope with applications to the theory of fuzzy measures

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Abstract

In this paper we study the adjacency structure of the order polytope of a poset. For a given poset, we determine whether two vertices in the corresponding order polytope are adjacent. This is done through filters in the original poset. We also prove that checking adjacency between two vertices can be done in quadratic time on the number of elements of the poset. As particular cases of order polytopes, we recover the adjacency structure of the set of fuzzy measures and obtain it for the set of p-symmetric measures for a given indifference partition; moreover, we show that the set of p-symmetric measures can be seen as the order polytope of a quotient set of the poset leading to fuzzy measures. From this property, we obtain the diameter of the set of p-symmetric measures. Finally, considering the set of p-symmetric measures as the order polytope of a direct product of chains, we obtain some other properties of these measures, as bounds on the volume and the number of vertices on certain cases.

Key words: Order polytope, adjacency, fuzzy measures, p-symmetric measures

1. Introduction

Fuzzy measures (also known as capacities or non-additive measures) are a generalization of probability distributions. More concretely, they are measures in which the additivity axiom has been relaxed to a monotonicity condition. This extension is needed in many practical situations, in which additivity is too restrictive. Fuzzy measures have proved themselves to be a powerful tool in many different fields, as Decision Theory, Game Theory, and many others (see [23] for a review of theoretical and practical applications of fuzzy measures).

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However, despite all these advantages, the practical application of fuzzy measures is limited by the increased complexity of the measure. If we have a finite space of cardinality n, only $n-1$ values are needed in order to completely determine a probability, while $2^n - 2$ coefficients are needed to define a fuzzy measure on the same referential. This exponential growth is the actual *Achilles' heel* of fuzzy measures.

With the aim of reducing this complexity several subfamilies have been defined. In these families some extra restrictions are added in order to decrease the number of coefficients but, at the same time, keep the modelling capabilities of the measures. Examples of subfamilies include the λ-measures [47], the k-intolerant measures [30], the k-additive measures [22], the p-symmetric measures [37], the decomposable measures [19], etc.

Let us focus on p-symmetric measures. It has been pointed out in [34] that p-symmetric measures and general fuzzy measures share similar properties; for example, both of them are convex polytopes, in both cases the vertices are $\{0, 1\}$-valued measures and the set of isometries follows the same structure. Thus, it makes sense to investigate the reason of these similarities. As it will be shown below, the reason lies in the fact that both families can be seen as order polytopes. Therefore, we can study general properties of order polytopes and the results apply to general fuzzy measures, p-symmetric measures and any other subfamily coming from an order polytope. The use of more general results to solve problems on capacities has been already used in [17], in which extensions on games to a vector space allow to decompose the core as a Minkowski difference of cores.

Our aim in this paper is to study the adjacency structure of order polytopes. These polytopes are associated with a finite partially ordered set (poset) and have been studied [44, 31] for their importance on the research of the complexity of enumeration problems [11, 8], and in sorting with partial information [26]. These polytopes are strongly related to the poset ideals and filters, which in turn have wide applications in distributed computing [25, 33, 38], algorithmic combinatorics [21], and discrete optimization and operations research [45].

As proved in [32, 40], the problem of determining non-adjacency of vertices of a polytope is, in some cases, NP-complete (see [20] for a definition of NP-complete problems and related notions). However, in this paper we give a necessary and sufficient condition of adjacency on order polytopes which proves that the problem of determining adjacency can be solved, in this case, in quadratic time on the number of elements of the poset (see Corollary 7 below).

In a previous paper [15], we have characterized the adjacency structure of the set of fuzzy measures. This was related to the problem of practical identification of fuzzy measures from sample information [14] and to the problem of determining the extreme points of the family of fuzzy measures being at most k-additive [35]. More concretely, in [35] it is shown that there are vertices of the family of fuzzy measures being at most k-additive that are not $\{0, 1\}$-valued measures for $k \geq 3$; these vertices come from convex combinations of vertices of general fuzzy measures that are not in the family. For example, it can be seen that the vertices of the family of fuzzy measures being at most $(n-1)$-additive (where

n is the cardinality of the referential set) that are not $\{0,1\}$-valued are convex combinations of exactly two adjacent vertices of the set of general measures that are not in the family [16].

The results of this work generalize those in [15] and provide the adjacency structure of the family of p-symmetric measures as a particular case. As an example of application, we obtain the diameter of the family of p-symmetric measures; we also derive some other properties of this subfamily seen as an order polytope.

The paper is organized as follows. In the following section, we introduce the concept of order polytope and other notions that we will need throughout the paper. In Section 3 we deal with the problem of characterizing adjacency; we also prove that this can be solved in quadratic time. In Section 4 we study the diameter of the order polytope. Next, in Section 5 we study the order polytope coming from the quotient under an equivalence relation. Finally, we apply these results for the special case of p-symmetric measures in Section 6; we obtain the diameter of this polytope, and study the number of vertices and its volume. We finish with the conclusions and open problems.

2. Basic concepts

In this section, we recall some usual notions from the theory of ordered sets and we fix some notation. For an in-depth study consult [6, 7].

In this paper we deal with a finite **poset** $(P, \preceq)$ of m elements. We will denote the subsets of P by capital letters $A, B, ...$ and also $A_1, A_2, ...$; elements of P are denoted x, y, and so on. In particular, if any two elements in the poset are comparable, then we are dealing with a total order and P is a **chain**. Reciprocally, if no pair of elements can be compared, the poset is called an **antichain**. The **width** of P is the size of the largest subset of P which forms an antichain.

An **upper (resp. lower) semilattice** is a poset in which any pair of elements has a least upper (resp. maximal lower) bound.

Given a poset $(P, \preceq)$, we define the **dual** poset $(\overline{P}, \preceq')$ as another poset with the same underlying set and satisfying

$$x \preceq y \text{ in } P \Leftrightarrow y \preceq' x \text{ in } \overline{P}.$$

If Q is a subset of P, it inherits a structure of poset from the restriction of $\preceq$ to Q. In this case, we say that Q is a **subposet** of P.

A subset F of P is a **filter** (or *upper set* [7] or *final segment* [1]) if for any $x \in F$ and any $y \in P$ such that $x \preceq y$, it follows that $y \in F$ (notice that the empty set is always a filter). For a filter F, there are elements $x \in F$ such that for any $y \in F, x \neq y$, then $y \not\preceq x$; these elements are called *minimal elements* of the filter. Remark that these minimal elements determine an antichain on P. Moreover, the set of minimal elements characterizes the filter. Then, there is a bijection between filters and antichains. The dual notion of a filter is an **ideal**, i.e., a set that contains all the lower bounds of its elements. Filters on a poset are used in [27] to determine a triangulation of the order polytope; from this result, it is possible to define a product of fuzzy measures on product spaces.

Given two filters F_1 and F_2 of P, we can define $F_1 \cup F_2$ and $F_1 \cap F_2$ as the usual union and intersection of subsets. It is trivial to check that $F_1 \cup F_2$ and $F_1 \cap F_2$ are also filters in P. In fact, the set of all filters of P forms a lattice under set inclusion called the **filter lattice** [6] or *tail lattice* [4] of P (see Figure 10 below). This structure plays a prominent role in the adjacency of vertices of the order polytope, as it will become clear in next section.

A special type of filters is the family of the so-called **principal filters**; these filters are those generated by an element. That is, for $x \in P$, the principal filter of x is defined by

$$F_x := \{y \in P : x \preceq y\}.$$

The **comparability graph** $G(P)$ associated to the poset P is the (undirected) graph whose nodes are the elements of P and with edges between comparable elements (i.e., there is an edge connecting x and y if either $x \prec y$ or $y \prec x$, where $x \prec y$ means that $x \preceq y$ and $x \neq y$). We say that P is **connected** if $G(P)$ is connected.

Given a poset $(P, \preceq)$ with m elements, it is possible to associate to P, in a natural way, a polytope $O(P)$ in $\mathbb{R}^m$, called the **order polytope** of P (cf. [44]). The polytope $O(P)$ is formed by the m-tuples f of real numbers indexed by the elements of P satisfying

- $0 \leq f(x) \leq 1$ for every x in P
- $f(x) \leq f(y)$ whenever $x \preceq y$ in P.

Thus, the polytope $O(P)$ consists in (the m-tuples of images of) the order-preserving functions from P to $[0, 1]$.

It is a well-known fact [44] that $O(P)$ is a 0/1-polytope, i.e., its extreme points are all in $\{0, 1\}^m$. In fact, it is easy to see that the extreme points of $O(P)$ (points of $O(P)$ that cannot be put as convex combination of two other points of $O(P)$) are exactly the (characteristic functions of the) filters of P [27].

Let us explain these concepts through two examples on fuzzy measures: Consider $X = \{x_1, ..., x_n\}$ a finite referential set. The set of **non-additive measures** [18], **fuzzy measures** [46] or **capacities** [12] over X, denoted by $\mathcal{FM}(X)$, is the set of functions $\mu : \mathcal{P}(X) \to [0, 1]$ satisfying

- $\mu(\emptyset) = 0, \mu(X) = 1.$
- $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{P}(X)$ such that $A \subseteq B$.

An example of fuzzy measures is the so-called *unanimity games* defined for $A \subseteq X$, $A \neq \emptyset$ by

$$u_A(B) := \begin{cases} 1 & \text{if A} \subseteq \text{B} \\ 0 & \text{otherwise} \end{cases}$$

With some abuse of notation, we also define

$$u_\emptyset(B) := \begin{cases} 1 & \text{if B} \neq \emptyset \\ 0 & \text{if B} = \emptyset \end{cases}$$

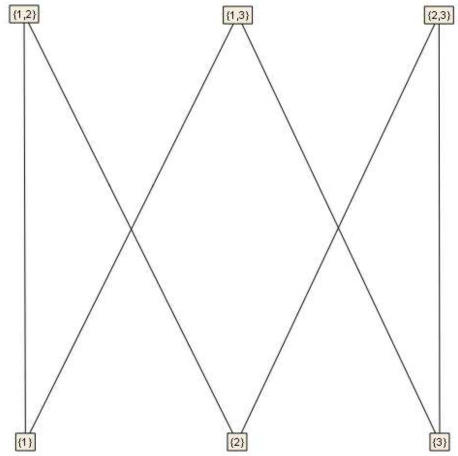

Figure 1: Poset structure when $|X| = 3$.

Figure 2: Filter F_1

From the point of view of order polytopes, $\mathcal{FM}(X)$ is the order polytope of the poset $(P, \preceq)$ where $P = \mathcal{P}(X) \backslash \{X, \emptyset\}$ and $\preceq$ is the inclusion between subsets. The unanimity game $u_A, A \neq \emptyset, X$ can be associated with the vertex given by the principal filter F_A, while $u_\emptyset$ corresponds to the filter P and u_X to $\emptyset$. Figure 1 shows the poset for $|X| = 3$. Figures 2, 3 and 4 show examples of other filters of the poset shown in Figure 1 that will serve us in order to clarify the results in the paper.

Another example is the family of p-symmetric measures [37]. The concept of p-symmetric measure appears as a middle term between symmetric measures

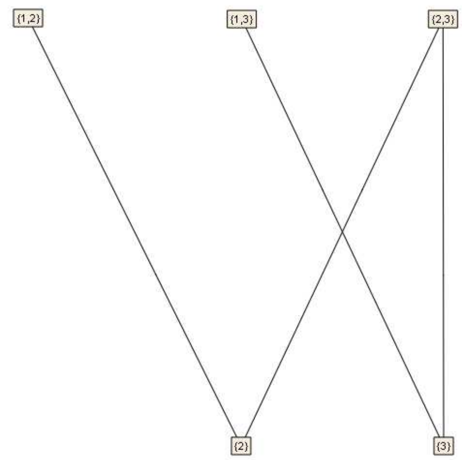

Figure 3: Filter F_2

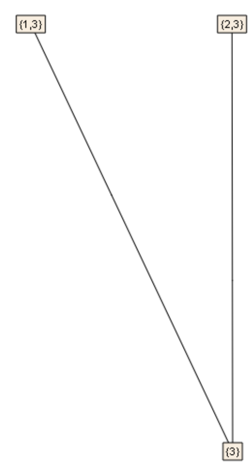

Figure 4: Filter F_3

and general fuzzy measures.

Symmetric measures are those whose values depend only on the cardinality, and they can be interpreted as all elements having the same behavior. The definition of p-symmetric measures is based on the concept of subsets of indifference. Subsets of indifference are subsets of X whose elements have the same behavior. From a mathematical point of view, A is a **subset of indifference** if $\forall B_1, B_2 \subseteq A$, $|B_1| = |B_2|$, we have

$$\mu(B_1 \cup C) = \mu(B_2 \cup C), \forall C \subseteq X \backslash A.$$

Thus, the idea of p-symmetric measures is to divide X in p subsets of indifference. More concretely, we say that a fuzzy measure μ is **p-symmetric** if the coarsest partition of the universal set X in subsets of indifference is $\{A_1, ..., A_p\}, A_i \neq \emptyset, \forall i \in \{1, ..., p\}$.

The existence and unicity of this partition has been proved in [36]. We will denote by $\mathcal{FM}(A_1, ..., A_p)$ the set of fuzzy measures for which A_i, $i = 1, ..., p$, is a subset of indifference (but not necessarily being p-symmetric! Indeed, any symmetric measure belongs to $\mathcal{FM}(A_1, ..., A_p)$). It can be easily seen that $\mathcal{FM}(A_1, ..., A_p)$ is a convex polytope for a fixed partition $\{A_1, ..., A_p\}$.

As all elements in a subset of indifference have the same behavior, when dealing with a fuzzy measure in $\mathcal{FM}(A_1, ..., A_p)$, we only need to know the number of elements of each A_i that belong to a given subset C of the universal set X. Therefore, the following result holds:

Lemma 1. *[37] If $\{A_1, ..., A_p\}$ is a partition of X, then in order to define a measure in $\mathcal{FM}(A_1, ..., A_p)$, any $C \subseteq X$ can be identified with a p-dimensional vector $(c_1, ..., c_p)$ with $c_i := |C \cap A_i|$.*

Then, the set $\mathcal{FM}(A_1, ..., A_p)$ can be seen as the order polytope of the poset $(P(A_1, ..., A_p), \preceq)$, where

$$P(A_1, ..., A_p) := \{(i_1, ..., i_p) : i_j \in \{0, ..., |A_j|\}, i, j \in \mathbb{Z}\} \backslash \{(0, ..., 0), (|A_1|, ..., |A_p|)\},$$

and $\preceq$ is given by $(c_1, ..., c_p) \preceq (b_1, ..., b_p) \Leftrightarrow c_i \leq b_i, i = 1, ..., p$.

Moreover, we will show in Section 6 that this poset is a quotient poset of the corresponding poset leading to $\mathcal{FM}(X)$.

As a consequence, the vertices in these two subfamilies are included in the set of $\{0, 1\}$-valued measures, thus recovering the results obtained in [35].

However, there are other subfamilies of fuzzy measures that are not order polytopes. This is the case of the 2-monotone measures; the results follows from the fact that there are vertices of the polytope of 2-monotone measures that are not $\{0, 1\}$-valued [42].

Given an order polytope $O(P)$, the **skeleton** of $O(P)$ is the graph whose vertices are the vertices of $O(P)$ and such that there is an edge connecting two vertices if they are adjacent vertices of $O(P)$. Figure 5 shows the skeleton of $\mathcal{FM}(X)$ when $|X| = 3$; in this figure, filters are noted by their corresponding antichains.

Given two vertices of $O(P)$, we define the distance between them as the number of edges of the minimal path connecting them in the skeleton of $O(P)$. The **diameter** of $O(P)$ is the maximal distance between two vertices.

3. Adjacency on the order polytope

This section is devoted to the study of sufficient and necessary conditions for two (characteristic functions of) filters to be adjacent as vertices of $O(P)$; it will be shown that they all derive from order-theoretic properties of P. From

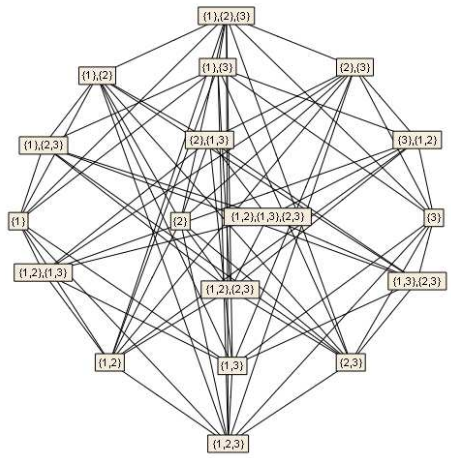

Figure 5: Adjacency structure of $\mathcal{FM}(X)$ ($|X| = 3$)

now on, in order to simplify the notation, we will identify the filters of P with their characteristic functions (and thus, with the vertices of $O(P)$). We start with the following lemma.

Lemma 2. *If F_1 and F_2 are filters of P, then*

$$\frac{1}{2}(F_1 + F_2) = \frac{1}{2}(F_1 \cup F_2) + \frac{1}{2}(F_1 \cap F_2).$$

Proof: Take x in P. If $x \in F_1 \cap F_2$ then $x \in F_1 \cup F_2$ and

$$\frac{1}{2}(F_1 \cup F_2)(x) + \frac{1}{2}(F_1 \cap F_2)(x) = \frac{1}{2} + \frac{1}{2} = \frac{1}{2}F_1(X) + \frac{1}{2}F_2(X).$$

If $x \notin F_1 \cup F_2$ then

$$\frac{1}{2}(F_1 \cup F_2)(x) + \frac{1}{2}(F_1 \cap F_2)(x) = 0 = \frac{1}{2}F_1(X) + \frac{1}{2}F_2(X).$$

If $x \in F_1$ but $x \notin F_2$ then

$$\frac{1}{2}(F_1 \cup F_2)(x) + \frac{1}{2}(F_1 \cap F_2)(x) = \frac{1}{2} + 0 = \frac{1}{2}F_1(X) + \frac{1}{2}F_2(X).$$

The case when $x \in F_2$ but $x \notin F_1$ is analogous. ■

Then, the following holds:

Corollary 1. *A necessary condition for F_1 and F_2 to be adjacent vertices in $O(P)$ is that either $F_1 \subset F_2$ or $F_2 \subset F_1$.*

Note however that the condition is not necessary. An example showing this can be found in [15] for the special case of fuzzy measures.

Now, we are in conditions of stating and proving the main theorem of this section.

Theorem 1. *If F_1 and F_2 are filters of P and $F_1 \subset F_2$, then F_1 and F_2 are adjacent vertices in $O(P)$ if and only if $F_2 \setminus F_1$ is a connected subposet of P.*

Proof: $\Rightarrow$) Suppose that $F_2 \setminus F_1$ is not connected. Then, it is possible to obtain A_1 and A_2 two disjoint subsets of $F_2 \setminus F_1$, both of them nonempty and determining a partition of $F_2 \setminus F_1$ such that if $x \in A_1$ and $y \in A_2$, then x and y are incomparable.

We define $H_1 := F_1 \cup A_1$ and $H_2 := F_1 \cup A_2$. Let us show that thus defined, H_1 and H_2 are filters.

Consider $x \in H_1$ and take y such that $x \preceq y$. If $x \in F_1$, then $y \in F_1 \subseteq H_1$ as F_1 is a filter. Otherwise, $x \in A_1 \subseteq F_2$, whence $y \in F_2$; if $y \in F_1$ we are done; otherwise, $y \in F_2 \setminus F_1$ and $y \notin A_2$ as $x \in A_1$ and elements in A_1 and A_2 are not comparable; thus, $y \in A_1 \subseteq H_1$. Similarly, H_2 is a filter.

Now, both H_1 and H_2 are different from F_1 and F_2 and

$$\frac{1}{2}(F_1 + F_2) = \frac{1}{2}(H_1 + H_2)$$

Consequently, F_1 and F_2 are not adjacent.

$\Leftarrow$) Suppose now that F_1 and F_2 are not adjacent. Then, there exist filters $H_1, \ldots, H_k$ not all equal to F_1 or F_2 and $\beta \in (0,1)$ such that

$$\beta F_1 + (1-\beta) F_2 = \sum_{i=1}^{k} \lambda_i H_i, \; \sum_{i=1}^{k} \lambda_i = 1, \; \lambda_i > 0, \; \forall i = 1, ..., k.$$

Clearly, $F_1 \subseteq H_i \subseteq F_2$ for all i. We can suppose without loss of generality that $H_1 \neq F_1, F_2$. Define $L_1 := H_1 \setminus F_1$ and $L_2 := F_2 \setminus H_1$. By construction, L_1 and L_2 are nonempty subsets. If $F_2 \setminus F_1$ is connected, then there exist $x \in L_1$ and $y \in L_2$ such that either $x \prec y$ or $y \prec x$.

If $x \prec y$, then $y \in H_1$, since H_1 is a filter, contradicting the fact that L_2 contains no elements from H_1. Hence, $y \prec x$.

Since $\lambda_1 > 0$, $x, y \in F_2 \setminus F_1$, $H_1(y) = 0$ and from the monotonicity of the filters H_i, we have

$$1 - \beta = (\beta F_1 + (1-\beta) F_2)(y) = (\sum_{i=1}^{k} \lambda_i H_i)(y) = (\sum_{i=2}^{k} \lambda_i H_i)(y) \leq$$

$$(\sum_{i=2}^{k} \lambda_i H_i)(x) < \lambda_1 + (\sum_{i=2}^{k} \lambda_i H_i)(x) = (\beta F_1 + (1-\beta) F_2)(x) = (1-\beta),$$

a contradiction. Therefore, $F_2 \setminus F_1$ is not connected and the result follows. ■

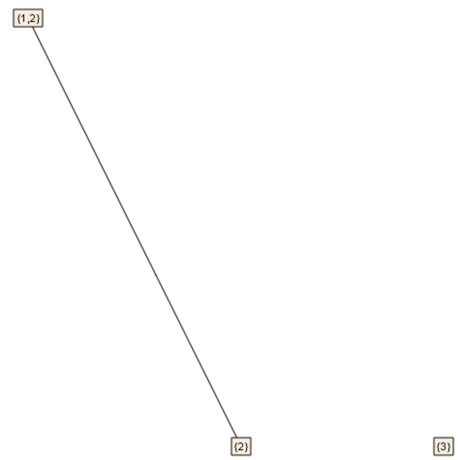

Figure 6: Poset $F_2 \setminus F_1$

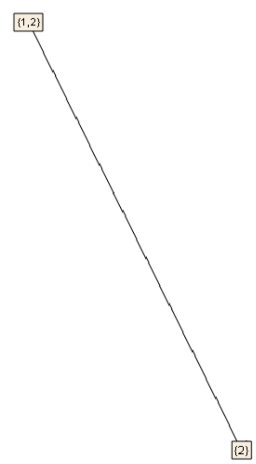

Figure 7: Poset $F_2 \setminus F_3$

Figures 6 and 7 show the posets $F_2 \setminus F_1$ and $F_2 \setminus F_3$. Then, F_2 and F_1 are not adjacent, while F_2 and F_3 are.

As a consequence, the following can be proved:

Corollary 2. *Two filters F_1 and F_2 of P are adjacent vertices in $O(P)$ if and only if their symmetric difference $(F_1 \setminus F_2) \cup (F_2 \setminus F_1)$ is connected.*

Proof: Clearly $(F_1 \setminus F_2) \cup (F_2 \setminus F_1)$ can only be connected if either $F_1 \subset F_2$ or $F_2 \subset F_1$. The result follows then from Theorem 1. ■

Notice that this condition of adjacency is analogous to that obtained by Chvátal (see [13]) for other types of 0/1-polytopes.

In [15], we have shown that, surprisingly, for $\mathcal{FM}(X)$, the unanimity games $u_\emptyset$ and u_X are adjacent if $|X| > 2$, but not when $|X| = 2$. This result can be explained with the following corollary, another immediate consequence of Theorem 1 (notice that $u_\emptyset$ corresponds to P while u_X corresponds to $\emptyset$).

Corollary 3. *The filters P and $\emptyset$ are adjacent if and only if P is connected.*

If we consider two principal filters, it follows from Theorem 1 that adjacency is equivalent to comparability, as stated in the following result.

Corollary 4. *If F_x and F_y are principal filters generated by two different elements x and y, then F_x and F_y are adjacent if and only if x and y are comparable.*

Proof: Consider $x, y \in P$; then F_x and F_y are comparable if and only if so are x and y. Moreover, if for instance $F_y \subset F_x$, then $F_x \setminus F_y = \{z \in P : x \preceq z, y \not\preceq z\}$, which is always connected (x is its minimum element). ■

From Lemma 2, Corollary 1 and from the "only if" part of the proof of Theorem 1, we also have the following important consequences.

Corollary 5. *If F_1 and F_2 are non-adjacent filters of P, then there exist two other filters H_1 and H_2 such that*

$$\frac{1}{2}(F_1 + F_2) = \frac{1}{2}(H_1 + H_2).$$

In [39], it is defined the concept of **combinatorial polytope**. A polytope is combinatorial if

1. its vertices are $\{0, 1\}$-valued.
2. for each pair of non-adjacent vertices a, b there exist two other vertices c, d such that $a + b = c + d$.

For these polytopes, it is proved in [39] that their corresponding skeleton is either Hamilton connected (there exists a Hamiltonian path between each pair of nodes) or a hypercube. Then, as a consequence of Corollary 5, the following can be stated regarding the skeleton of $O(P)$.

Corollary 6. *The skeleton of $O(P)$ is either Hamilton connected or the graph of a hypercube.*

Figure 8 shows a Hamiltonian path in the skeleton given in Figure 5.

Let us now characterize the posets whose order polytope is a hypercube.

Proposition 1. *The skeleton of $O(P)$ is a hypercube if and only if P is an antichain.*

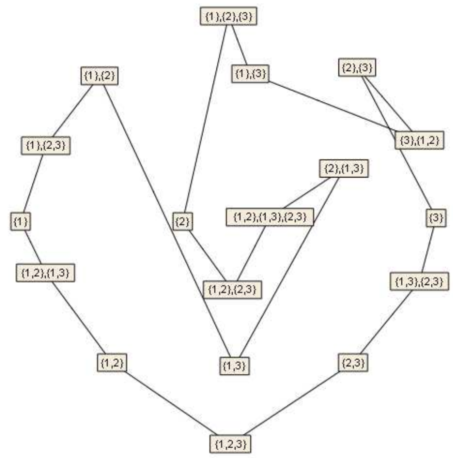

Figure 8: Example of Hamiltonian path on $\mathcal{FM}(X)$ when $|X| = 3$.

Proof: The "if" part is trivial, just considering that $O(P)$ is the set of monotone functions $f : P \to [0, 1]$. Since P is an antichain, the vertices are just $\{0, 1\}$-functions from P to $[0, 1]$ and the result holds.

Now, suppose that P is not an antichain. Consider an element x in P such that there exists y satisfying $y \prec x$. Consider F_x and F_y the principal filters generated by x and y respectively. Then $\emptyset \subset F_x \subset F_y$ and clearly $F_x \setminus \emptyset = F_x$ and $F_y \setminus \emptyset = F_y$, so $\emptyset$ is adjacent to F_x and F_y (principal filters are connected since they have a minimum). Also, F_x and F_y are adjacent by Corollary 4. Thus, there is cycle of length 3 in the skeleton of $O(P)$ which cannot be a hypercube since hypercubes have not cycles of odd length. ■

Moreover, we can prove the following dual result.

Proposition 2. *The skeleton of $O(P)$ is a complete graph (there is an edge connecting each pair of nodes) if and only if P is a chain.*

Proof: $\Rightarrow$) If P is not a chain, then there exists a pair of elements x and y such that neither $x \preceq y$ nor $y \preceq x$. Then, the principal filters F_x and F_y are incomparable and from Lemma 1 cannot be adjacent, so that there is not an edge connecting these two vertices in the skeleton of $O(P)$.

$\Leftarrow$) If P is a chain, then all filters F of P are principal. Since all elements in P are comparable, it follows from Corollary 4 that they are all adjacent. ■

Theorem 1 also allows to study the complexity of determining adjacency in $O(P)$.

Corollary 7. *Checking whether two filters F_1 and F_2 are adjacent can be done in quadratic time in the size m of P.*

Proof: We can check whether F_1 and F_2 are comparable in quadratic time on m (taking the cost of operations to check comparability of elements in P as unitary). If they are not comparable, then we are done. Otherwise, if for instance $F_1 \subset F_2$, we can construct the subposet $F_2 \setminus F_1$ also in quadratic time. Checking whether $F_2 \setminus F_1$ is connected can be done in quadratic time, for example with Prim or Kruskal algorithms (cf. [3]), whence the result. ■

We now give an alternative form of Theorem 1 in terms of antichains. This result generalizes the results found for fuzzy measures and monotone Boolean functions in [15]. We first introduce the notion of decomposability.

If P is a poset and C and D are antichains in P, we say that D is **C-decomposable** [15] if there exists a partition $\{A, B\}$ of D such that

1. $A, B \not\subseteq C$,
2. For every $a \in A, b \in B$ and d such that $a \preceq d$ and $b \preceq d$, there exists $c \in C$ such that $c \preceq d$.

Theorem 2. *Suppose P is a poset. Let $F_1 \subset F_2$ be two filters in P and let A_1 and A_2 be the antichains of minimal elements of F_1 and F_2, respectively. Then, F_1 and F_2 are adjacent if and only if A_2 is not A_1-decomposable.*

Proof: $\Rightarrow$) Suppose that A_2 is A_1-decomposable. Then, we have a partition $\{A, B\}$ of A_2 such that $A, B \not\subseteq A_1$ and for every $a \in A, b \in B$ and d such that $a \preceq d$ and $b \preceq d$, there exists $c \in A_1$ such that $c \preceq d$. Consider

$$H_A := \{x \in F_2 \setminus F_1 : e \preceq x \text{ for some } e \in A \setminus A_1\}$$

$$H_B := (F_2 \setminus F_1) \setminus H_A.$$

Clearly, H_A and H_B are both nonempty since $A, B \not\subseteq A_1$; moreover, they form a partition of $F_2 \setminus F_1$.

If $F_2 \setminus F_1$ is connected, then there exist $x \in H_A$ and $y \in H_B$ such that either $x \prec y$ or $y \prec x$. Suppose that $x \prec y$. Then, there exist $a \in A$ and $b \in B$ such that $a \preceq x \prec y$ and $b \preceq y$. Hence, by decomposability there exists c in A_1 such that $c \preceq y$. But $c \in F_1$ which is a filter, so $y \in F_1$ contradicting the fact that $H_B \subseteq F_2 \setminus F_1$. If $y \prec x$, then there exist $a \in A$ and $b \in B$ such that $a \preceq x$ and $b \preceq y \prec x$. By decomposability there exists c in A_1 such that $c \preceq x$. But $c \in F_1$ which is a filter, so $x \in F_1$ contradicting the fact that $H_A \subseteq F_2 \setminus F_1$. In either case, we conclude that $F_2 \setminus F_1$ is not connected and, by Theorem 1, F_1 and F_2 are not adjacent.

$\Leftarrow$) Now suppose that F_2 and F_1 are not adjacent. From Theorem 1 it follows that $F_2 \setminus F_1$ is not connected. Take H and K a partition of $F_2 \setminus F_1$ in nonempty subsets such that every pair of elements $x \in H$ and $y \in K$ are incomparable.

Consider H' and K' the sets of minimal elements of H and K respectively, and define

$$A := H' \cup (A_1 \cap A_2), B := K'.$$

Let us prove that $\{A, B\}$ define a A_1-decomposition of A_2.

Clearly, $A, B \not\subseteq A_1$ and $A \cup B = A_2$. Take $a \in A$ and $b \in B$. If $a \in A_1$, then $c := a \preceq d$ for any d such that $a \preceq d$ and $b \preceq d$, as required for A, B to be a A_1-decomposition of A_2. If $a \notin A_1$, then $a \in H'$ and for any d such that $a \preceq d$ and $b \preceq d$, $d \notin F_2 \setminus F_1$ (if $d \in F_2 \setminus F_1$ then d is either in H or in K, but there are not comparable pairs of elements from H and K and d is comparable to both a and b). Since F_2 is a filter, we have then that $d \in F_2 \cap F_1 \subseteq F_1$. Take c a minimal element of F_1 such that $c \preceq d$ and the result follows. ■

We can state a dual result in terms of antichains of maximal elements.

Corollary 8. *Suppose P is a poset. Let $F_1 \subset F_2$ be two filters in P and A_1, A_2 be the corresponding antichains of maximal elements of $P \setminus F_1$ and $P \setminus F_2$, respectively. Then, F_1 and F_2 are adjacent if and only if there exists a partition $\{A, B\}$ of A_1 such that*

1. *$A, B \not\subseteq A_2$,*
2. *For every $a \in A, b \in B$ and d such that $d \preceq a$ and $d \preceq b$, there is $c \in A_2$ such that $d \preceq c$.*

Proof: Consider $\overline{P}$ the dual poset of P. Then, it is straightforward to check that $P \setminus F_1$ and $P \setminus F_2$ are filters in $\overline{P}$, $P \setminus F_2 \subset P \setminus F_1$ and A_1 and A_2 are the minimal elements in $\overline{P}$ of $P \setminus F_1$ and $P \setminus F_2$, respectively. Moreover, $(P \setminus F_1) \setminus (P \setminus F_2) = F_2 \setminus F_1$. The result follows from Theorems 1 and 2. ■

For the special case of P being an upper semilattice, these results turn into the following form:

If P is an upper semilattice and C and D are antichains in P, we say that D is C-decomposable if there exists a partition A, B of D such that

1. $A, B \not\subseteq C$,
2. For every $a \in A$ and $b \in B$ there exists $c \in C$ such that $c \preceq a \vee b$.

Corollary 9. *Suppose P is an upper semilattice. Let $F_1 \subseteq F_2$ be two filters in P and A_1 and A_2 the antichains of minimal elements of F_1 and F_2, respectively. Then, F_1 and F_2 are adjacent if and only if A_2 is not A_1-decomposable.*

A similar result can be obtained for lower semilattices. Notice that this implies (cf. [15]) that if P is an upper (resp. lower) semilattice, as filters are given by the antichains of their minimal elements (resp. maximal elements of their complementaries), then we can test adjacency in polynomial (in fact quadratic) time on the width of P.

4. On the diameter of $O(P)$

In this section we will give some results regarding the diameter of $O(P)$.

Proposition 3. *The diameter of $O(P)$ is at most the width of P.*

Proof: Take F_1 and F_2 two filters in P. Consider the subposets $H := F_1 \setminus F_2$ and $K := F_2 \setminus F_1$. Let $H_1, \ldots, H_s$ be the connected components of H and $K_1, \ldots, K_t$ the connected components of K.

If $x \in H_i$ for some i and $y \in K_j$ for some j, then x and y are incomparable; for if, for instance, $x \prec y$ then $y \in F_1$ since $x \in F_1$ and F_1 is a filter, and we have a contradiction. Hence the width of P is at least $s+t$, since we can take an element from each $H_1, \ldots, H_s, K_1, \ldots, K_t$ to form an antichain.

Define $I_i := (F_1 \cap F_2) \cup (H_1 \cup \ldots \cup H_i)$ for $i = 1, \ldots, s$ $I_0 := F_1 \cap F_2$, and $L_j := (F_1 \cap F_2) \cup (K_1 \cup \ldots \cup K_j)$ for $j = 1, \ldots, t, L_0 := F_1 \cap F_2$. Let us prove that every I_i is a filter. Suppose $x \in I_i$ and take y such that $x \preceq y$. Then, $y \in F_1$ since $x \in I_i \subseteq F_1$ and F_1 is a filter. If $y \in F_2$, then $y \in (F_1 \cap F_2) \subseteq I_i$ and we are done. If $y \notin F_2$, then $y \in H_l$ for some l. But if $l > i$, we have a contradiction with $x \preceq y$ since H_l is disconnected from $H_1, \ldots, H_i$. Hence, $y \in I_i$. Analogously, every L_j is also a filter.

Remark that I_i and $I_{i+1}, i = 0, ..., s-1$ are adjacent by Theorem 1. The same holds for L_j and $L_{j+1}, j = 0, ..., t-1$. Clearly, $F_1 = I_s$, $F_2 = L_t$, $I_0 = F_1 \cap F_2 = L_0$ and $I_s \ldots, I_0, L_1, \ldots, L_t$ is a path of adjacent filters with length $s+t$. Thus, the distance between F_1 and F_2 is at most $s+t$ and the result is proved. ■

Notice that this upper bound can be far away the real diameter of $O(P)$ in some cases. For instance, for the set of fuzzy measures $\mathcal{FM}(X)$, it has been proved in [15] that the diameter is exactly 3 when $n = |X| \geq 3$, while the width is $\binom{n}{[\frac{n}{2}]}$ by the Sperner theorem [29]. On the other hand, this bound is attained for some posets. This is the case, for instance, when the poset is a disjoint union of chains, as the following result (of more general interest) proves.

Proposition 4. *If P has l connected components $P_1, \ldots, P_l$ and the diameter of $O(P_i)$ is d_i, then the diameter of $O(P)$ is $\sum_{i=1}^{l} d_i$.*

Proof:[1] If H is a filter of P then we can represent it by a tuple $(H \cap P_1, \ldots, H \cap P_l)$. We will consider, then, $(H_1, \ldots, H_l)$ with $H_i := H \cap P_i$. It is easy to see that each element H_i in the tuple is a filter of the subposet P_i. If $K = (K_1, \ldots, K_l)$ is another filter it follows easily that H and K are adjacent if and only there exists j such that $H_i = K_i$ for $i \neq j$ and H_j and K_j are adjacent in $O(P_j)$ (notice that, in fact, this shows that $O(P)$ is the Cartesian product of the graphs $O(P_i)$, see [24]). It follows readily that the diameter of $O(P)$ is the sum of the diameters of the graphs $O(P_i), i = 1, \ldots, j$. ■

In some cases it is interesting to know the distance between the maximum and minimum filters (P and $\emptyset$ respectively). Our next result deals with this particular situation.

Proposition 5. *If P has l connected components, then the distance between $\emptyset$ and P is l.*

[1] We are in debt with one of the anonymous referees for suggesting this line of proof.

Proof: Consider $P_1, \ldots, P_l$ the connected components of P. Define $Q_i := \cup_{j=1}^{i} P_j$ for $i = 0, \ldots, l$. Every Q_i is a filter, $Q_0 = \emptyset$, $Q_l = P$ and Q_i is adjacent to Q_{i+1} for $i = 0, \ldots, l-1$ by Theorem 1. Hence, $\emptyset$ and P can be connected by l edges and the distance between P and $\emptyset$ is bounded by l.

On the other hand, if we have a sequence of adjacent filters $\emptyset = F_0, \ldots, F_k = P$ with $k < l$, then there exists i such that one of the sets $F_{i+1} \setminus F_i$ (if $F_i \subset F_{i+1}$; otherwise, it should be $F_i \setminus F_{i+1}$) contains elements from more than one connected component and consequently they cannot be connected. Thus, not every F_i is adjacent to F_{i+1}, a contradiction. Therefore, the distance between P and $\emptyset$ is at least l. ■

For connected posets we can give exact values of the diameter under some assumptions.

Lemma 3. *P has a minimum element if and only if the filter P is adjacent to any other filter. Similarly, P has a maximum element if and only if the filter $\emptyset$ is adjacent to any other filter.*

Proof: We only prove the first statement in the Lemma (the other being dual). Suppose that P has a minimum element x; then, $P = F_x$. If F is a filter of P, then we have $F \subseteq P$. Also, $x \preceq y$ for every y in $P \setminus F$. Hence, $P \setminus F$ is connected and from Theorem 1 it follows that P and F are adjacent.

If P has no minimum element, then there exist two minimal elements x and y which are incomparable. Then, the filter $P \setminus \{x, y\}$ is not adjacent to P. ■

As a corollary, the following can be stated.

Corollary 10. *If P has a maximum or a minimum element, then the diameter of $O(P)$ is at most 2. In addition, if there are two incomparable elements in P, then the diameter is exactly 2.*

Proof: From the previous Lemma we have that the diameter is at most 2 (all filters are adjacent to either P or $\emptyset$). If there are two incomparable elements x and y, then their principal filters F_x and F_y are also incomparable and applying Corollary 1, they are not adjacent, whence the diameter is at least 2. ■

This last result generalizes the result obtained in [15] regarding the diameter of the polytope of monotone Boolean functions. Notice, however, that the converse of the previous result is not true. For instance, if P is an antichain of size 2, then its order polytope is the unit square which has diameter 2, but P has neither a maximum nor a minimum element.

5. On quotients of posets

Let us now deal with quotients of posets. Take a poset P. Let us consider an equivalence relation $\mathcal{R}$ on P; we denote the corresponding quotient set by Q

and the classes of Q by $[a], [b]$, and so on. The order relation $\preceq$ induces another relation $\preceq'$ on Q defined by

$$[a] \preceq' [b] \Leftrightarrow \exists a \in [a], b \in [b], a \preceq b.$$

Let us consider the transitive closure (cf. [7]) of $\preceq'$, that we will denote by $\preceq_Q$. Thus, $\preceq_Q$ is reflexive and transitive. In general, $\preceq_Q$ does not determine a partial order on Q, as it could be the case that $[a] \preceq_Q [b], [b] \preceq_Q [a]$ and $[a] \neq [b]$.

We say that $\mathcal{R}$ is **regular** if Q can be ordered in such a way that the canonical map $\pi : P \to Q$ is isotone.

Now, the following can be proved ([7], Theorem 3.1):

Theorem 3. *If $\mathcal{R}$ is an equivalence relation on P, the following statements are equivalent:*

- *$\preceq_Q$ is a partial order on Q.*
- *$\mathcal{R}$ is regular.*

From now on, we will assume that $\preceq_Q$ is in the conditions of Theorem 3, so that Q is a poset and we can build the corresponding order polytope $O(Q)$.

Let us study the relations between filters in P and Q. Consider a filter F on Q. We associate to F the subset $G := \pi^{-1}(F)$ of P. Indeed, G is a filter. For if $a \in G$ and $a \preceq b$, then $[a] \preceq_Q [b]$, whence $[b] \in F$ as F is a filter; hence, $b \in G$.

Lemma 4. *Consider F_1, F_2 two filters in Q and let us define the corresponding filters $G_1 := \pi^{-1}(F_1), G_2 := \pi^{-1}(F_2)$ in P. Then, if G_1 and G_2 are adjacent in $O(P)$, so are F_1 and F_2 in $O(Q)$.*

Proof: Suppose without loss of generality that $F_1 \subset F_2$. Then, $G_1 \subset G_2$. By Theorem 1, if G_1 and G_2 are adjacent, then $G_2 \setminus G_1$ is connected. Let us prove that $F_2 \setminus F_1$ is connected.

Take $[a], [b] \in F_2 \setminus F_1$. Consider $a \in [a], b \in [b]$; then, $a, b \in G_2 \setminus G_1$, whence there is a sequence of comparable elements connecting them. Let us denote the elements in the sequence by $c_1, ..., c_r$. As $c_i \in G_2 \setminus G_1$, $[c_i] \in F_2 \setminus F_1$. Moreover, for $i = 1, ..., r-1$, if, for instance, $c_i \preceq c_{i+1}$, it is $[c_i] \preceq_Q [c_{i+1}]$, so that we have obtained a sequence in $F_2 \setminus F_1$ connecting $[a]$ and $[b]$. ■

Notice that the reciprocal is not true. As we will show in next section, the set $\mathcal{FM}(A_1, ..., A_p)$ can be seen as the order polytope of a quotient set of $\mathcal{P}(X) \setminus \{X, \emptyset\}$. We will show in that section that the set of 1-symmetric measures has diameter 1, so every pair of filters are adjacent to each other. However, the vertices of $\mathcal{FM}(X)$ given by $\mu(A) = 1 \, \forall A \neq \emptyset$ and $\mu'(A) = 1$, $\forall |A| > 1$ are not adjacent if $|X| > 1$, whence the corresponding G and G' are such that $G \setminus G'$ is not connected.

Notice also that if P is a poset and Q is a quotient of P, it might be the case that the image of a filter of P is not a filter of Q. Consider, for example, $P = \{a, b, c, d\}$ with $a \prec b$ and $c \prec d$ as the only comparable pairs of elements.

Then, if we identify b with c we obtain $Q = \{[a], [b, c], [d]\}$, which is a chain of three elements with $[d]$ as a maximum. The set $\{a, b\}$ is a filter in P but $\pi(\{a, b\}) = \{[a], [b, c]\}$ is not a filter in Q.

Even in the case when the image of every filter of P is a filter in the quotient Q, it can happen that the images of adjacent filters in P are not adjacent in Q. For instance, consider P the poset underlying $\mathcal{FM}(\{1, 2, 3\})$ (see Figure 1) and the quotient Q obtained by identifying $\{2\}$ with $\{3\}$ and $\{1, 2\}$ with $\{1, 3\}$ in P. The corresponding poset Q is depicted in Figure 9. Then, $F_1 = P$ and $F_2 = \{\{1, 2\}\}$ are adjacent filters in P (applying Theorem 1), but $= \pi(F_1) = Q$ and $\pi(F_2) = \{(1, 1)\}$ are non-adjacent filters in Q (again by Theorem 1).

However, with the help of Lemma 4 we can obtain a result relating sequences of adjacent filters in P to sequences of adjacent filters in Q. Consider P a poset, Q a quotient of P and π the projection of P onto Q. Define the **restricted diameter** of $O(P)$ by π as the diameter of the subgraph of the skeleton of $O(P)$ formed by those vertices which are pre-image by π of filters of Q. Then we can prove the following result.

Proposition 6. *The diameter of $O(Q)$ is less than or equal to the restricted diameter of $O(P)$.*

Proof: Consider F_1 and F_2 two filters in Q. Define $G_1 := \pi^{-1}(F_1)$ and $G_2 := \pi^{-1}(F_2)$. If the restricted diameter of $O(P)$ is l then there exists a sequence $H_1, \ldots, H_j$ of adjacent filters in P such that $H_1 = G_1$ and $H_j = G_2$, with $j \leq l$ and every H_i is the preimage of some filter in Q. By Lemma 4, the sequence $\pi(H_1), \ldots, \pi(H_j)$ consists in adjacent filters of Q and connects F_1 and F_2. Thus, the distance between F_1 and F_2 is at most the same as the distance between G_1 and G_2. As it could be the case that there are sequences of adjacent filters connecting F_1 and F_2 in Q such that the corresponding preimages are not adjacent in P, the result holds. ■

6. The case of $\mathcal{FM}(A_1, ..., A_p)$

Consider a finite referential set $X = \{x_1, ..., x_n\}$ and consider a partition $\{A_1, ..., A_p\}$ of X. Consider the poset $P = \mathcal{P}(X) \backslash \{X, \emptyset\}$ and let us define $\preceq$ as the subset relation between subsets. As we have seen, the corresponding order polytope is $\mathcal{FM}(X)$. On P we consider the following equivalence relation:

$$B \mathcal{R} C \Leftrightarrow |B \cap A_i| = |C \cap A_i|, \forall i = 1, ..., p.$$

In this case, for a subset B of P, the equivalence class $[B]$ is given by the subsets C of X satisfying

$$|B \cap A_i| = |C \cap A_i|, i = 1, ..., p.$$

Thus, Q can be identified with the set of p-uples $\vec{b} = (b_1, ..., b_p) \in \mathbf{Z}^p$ with $|A_i| \geq b_i \geq 0, i = 1, ..., p$, and $\vec{b} \neq \vec{0}, (|A_1|, ..., |A_p|)$. The relation $\preceq_Q$ induced in Q is given by

$$(a_1, \ldots, a_p) \preceq_Q (b_1, \ldots, b_p) \Leftrightarrow a_i \leq b_i, i = 1, ..., p,$$

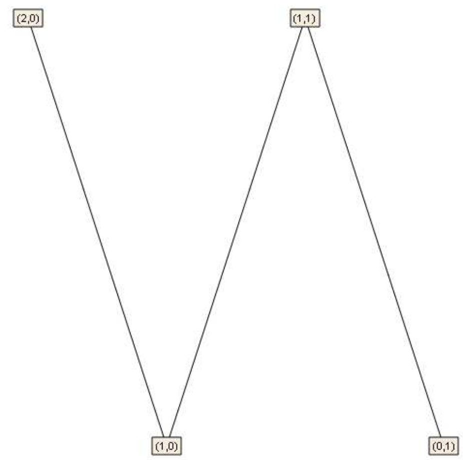

Figure 9: Example of quotient poset

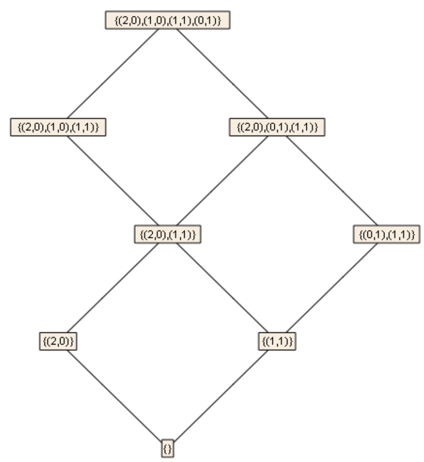

Figure 10: Example of lattice of filters

so that it satisfies the conditions of Theorem 3. Therefore, $(Q, \preceq_Q)$ is a poset and we can define the corresponding order polytope of Q. This order polytope is the set $\mathcal{FM}(A_1, ..., A_p)$. Then, p-symmetric measures for a given indifference partition $\{A_1, \ldots, A_p\}$ can be seen as the order polytope of a quotient of the poset whose corresponding order polytope is $\mathcal{FM}(X)$. Figure 9 shows the poset for $\mathcal{FM}(A_1, A_2)$ when $|A_1| = 2$ and $|A_2| = 1$. Figure 10 shows the corresponding lattice of filters.

As $\mathcal{FM}(A_1, ..., A_p)$ can also be seen as an order polytope, we can study its adjacency structure through Theorem 1.

In the following results, we deal with the diameter of $\mathcal{FM}(A_1, ..., A_p)$. We start with the simpler cases.

If $p = 1, |X| > 1$, then $\mathcal{FM}(A_1)$ is a chain. By Proposition 2, the diameter is 1. If $|X| = 1$, the diameter is 0.

Proposition 7. *If $p = 2$ and either $|A_1| = 1$ or $|A_2| = 1$, then the diameter of $\mathcal{FM}(A_1, A_2)$ is exactly 2.*

Proof: In this case, the width of the poset is clearly 2. By Proposition 3, its diameter is at most 2. Since the poset is not a chain, from Proposition 2 it follows that the diameter is exactly 2. ■

Lemma 5. *Consider $\mathcal{FM}(A_1, \ldots, A_p)$ with $|A_1| + \ldots + |A_p| = 3$. Then, its diameter is at most 3.*

Proof: We have three cases.

- If $p = 3$, then $\mathcal{FM}(A_1, \ldots, A_p) = \mathcal{FM}(\{1, 2, 3\})$ which has diameter 3 (see [15]).
- If $p = 2$, then we are in the conditions of Proposition 7, and the diameter is 2.
- If $p = 1$, then $\mathcal{FM}(A_1, \ldots, A_p)$ is a chain and has diameter 1 (see Proposition 2). ■

Proposition 8. *Consider $\mathcal{FM}(X)$ with $|X| > 3$. Consider the projection π to $\mathcal{FM}(A_1, \ldots, A_p)$. Then the restricted diameter of $\mathcal{FM}(X)$ is at most 3.*

Proof: Take two vertices F_1 and F_2 in $\mathcal{FM}(X)$ which are preimage by π of vertices of $\mathcal{FM}(A_1, \ldots, A_p)$. Denote by P the poset underlying $\mathcal{FM}(X)$. If F_1 and F_2 are both adjacent to either $\emptyset$ or P, then we are done, since $\emptyset, P$ are adjacent (P is connected since $|X| > 3$) and both are, clearly, preimage of the vertices $\emptyset, Q$ of $\mathcal{FM}(A_1, \ldots, A_p)$.

Suppose that F_1 is not adjacent to $\emptyset$ nor P. Let us prove that its restricted distance to $\emptyset$ and P is 2. In [15], we have proved that in this case there exists $i \in X$ such that the minimal subsets of F_1 are exactly $\{i, j\}$ for $i \neq j$ and $X \setminus \{i\}$. Since F_1 is the preimage of a vertex of $\mathcal{FM}(A_1, \ldots, A_p)$, then it must be the case that $\{i\}$ is one of the sets of indifference; for if i is equivalent to other element j, for any k different from i and j, it follows that $\{i, k\}$ and $\{j, k\}$ would be equivalent, whence $\{j, k\}$ would be a minimal subset of F_1, a contradiction.

Thus, if we consider the filter F_1' whose minimal subsets are $\{i\}$ and $X \setminus \{i\}$, we conclude that it is adjacent to F_1, as their difference is just one element, hence connected. Moreover, F_1' is adjacent to P by Theorem 2, as the minimal subsets of P are $\{j\}$ with $j \in X$ and $|X| > 3$. Finally, F_1' is a preimage of a vertex of $\mathcal{FM}(A_1, \ldots, A_p)$, since $\{i\}$ forms a class by itself. Thus, the restricted distance between F_1 and P is 2.

Consider now F_1'' the filter whose minimal subsets are $\{i,j\}$ for $i \neq j$. Again, the difference between F_1'' and F_1 is exactly one element and thus, they are adjacent. Also, F_1'' is adjacent to $\emptyset$ by Theorem 2. Finally, it is the preimage of a vertex of $\mathcal{FM}(A_1, \ldots, A_p)$ (namely, the filter of those subsets which contain $[i]$ and at least one other element). Thus, the restricted distance between F_1 and $\emptyset$ is 2.

Therefore, if F_2 is adjacent to either $\emptyset$ or P, we are done. Otherwise, there exists an element $k \in X$ different from i such that the minimal subsets of F_2 are $\{k,j\}$ for $k \neq j$ and $X \setminus \{k\}$. Consider the filter $G_{i,k}$ whose minimal sets are $\{\{i,j\}, \{k,j\}, j \neq i,k\}$. This filter is adjacent to both F_1 and F_2 by Theorem 2, taking into account that $|X| > 3$; moreover, $G_{i,k}$ is the preimage of a vertex of $\mathcal{FM}(A_1, \ldots, A_p)$, since it is the union of F_1 and F_2 and $\mathcal{FM}(A_1, \ldots, A_p)$ is an order polytope. Consequently, the distance between F_1 and F_2 in the restricted skeleton is 2 are the result follows. ∎

Thus, as a consequence of Proposition 6, the following can be stated.

Corollary 11. *The diameter of $\mathcal{FM}(A_1, ..., A_p)$ is at most 3.*

We study now the concrete value of the diameter for different situations. We will use the vectorial notation for the elements of the quotient set. In order to avoid hard notation, we will denote $a_i := |A_i|$.

Proposition 9. *Consider $\mathcal{FM}(A_1, ..., A_p)$. If $p \geq 3$, then the diameter is exactly 3.*

Proof: It suffices to find two filters that are at distance 3. Let us define $\vec{v}_{1i}, \vec{v}_{2i}, i = 3, ..., p$ by

$$\vec{v}_{1i}(j) := \begin{cases} 1 & \text{if } j \in \{1,i\} \\ 0 & \text{otherwise} \end{cases} \quad \vec{v}_{2i}(j) := \begin{cases} 1 & \text{if } j \in \{2,i\} \\ 0 & \text{otherwise} \end{cases}$$

Now, consider the filters

$$F_1 := \{\vec{b} | \vec{b} \geq (a_1, a_2, 0, ..., 0) \text{ or } \vec{b} \geq (0, 0, a_3, ..., a_p)\}.$$

$$F_2 := \{\vec{b} \,|\, \vec{b} \geq \vec{v}_{1i} \text{ or } \vec{b} \geq \vec{v}_{2i} \text{ for some } i \in \{3, ..., p\}\}.$$

If $p = 3$ and $|A_i| = 1, i = 1,2,3$, then $\mathcal{FM}(A_1, A_2, A_3)$ is the set of fuzzy measures on a referential of cardinality 3. In this case, we have proved in [15] that the diameter is 3.

Otherwise, F_1 and F_2 are not comparable (if necessary, we can reorder A_1, A_2, A_3 so $|A_3| > 1$). Consequently, the distance is at least 2. We will prove that these filters are at distance 3. For this, it suffices to show that for any other filter F, it cannot be adjacent to both F_1 and F_2.

Suppose that F is adjacent to F_1. We will show that F is not adjacent to F_2. As F is adjacent to F_1, then either $F_1 \subset F$ or $F \subset F_1$.

- Suppose $F_1 \subset F$. Then, $F \setminus F_1$ is connected as they are adjacent.

  Suppose F_2 and F are comparable (otherwise we are done). Remark that $F_2 \subset F$. Otherwise, $F_1 \subset F \subset F_2$, a contradiction because F_1 and F_2 are not comparable. We will show that $F \setminus F_2$ is not connected. Let us define

  $$\mathcal{C}_1 := \{\vec{c} \in F \setminus F_2 \,|\, \vec{c} \leq (a_1, a_2, 0, ..., 0)\}$$

  $$\mathcal{C}_2 := (F \setminus F_2) \setminus \mathcal{C}_1.$$

  As $F_1 \subset F$, $(a_1, a_2, 0, ..., 0) \in F \setminus F_2$, whence $\mathcal{C}_1 \neq \emptyset$. Similarly, $(0, 0, a_3, ..., a_p) \in \mathcal{C}_2$. Thus, neither $\mathcal{C}_1$ nor $\mathcal{C}_2$ are empty.

  Consider $\vec{b} \in \mathcal{C}_1$. Then, $b_3 = ... = b_p = 0$. On the other hand, for $\vec{c} \in \mathcal{C}_2$ we have that there exists $c_i > 0$ for $i \geq 3$ (otherwise $\vec{c} \in \mathcal{C}_1$, a contradiction).

  Let us show that $c_1 = c_2 = 0$. Suppose without loss of generality that $c_1 > 0$. Then, if $c_i > 0$, we have $\vec{c} \geq \vec{v}_{1i}$, whence $\vec{c} \in F_2$, a contradiction.

  But if $c_1 = c_2 = 0$, then $\vec{b}$ and $\vec{c}$ are not comparable and thus $F \setminus F_2$ is not connected.

- Suppose $F \subset F_1$. Then, $F_1 \setminus F$ is connected. Let us define

  $$\mathcal{C}_1 := \{\vec{c} \in F_1 \,|\, \vec{c} \geq (a_1, a_2, 0, ..., 0)\},\ \mathcal{C}_2 := F_1 \setminus \mathcal{C}_1.$$

  We have three possibilities:

  - Suppose $\mathcal{C}_1 \subseteq F$. Then $(a_1, a_2, 0, ..., 0) \in F$. As $(a_1, a_2, 0, ..., 0) \notin F_2$, we conclude that $F \not\subseteq F_2$. On the other hand, in order to be adjacent, they are necessarily comparable, whence it must be $F_2 \subset F$. But then, $F_2 \subset F \subset F_1$, a contradiction.
  - The case $\mathcal{C}_2 \subseteq F$ is analogous.
  - Thus, we necessarily have $\mathcal{C}_1 \not\subseteq F$ and $\mathcal{C}_2 \not\subseteq F$. We will show that in this case, F_1 and F cannot be adjacent. Let us define

    $$\mathcal{D}_1 := \{\vec{c} \in \mathcal{C}_1 \setminus F\},\ \mathcal{D}_2 := (F_1 \setminus F) \setminus \mathcal{D}_1.$$

    Notice that $(a_1, a_2, 0, ..., 0) \in \mathcal{D}_1$ as $\mathcal{C}_1 \not\subseteq F$. Moreover, $(0, 0, a_3, ..., a_p) \in \mathcal{D}_2$ as $\mathcal{C}_2 \not\subseteq F$. Then, $\mathcal{D}_1, \mathcal{D}_2 \neq \emptyset$.
    For $\vec{b} \in \mathcal{D}_1$, it is $b_1 = a_1, b_2 = a_2$. Analogously, for $\vec{c} \in \mathcal{D}_2$, it is $c_i = a_i, i = 3, ..., p$. Then, $\vec{b}$ and $\vec{c}$ cannot be compared and $F_1 \setminus F$ is not connected, a contradiction. This finishes the proof. ■

Proposition 10. *If $p = 2$ and $|A_i| \geq 2, i = 1, 2$, then the diameter of $\mathcal{FM}(A_1, A_2)$ is 3.*

Proof: As in the previous proposition, it suffices to find two filters such that no other filter is adjacent to both of them. Consider

$$F_1 := \{(b_1, b_2) \,|\, b_1 = a_1 \text{ or } b_2 = a_2\},$$

$$F_2 := \{(b_1, b_2) \,|\, b_i \geq 1,\, i = 1, 2\}.$$

If $|A_i| \geq 2, i = 1, 2$, then F_1 and F_2 are not comparable, so $d(F_1, F_2) \geq 2$.

Suppose that F is adjacent to F_1. We will show that it cannot be adjacent to F_2. As F is adjacent to F_1, then either $F_1 \subset F$ or $F \subset F_1$.

- Suppose $F_1 \subset F$. Then, $F \setminus F_1$ is connected as they are adjacent. If F and F_2 are adjacent, then $F_2 \subset F$. Otherwise, $F_1 \subset F \subset F_2$, a contradiction. We will show that $F \setminus F_2$ is not connected. Let us define

  $$\mathcal{C}_1 := \{(c_1, c_2) \in F \setminus F_2 \,|\, c_2 = 0\}, \mathcal{C}_2 := (F \setminus F_2) \setminus \mathcal{C}_1.$$

  As in the proof of the previous proposition, $(a_1, 0) \in \mathcal{C}_1$ and $(0, a_2) \in \mathcal{C}_2$, whence none of these subsets is empty.

  If $\vec{b} \in \mathcal{C}_1$, then $b_2 = 0$, and if $\vec{c} \in \mathcal{C}_2$, we have $c_2 > 0$; thus, $c_1 = 0$ (otherwise, $\vec{c} \in F_2$). Then, $\vec{b}$ and $\vec{c}$ are not comparable and therefore, $F \setminus F_2$ is not connected.

- Suppose $F \subset F_1$. Then, $F_1 \setminus F$ is connected. Let us define

  $$\mathcal{C}_1 := \{\vec{c} \in F_1 \,|\, c_1 = a_1\}, \mathcal{C}_2 := F_1 \setminus \mathcal{C}_1.$$

  We have three possibilities:

  - Suppose $\mathcal{C}_1 \subseteq F$. Then $(a_1, 0) \in F$, whence $F \not\subset F_2$. Thus, if they are adjacent, it must be $F_2 \subset F$. But then, $F_2 \subset F \subset F_1$, a contradiction.
  - The case $\mathcal{C}_2 \subseteq F$ is analogous.
  - Thus, we necessarily have $\mathcal{C}_1 \not\subseteq F$ and $\mathcal{C}_2 \not\subseteq F$. Let us define

    $$\mathcal{D}_1 := \{\vec{c} \in F_1 \setminus F \,|\, c_1 = a_1\}, \mathcal{D}_2 := (F_1 \setminus F) \setminus \mathcal{D}_1.$$

    Notice that $(a_1, 0) \in \mathcal{D}_1$ as $\mathcal{C}_1 \not\subseteq F$. Moreover, $(0, a_2) \in \mathcal{D}_2$ as $\mathcal{C}_2 \not\subseteq F$. Then, $\mathcal{D}_1, \mathcal{D}_2 \neq \emptyset$.

    For $\vec{b} \in \mathcal{D}_1$, it is $b_1 = a_1$. Analogously, for $\vec{c} \in \mathcal{D}_2$, it is $c_2 = a_2$ and $c_1 \neq a_1$. Then, $\vec{b}$ and $\vec{c}$ cannot be compared and $F_1 \setminus F$ is not connected. This finishes the proof. ■

The set of p-symmetric measures for a fixed indifference partition can also be seen as the set of order-preserving functions from a product of p chains (a chain for each subset of indifference) minus two elements ($\vec{0}$ and $(a_1, ..., a_p)$) to $[0, 1]$. Then, we can deduce several other properties from the abundant literature on this topic. For instance, a generalization of the well-known Sperner's Theorem (see [29]) gives the maximum size of minimal elements in the extreme points of the p-symmetric measures.

Theorem 4. *[2] The width of the product of chains of sizes* $|A_1|+1,\ldots,|A_p|+1$ *(and hence, the maximum number of minimal elements in a vertex of* $\mathcal{FM}(A_1,\ldots,A_p)$*) is equal to the number of t-uples of integers* $(b_1,\ldots,b_p)$ *such that* $0 \leq b_i \leq |A_i|$ *and* $b_1+\ldots+b_p = [\frac{n}{2}]$ *where* $n = |A_1|+\ldots+|A_p|$*. That width is also the coefficient of* $x^{[\frac{n}{2}]}$ *in the polynomial*

$$(1+x+x^2+\ldots+x^{|A_1|}) \times \cdots \times (1+x+x^2+\ldots+x^{|A_p|}).$$

We can also give the values of the number of vertices of $\mathcal{FM}(A_1,\ldots,A_p)$ for small values of p (the general case is a long standing open problem in Combinatorics [5]). We only have to note that these numbers are, in fact, two less (we have to exclude the antichains $\{(a_1,\ldots,a_p)\}$ and $\{(0,\ldots,0)\}$) than the numbers of antichains in the product of chains of sizes $|A_1|+1,\ldots,|A_p|+1$. Then, from the results in [5, 43] we can deduce the following result.

Theorem 5.

- *The number of 1-symmetric measures on a referential set of* n *elements is* n*.*
- *The number of vertices in* $\mathcal{FM}(A,B)$ *is given by*

$$\binom{a+b}{a} - 2$$

where $a = |A|+1$ *and* $b = |B|+1$*.*

- *The number of vertices in* $\mathcal{FM}(A,B,C)$ *is*

$$\left(\Pi_{j=0}^{a-1} \frac{\binom{c+b+j}{b}}{\binom{b+j}{b}}\right) - 2$$

where $a = |A|+1$*,* $b = |B|+1$ *and* $c = |C|+1$*.*

- *The number of vertices of* $\mathcal{FM}(A,B,C,D)$ *with* $|B| = |C| = |D| = 1$ *is*

$$48\binom{a+8}{8} - 96\binom{a+7}{7} + 63\binom{a+6}{6} - 15\binom{a+5}{5} + \binom{a+4}{4} - 2$$

where $a = |A|+1$*.*

Finally, we can estimate the volume of $\mathcal{FM}(A_1,\ldots,A_p)$ from a result by Stanley [44] that relates the volume of an order polytope $O(P)$ with the number of linear extensions of P (injective order-preserving functions from P to a chain of $|P|$ elements).

Theorem 6. *[44] The volume of* $O(P)$ *is*

$$\frac{LE(P)}{|P|!}$$

where $LE(P)$ *is the number of linear extensions of* P*.*

The number of linear extensions of the Boolean lattice and of products of chains have been studied, for instance, in [9, 10]. From the results therein we can deduce the following result.

Theorem 7. *The volume $v_{A_1,\ldots,A_p}$ of $\mathcal{FM}(A_1,\ldots,A_p)$ satisfies*

$$\frac{\Pi_{i=1}^{n-1} c_i!}{((\Pi_{i=1}^{p}|A_i|+1)-2)!} \leq v_{A_1,\ldots,A_p} \leq \frac{\Pi_{i=1}^{n-1} c_i^{c_i}}{((\Pi_{i=1}^{p}|A_i|+1)-2)!}$$

where $n = |A_1| + \ldots + |A_p|$ and c_i is the coefficient of x^i in the polynomial

$$(1 + x + x^2 + \ldots + x^{|A_1|}) \times \cdots \times (1 + x + x^2 + \ldots + x^{|A_p|}).$$

Notice that recently there has been progress in the study of algorithms for counting the number of linear extensions of a poset (see [28]) which could be used to obtain better approximations of these volumes.

7. Conclusions and open problems

In this paper we have studied the adjacency structure for order polytopes. In particular, the set of fuzzy measures can be seen as the order polytope of the poset $\mathcal{P}(X)\backslash\{X,\emptyset\}$; similarly, the set of p-symmetric measures for a fixed partition of indifference is also an order polytope.

We have obtained a characterization of adjacency for order polytopes in terms of the connection with certain graphs. A similar characterization can be obtained in terms of antichains. Therefore, the results in the paper generalize those in [15] for the polytope of fuzzy measures and the convex hull of monotone Boolean functions. We have also proved that determining whether two vertices of the order polytope are adjacent can be done in quadratic time on the number of elements of the poset.

We have also studied the diameter of the order polytope and some properties related to quotients of posets.

As an application of these results we can obtain the adjacency structure of $\mathcal{FM}(A_1,...,A_p)$. Considering $\mathcal{FM}(A_1,...,A_p)$ as an order polytope coming from a quotient set, we have obtained the diameter of the polytope. Moreover, seen as the order polytope of a product of chains, we have derived some other properties.

For future research, we have the problem of determining the adjacency graph of other subfamilies of fuzzy measures that cannot be seen as order polytopes, as the set of k-additive measures. We would also like to study which families of fuzzy measures can be written as quotients by an equivalence relation. We are also interested in studying explicit Hamiltonian paths of adjacent fuzzy measures and p-symmetric measures.

In a previous paper, we have shown the interest in determining the group of isometries of polytopes in the problem of identification [34]. We feel that these results can be applied in determining this set for order polytopes.

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