# Characterizing isometries on the order polytope with an application to the theory of fuzzy measures

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**Abstract**

In this paper we study the group of isometries over the order polytope of a poset. We provide a result that characterizes any isometry based on the order structure in the original poset. From this result we provide upper bounds for the number of isometries over the order polytope in terms of its number of connected components. Finally, as an example of application, we recover the set of isometries for the polytope of fuzzy measures and the polytope of $p$-symmetric measures when the indifference partition is fixed.

*Keywords:* Order polytope, isometries, fuzzy measures, $p$-symmetric measures.

## 1 Introduction

In this paper we deal with a special type of polytopes, the so-called order polytopes. These polytopes are associated with a finite partially ordered set (poset) and have been studied [23, 16] for their importance on the research of the complexity of enumeration problems [4, 3], and in sorting with partial information [15]. These polytopes are strongly related to the poset ideals and filters, which in turn have wide applications in distributed computing [14, 17, 11], algorithmic combinatorics [10], and discrete optimization and operations research [24]. In particular, some families of fuzzy measures can be seen as order polytopes. In fact, order polytopes are a natural generalization of the set of fuzzy measures and the set of fuzzy measures being at most $p$-symmetric with respect to a fixed partition.

The aim of this paper is to study the set of isometries of an order polytope. To our knowledge, this problem has not been addressed yet for this general case. In the case of fuzzy measures, the problem of obtaining the set of isometries is related to the problem of identification of fuzzy measures from sample information [6] and has been solved for some subfamilies in [18]. The results on this paper generalize those results.

We will prove that the set of isometries of an order polytope is strongly related to the structure of the subjacent poset. Thus, it is possible to derive any isometry in a very simple way. Besides, these results can be applied to any order polytope, no matter the structure of the corresponding poset.

The paper is organized as follows. In next section we introduce the concept of order polytope and other notions that we will need throughout the paper. In Section 3 we deal with the problem of characterizing the isometries for order polytopes; we also derive some consequences of this characterization. In Section 4 we study the subgroup of isometries satisfying $h(\emptyset) = \emptyset$. Next, in Section 5 we apply these results for the special case of $p$-symmetric measures and general fuzzy measures. Finally, we study upper bounds for the number of isometries on the order polytope of a poset in terms of the number of connected components in Section 6. We finish with the conclusions and open problems.

## 2 Order polytopes

In this sectition we recall some usual notions from the theory of ordered sets and we fix some notation. For an in-depth study, consult [1, 2].

Throughout the paper, we deal with a finite **poset** $(P, \preceq)$ (or $P$ for short) of $p$ elements. We will denote the subsets of $P$ by capital letters $A, B, ...$ and also $A_1, A_2, ...$; elements of $P$ are denoted $i, j$, and so on, and also $x, y, z, ...$ In particular, if every pair of elements in the poset are comparable, then we are dealing with a total order and $P$ is a **chain**. Reciprocally, if no pair of elements can be compared, the poset is called an **antichain**.

Given a poset $(P, \preceq)$, we define the **dual** poset $(\overline{P}, \preceq')$ as another poset with the same underlying set and satisfying

$$i \preceq j \text{ in } P \Leftrightarrow j \preceq' i \text{ in } \overline{P}.$$

If $(P, \preceq)$ is isomorphic to $(\overline{P}, \preceq')$, we say that $P$ is **autodual**.

If $A$ is a subset of $P$, it inherits a structure of poset from the restriction of $\preceq$ to $A$. In this case, we say that $A$ is a **subposet** of $P$.

A subset $F$ of $P$ is a **filter** if for any $x \in F$ and any $y \in P$ such that $x \preceq y$, it follows that $y \in F$. We will denote filters by $F_1, F_2, ...$ and also $G_1, G_2, ...$ The dual notion of a filter is an **ideal**, i.e., a set that contains all lower bounds of its elements.

Given two filters $F_1$ and $F_2$ of $P$, we can define $F_1 \cup F_2$ and $F_1 \cap F_2$ as the usual union and intersection of subsets. It is trivial to check that $F_1 \cup F_2$ and $F_1 \cap F_2$ are also filters in $P$. In fact, the set of all filters of $P$ forms a lattice under set inclusion called the **filter lattice** of $P$ (see [1]). A filter $F$ is said to be **join-irreducible** if whenever $F = G_1 \cup G_2$ for two other filters $G_1, G_2$, it implies $F = G_1$ or $F = G_2$.

A special type of filters is the family of the so-called **principal filters**; these filters are those generated by an element. That is, for $i \in P$, the principal filter of $i$ is defined by

$$i^{\uparrow} := \{j \in P : i \preceq j\}.$$

Notice that, in finite posets, principal filters are join-irreducible and any join-irreducible filter is principal. For $i \in P$, let us denote by $i^{\downarrow}$ the principal ideal of $P$ given by those elements $j$ of $P$ such that $j \preceq i$. Principal ideals are the only join-irreducible ideals in $P$.

Let us now turn to order polytopes. Given a poset $(P, \preceq)$, it is possible to associate to $P$, in a natural way, a polytope $\mathcal{O}(P)$ in $\mathbb{R}^p$, called the **order polytope** of $P$ (cf. [23]). The polytope $\mathcal{O}(P)$ is formed by the $p$-uples $f$ of real numbers indexed by the elements of $P$ satisfying

- $0 \leq f(i) \leq 1$ for every $i$ in $P$
- $f(i) \leq f(j)$ whenever $i \preceq j$ in $P$.

Thus, the polytope $\mathcal{O}(P)$ consists in (the $p$-uples of images of) the order-preserving functions from $P$ to $[0, 1]$. It is a well-known fact [23] that $\mathcal{O}(P)$ is a 0/1-polytope, i.e. its extreme points are all in $\{0, 1\}^p$. In fact, it is easy to see that the extreme points of $\mathcal{O}(P)$ are exactly the (characteristic functions of the) filters of $P$. In this sense, the extreme point whose value is 1 for any element of $P$ is identified with the filter $P$, while the extreme point whose value is 0 for any element of $P$ is identified with the filter $\emptyset$.

We can find several examples of order polytopes in the theory of fuzzy measures. Consider $X = \{x_1, ..., x_n\}$ a finite referential set. The set of **non-additive measures** [8], **fuzzy measures** [25] or **capacities** [5] over $X$, denoted by $\mathcal{FM}(X)$, is the set of functions $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ satisfying

- $\mu(\emptyset) = 0, \mu(X) = 1$.
- $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{P}(X)$ such that $A \subseteq B$.

Fuzzy measures have been applied to many different fields, as Multicriteria Decision Making, Decision Under Uncertainty and Under Risk, Game Theory, Welfare Theory or Combinatorics (see [13] for a review of theoretical and practical applications of fuzzy measures).

From the point of view of order polytopes, $\mathcal{FM}(X)$ is the order polytope of the poset $(P, \preceq)$ where $P = \mathcal{P}(X) \backslash \{X, \emptyset\}$ and $\preceq$ is the inclusion between subsets. The principal filter $A^{\uparrow}$ is given by the measure

$$u_A(B) := \begin{cases} 1 & \text{if A} \subseteq \text{B} \\ 0 & \text{otherwise} \end{cases}$$

This measure is known as the *unanimity game* on $A$. Filter $P$ is given by

$$u_{\emptyset}(B) := \begin{cases} 1 & \text{if B} \neq \emptyset \\ 0 & \text{if B} = \emptyset \end{cases}$$

Finally, filter $\emptyset$ is given by $u_X$.

Another example of order polytope is the set of $p$-symmetric measures [21], a special class of fuzzy measures. This concept appears as a middle term between symmetric measures and general fuzzy measures. A fuzzy measure $\mu$ is said to be **symmetric** if it satisfies that for any $A, B \in \mathcal{P}(X)$ such that $|A| = |B|$, it is $\mu(A) = \mu(B)$.

The definition of $p$-symmetric measure is based on the concept of subsets of indifference. Given a subset $A$ of $X$ and $\mu \in \mathcal{FM}(X)$, we say that $A$ is a **subset of indifference** for $\mu$ if and only if $\forall B_1, B_2 \subseteq A$, $|B_1| = |B_2|$ and $\forall C \subseteq X \backslash A$, it is

$$\mu(B_1 \cup C) = \mu(B_2 \cup C).$$

With these definitions, given a fuzzy measure $\mu$, we say that $\mu$ is a **$p$-symmetric measure** if and only if the coarsest partition of the universal set in subsets of indifference is $\{A_1, ..., A_p\}, A_i \neq \emptyset, \forall i \in \{1, ..., p\}$.

The existence and unicity of this partition has been proved in [20]. We will denote by $\mathcal{FM}(A_1, ..., A_p)$ the set of fuzzy measures for which $A_i$, $i = 1, ..., p$, is a subset of indifference (but not necessarily $p$-symmetric! Indeed, any symmetric measure belongs to $\mathcal{FM}(A_1, ..., A_p)$).

As all the elements in the same subset of indifference have the same behavior, when dealing with a fuzzy measure $\mu \in \mathcal{FM}(A_1, ..., A_p)$, for a given subset $B$ of the universal set $X$, we only need to know the number of elements of each $A_i$ that belong to $B$. Therefore, the following result holds:

**Lemma 1** *[21] If $\{A_1, ..., A_p\}$ is a partition of $X$, then in order to define a measure in $\mathcal{FM}(A_1, ..., A_p)$, any $B \subseteq X$ can be identified with a $p$-dimensional vector $(b_1, ..., b_p)$ with $b_i := |B \cap A_i|$.*

More properties about $p$-symmetric measures can be found in [21, 20].

Then, the set $\mathcal{FM}(A_1, ..., A_p)$ can be seen as the order polytope of the poset $(P(A_1, ..., A_p), \preceq)$, where

$$P(A_1, ..., A_p) := \{(i_1, ..., i_p) : i_j \in \{0, ..., |A_j|\}, i, j \in \mathbb{Z}\} \backslash \{(0, ..., 0), (|A_1|, ..., |A_p|)\},$$

and $\preceq$ is given by $(c_1, ..., c_p) \preceq (b_1, ..., b_p) \Leftrightarrow c_i \leq b_i, i = 1, ..., p$.

Remark that $\mathcal{FM}(X)$ can be seen as a special case of $p$-symmetric measures; more concretely, $\mathcal{FM}(X)$ can be identified with $\mathcal{FM}(\{x_1\}, ..., \{x_n\})$.

In next sections we will study the group of **isometries** on $O(P)$, i.e. the group of bijective functions $h : O(P) \to O(P)$ keeping distances.

# 3 Characterizing isometries on order polytopes

Suppose $f : P \to P$ is a bijection such that there exist $F, F'$ two disjoint filters (we allow one of them to be empty) such that $P = F \cup F'$ and

1. $f(F)$ and $f(F')$ are filters in $P$.
2. If $i, j \in F$, then $i \preceq j$ if and only if $f(i) \preceq f(j)$.
3. If $i, j \in F'$, then $i \preceq j$ if and only if $f(j) \preceq f(i)$.

That is, $f$ is isotone on $F$ and antitone on $F'$. For a given $f$ in these conditions, we define $h_{f,F,F'} : O(P) \to O(P)$ by

$$h_{f,F,F'}(a_1, \ldots, a_m) := (b_1, \ldots, b_m)$$

where

$$b_{f(i)} := \begin{cases} a_i & \text{if i} \in \text{F} \\ 1 - a_i & \text{if i} \in \text{F}' \end{cases} \tag{1}$$

Let us first show that $h_{f,F,F'}$ is well-defined. Consider $(a_1, ..., a_m) \in O(P)$ and let us show that $(b_1, ..., b_m) \in O(P)$. Clearly, $0 \leq b_i \leq 1$, so it suffices to show that for any $f(i), f(j) \in P$ such that $f(i) \preceq f(j)$, it is $b_{f(i)} \leq b_{f(j)}$. We have two possibilities:

- If $f(i) \in f(F)$, then $f(j) \in f(F)$ as $f(F)$ is a filter. By the second condition, this implies that $i \preceq j$. Moreover, as $(a_1, ..., a_m) \in O(P)$ and Eq. (1), it follows that $b_{f(i)} = a_i \leq a_j = b_{f(j)}$.
- If $f(i) \in F'$, then $f(j)$ is in $f(F')$ as $f(F')$ is a filter. Now, as $f(i) \preceq f(j)$, it follows that $j \preceq i$ by definition of $f$. As $(a_1, ..., a_m) \in O(P)$ and Eq. (1), $b_{f(i)} = 1 - a_i \leq 1 - a_j = b_{f(j)}$.

Consequently, $(b_1, \ldots, b_m)$ is in $O(P)$ and $h_{f,F,F'}$ is well-defined.

Remark that if we consider the Euclidean distance[1], then $h_{f,F,F'}$ clearly keeps distances; therefore, $h_{f,F,F'}$ is injective. On the other hand, from the conditions on $f, F$ and $F'$, it follows that $h_{f^{-1},f(F),f(F')}$ can be built. Moreover, $h_{f,F,F'} \circ h_{f^{-1},f(F),f(F')}$ and $h_{f^{-1},f(F),f(F')} \circ h_{f,F,F'}$ are the identity map, whence we conclude that $h_{f,F,F'}$ is onto.

Consequently, the mapping $h_{f,F,F'}$ is an isometry on $O(P)$. We will say that $h_{f,F,F'}$ thus built is the **isometry induced by $f$ and the filters $F, F'$**. Note that indeed, from the definition of $h_{f,F,F'}$, this mapping can be seen as the restriction to $O(P)$ of an affine map whose associated linear mapping has determinant 1 or -1.

**Example 1** *Let us find explicitly the isometries $h_{f,F,F'}$ on $O(P)$ when $P = \{i, j\}$ is the antichain of two elements. Remark that for this choice of $P$, $O(P)$ can be identified with $\mathcal{FM}(X)$ for a referential $X$ of two elements.*

*In this case, there are two possibilities for $f$, namely:*

$$f_1(i) = i, f_1(j) = j, \quad \textit{and} \quad f_2(i) = j, f_2(j) = i.$$

*On the other hand, we have four different possibilities for $F$, namely: $\emptyset, \{i\}, \{j\}, \{i, j\}$.*

*The isometries $h_{f,F,F'}$ are given in next table:*

| *Vertex* | $F = \emptyset$ | | $F = \{i\}$ | | $F = \{j\}$ | | $F = \{i, j\}$ | |
|---|---|---|---|---|---|---|---|---|
| | $f_1$ | $f_2$ | $f_1$ | $f_2$ | $f_1$ | $f_2$ | $f_1$ | $f_2$ |
| $(0,0)$ | $(1,1)$ | $(1,1)$ | $(0,1)$ | $(1,0)$ | $(1,0)$ | $(0,1)$ | $(0,0)$ | $(0,0)$ |
| $(0,1)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ | $(1,1)$ | $(0,0)$ | $(0,0)$ | $(1,0)$ | $(0,1)$ |
| $(1,0)$ | $(1,0)$ | $(0,1)$ | $(0,0)$ | $(0,0)$ | $(1,1)$ | $(1,1)$ | $(0,1)$ | $(1,0)$ |
| $(1,1)$ | $(0,0)$ | $(0,0)$ | $(1,0)$ | $(0,1)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ | $(1,1)$ |

*Note that in this case $O(P)$ can be identified with the unit square and thus, the set of isometries is the dihedral group $D_4$. Therefore, the set of induced isometries recovers the whole set of isometries. Notice also that the definition of $h_{f,F,F'}$ is not independent of the choice of the filters $F$ and $F'$; in this case, if $f$ is the identity map $f_1$, then $F$ and $F'$ can be chosen to be any partition of $P$ in two subsets, each selection giving raise to different isometries.*

**Remark 1** *Remark that for any isometry $h_{f,F,F'}$, the subposets $F$ and $f(F)$ are order isomorphic via $f$. Similarly, the subposet $F'$ is order isomorphic to $\overline{f(F')}$. Moreover, if $A \subseteq F$, then $A$ and $f(A)$ are order isomorphic; similarly, if $A \subseteq F'$, then $A$ and $\overline{f(A)}$ are order isomorphic. Notice however that if $A \not\subseteq F, A \not\subseteq F'$, then $A$ is not order isomorphic in general to $f(A)$ nor $\overline{f(A)}$.*

[1] The results presented in the paper also hold if, instead of the Euclidean distance, a $p$-norm-metric ($1 \leq p < \infty$) is used.

**Lemma 2** *[18] Suppose $\mathcal{F} \subseteq \mathbb{R}^p$ is a convex set. Let $h$ be an isometry on $\mathcal{F}$ and consider $v_1, v_2 \in \mathcal{F}$. Then,*

$$h(\lambda v_1 + (1-\lambda)v_2) = \lambda h(v_1) + (1-\lambda)h(v_2), \forall \lambda \in [0,1].$$

As a consequence, we have

**Corollary 1** *If $\mathcal{F} \subseteq \mathbb{R}^p$ is a convex polyhedron and $h$ is an isometry on $\mathcal{F}$, then $f$ maps vertices in vertices.*

On the other hand, as we have seen in Section 2, vertices of $O(P)$ can be identified with filters. With this identification, for a filter $L$, $h_{f,F,F'}(L)$ is another filter given by

$$h_{f,F,F'}(L)_{f(i)} = \begin{cases} 1 & \text{if } i \in L \cap F \text{ or } i \in L^c \cap F' \\ 0 & \text{if } i \in L^c \cap F \text{ or } i \in L \cap F' \end{cases}$$

Otherwise said, if we define $G' := \{f(i) : i \in F'\}$, then $h_{f,F,F'}(L)$ is the filter whose elements are

$$\{f(i) : i \in L \cap F\} \cup G' \backslash \{f(i) : i \in L \cap F'\}. \qquad (2)$$

As $h_{f,F,F'}$ is defined by the images of the vertices, this characterizes $h_{f,F,F'}$.

Next theorem, which is the main result in the paper, shows that, in fact, all isometries of $O(P)$ arise from induced isometries.

**Theorem 1** *Let $h : O(P) \to O(P)$ be an isometry; then, there exists a bijection $f : P \to P$ and two filters $F, F'$ determining a partition on $P$ and satisfying conditions 1-3 such that $h = h_{f,F,F'}$.*

**Proof:** It suffices to prove the result for vertices, i.e. for filters on $P$. We will prove the theorem in several lemmas.

First, remark that as $h$ takes vertices in vertices, there exists a filter $F$ such that $h(F) = P$. Also, there exists a filter $F'$ such that $h(F') = \emptyset$. Consider the filters $G := h(P)$ and $G' := h(\emptyset)$, too. Since $h$ keeps distances,

$$|P| = d^2(P, \emptyset) = d^2(G, G') = |G \setminus G'| + |G' \setminus G|,$$

whence we conclude that $\{G, G'\}$ is a partition of $P$.

**Lemma 3** *Consider two filters $K_1, K_2 \subseteq P$ such that $K_1 \subseteq K_2$. Then,*

$$h(K_1) \cap G \subseteq h(K_2) \cap G. \qquad (3)$$

$$h(K_2) \cap G' \subseteq h(K_1) \cap G'. \qquad (4)$$

**Proof:** Let us denote $|K_1| = k_1, |K_2| = k_2$. As $K_1 \subseteq K_2$, $d^2(K_1, K_2) = k_2 - k_1$, whence $d^2(h(K_1), h(K_2)) = k_2 - k_1$. On the other hand,

$$d^2(h(K_1), h(K_2)) = x_1 + x_2 + y_1 + y_2, \qquad (5)$$

where

$$x_1 := |(h(K_1) \setminus h(K_2)) \cap G|,\ x_2 := |(h(K_2) \setminus h(K_1)) \cap G|,$$
$$y_1 := |(h(K_1) \setminus h(K_2)) \cap G'|,\ y_2 := |(h(K_2) \setminus h(K_1)) \cap G'|.$$

Define also

$$t := |h(K_1) \cap h(K_2) \cap G|,\ u := |h(K_1) \cap h(K_2) \cap G'|.$$

Then,

$$k_1 = d^2(\emptyset, K_1) = d^2(h(\emptyset), h(K_1)) = d^2(G', h(K_1)) = |G' \setminus h(K_1)| + |h(K_1) \setminus G'| = (|G'| - y_1 - u) + (x_1 + t),$$

since $G$ and $G'$ are complementary. Analogously,

$$k_2 = (|G'| - y_2 - u) + (x_2 + t).$$

Consequently,

$$k_2 - k_1 = (|G'| - y_2 - u) + (x_2 + t) - (|G'| - y_1 - u) - (x_1 + t) = y_1 - y_2 + x_2 - x_1,$$

and, since $k_2 - k_1 = x_1 + x_2 + y_1 + y_2$ by Eq. (5) and $x_1, x_2, y_1, y_2 \geq 0$, we conclude that $x_1 = 0$ and $y_2 = 0$. This implies that

$$h(K_1) \cap G \subseteq h(K_2) \cap G, \;\; h(K_2) \cap G' \subseteq h(K_1) \cap G'.$$

This finishes the proof. ■

**Lemma 4** *The isometry $h$ induces an order isomorphism between the lattice of filters contained in $F$ and the lattice of filters containing $G'$.*

**Proof:** As a particular case of the previous lemma, if $K$ is a filter such that $K \subseteq F$, applying Eq. (4), $G' = h(F) \cap G' \subseteq h(K) \cap G'$, whence $G' \subseteq h(K)$. Consequently, if $K_1 \subseteq K_2 \subseteq F$, then $G' = h(K_1) \cap G' = h(K_2) \cap G'$; on the other hand, by Eq. (3), $h(K_1) \cap G \subseteq h(K_2) \cap G$, so we conclude that $h(K_1) \subseteq h(K_2)$.

If $h$ is an isometry, the inverse of $h$ is also an isometry. For $h^{-1}$, filter $G$ plays the role of $F$, $G'$ the role of $F'$, $F$ that of $G$ and $F'$ that of $G'$. Applying the previous lemma to $h^{-1}$, we can conclude that for two filters $L_1, L_2$ such that $L_1 \subseteq L_2$, it is

$$h^{-1}(L_1) \cap F \subseteq h^{-1}(L_2) \cap F, \;\; h^{-1}(L_2) \cap F' \subseteq h^{-1}(L_1) \cap F'.$$

Consider two filters $L_1$ and $L_2$ such that $G' \subseteq L_1 \subseteq L_2$. Then, $h^{-1}(L_i) \cap F' \subseteq h^{-1}(G') \cap F' = \emptyset$, $i = 1, 2$, since $h^{-1}(G') = \emptyset$. Thus, $h^{-1}(L_i) \subseteq F$, $i = 1, 2$, and, as $L_1 \subseteq L_2$, it follows that $h^{-1}(L_1) \subseteq h^{-1}(L_2) \subseteq F$.

Therefore, $h$ induces an order isomorphism between the lattice of filters contained in $F$ and the lattice of filters containing $G'$. ■

**Lemma 5** *Suppose $A_1, ..., A_n$ are filters in $F$. Then,*

$$h(A_1 \cup ... \cup A_n) = h(A_1) \cup ... \cup h(A_n), \; h(A_1 \cap ... \cap A_n) = h(A_1) \cap ... \cap h(A_n).$$

**Proof:** It suffices to prove the result for $n = 2$ and unions. The result follows by induction for general $n$ and the same can be done for intersections.

Suppose then $A_1, A_2 \in F$. As $A_i \subseteq A_1 \cup A_2 \subseteq F, i = 1, 2$, we conclude that $h(A_i) \subseteq h(A_1 \cup A_2)$, whence $h(A_1) \cup h(A_2) \subseteq h(A_1 \cup A_2)$.

On the other hand, as $h(A_i) \subseteq h(A_1) \cup h(A_2), i = 1, 2$, it is $A_i \subseteq h^{-1}(h(A_1) \cup h(A_2)), i = 1, 2$, whence $A_1 \cup A_2 \subseteq h^{-1}(h(A_1) \cup h(A_2))$ and $h(A_1 \cup A_2) \subseteq h(A_1) \cup h(A_2)$. ■

Consider an element $x \in F$ and $x^{\uparrow}$ the principal filter generated by $x$. Then, $x^{\uparrow} \subseteq F$, whence $G' \subseteq h(x^{\uparrow})$. Notice that $x^{\uparrow}$ is a join-irreducible filter, so $h(x^{\uparrow})$ is also join-irreducible among the filters containing $G'$. Hence, $h(x^{\uparrow})$ is of the form $G' \cup y^{\uparrow}$ with $y \in G$. Moreover, for any filter of the form $G' \cup y^{\uparrow}, y \in G$, it is $h^{-1}(G' \cup y^{\uparrow}) = x^{\uparrow}$, for some $x \in F$. For otherwise, $h^{-1}(G' \cup y^{\uparrow})$ would not be join-irreducible, contradicting that $h$ (and $h^{-1}$) induces an order isomorphism between the lattice of filters contained in $F$ and the filter of lattices containing $G'$.

Define $f_1 : F \to G$ by $f_1(x) := y$, where $y \in G$ is such that $h(x^{\uparrow}) = G' \cup y^{\uparrow}$. By construction, $f_1$ is bijective and by Lemma 4, it satisfies that for $x_1, x_2 \in F, x_1 \preceq x_2 \Leftrightarrow f_1(x_1) \preceq f_1(x_2)$, i.e. $f_1$ is isotone on $F$.

**Lemma 6** *If $K$ is a filter contained in $F$, then*

$$h(K) = G' \cup \{f_1(x) : x \in K\}.$$

**Proof:** We know that $K = \cup_{x \in K} x^{\uparrow}$. Applying the previous lemma,

$$h(K) = h(\bigcup_{x \in K} x^{\uparrow}) = \bigcup_{x \in K} h(x^{\uparrow}) = \bigcup_{x \in K} (G' \cup f_1(x)^{\uparrow}) = (\bigcup_{x \in K} f_1(x)^{\uparrow}) \cup G' = G' \cup \{f_1(x) : x \in K\}. \tag{6}$$

This finishes the proof. ■

**Lemma 7** *The isometry $h$ induces a reverse-order isomorphism between the lattice of filters contained in $F'$ and the lattice of filters contained in $G'$.*

**Proof:** Take a filter $K$ contained in $F'$. We have by Eq. (3) that $h(K) \cap G \subseteq h(F') \cap G = \emptyset$ by definition of $F'$; thus, $h(K) \subseteq G'$. Therefore, if $K_1 \subseteq K_2 \subseteq F'$ we will have by Eq. (4) $h(K_2) \subseteq h(K_1) \subseteq G'$. By a similar argument $h^{-1}$ takes filters contained in $G'$ to filters contained in $F'$, also reversing the order. Therefore, $h$ induces a reverse-order isomorphism between the lattice of filters contained in $F'$ and the lattice of filters contained in $G'$. ■

The same as Lemma 5, the following can be proved:

**Lemma 8** *Suppose $A_1, ..., A_n$ are filters in $F'$. Then,*

$$h(A_1 \cup ... \cup A_n) = h(A_1) \cap ... \cap h(A_n),\ h(A_1 \cap ... \cap A_n) = h(A_1) \cup ... \cup h(A_n).$$

Take $x$ in $F'$. Consider $x^{\uparrow}$ the principal filter generated by $x$. Clearly $x^{\uparrow} \subseteq F'$. Since $x^{\uparrow}$ is a join-irreducible filter, then $h(x^{\uparrow})$ is a meet-irreducible filter; for otherwise, $h(x^{\uparrow}) = A \cap B$, whence $x^{\uparrow} = h^{-1}(A \cap B) = h^{-1}(A) \cup h^{-1}(B)$ and $x^{\uparrow}$ would not be join-irreducible, a contradiction. Therefore, $G' \setminus h(x^{\uparrow})$ is a join-irreducible ideal and thus it is principal, generated by an element $y$ of $G'$, i.e. $G' \setminus h(x^{\uparrow}) = y^{\downarrow}$.

Define $f_2 : F' \to G'$ by $f_2(x) := y$, where $y \in G'$ is such that $G' \setminus h(x^{\uparrow}) = y^{\downarrow}$. As for $f_1$, $f_2$ is bijective.

**Lemma 9** *For $x_1, x_2 \in F', x_1 \preceq x_2 \Leftrightarrow f_2(x_2) \preceq f_2(x_1)$, i.e. $f_2$ is antitone on $F'$.*

**Proof:** For $x_1, x_2 \in F'$, it is $x_1 \preceq x_2$ if and only if $x_2^{\uparrow} \subseteq x_1^{\uparrow}$. This is equivalent to $h(x_1^{\uparrow}) \subseteq h(x_2^{\uparrow})$ and $G' \setminus h(x_2^{\uparrow}) \subseteq G' \setminus h(x_1^{\uparrow})$, so it is equivalent to $f_2(x_2) = y_2 \preceq y_1 = f_2(x_1)$. ■

**Lemma 10** *If $K$ is a filter contained in $F'$, then*

$$h(K) = G' \setminus \{f_2(x) : x \in K\}.$$

**Proof:** Consider $K \subseteq F'$. Clearly, $K = \cup_{x \in K} x^{\uparrow}$. Therefore,

$$h(K) = h(\bigcup_{x \in K} x^{\uparrow}) = \bigcap_{x \in K} h(x^{\uparrow}) = \bigcap_{x \in K} (G' \setminus \{y \in G' : y \preceq f_2(x)\}) = G' \setminus \bigcup_{x \in K} \{y \in G' : y \preceq f_2(x)\} = G' \setminus \{f_2(x) : x \in K\}. \tag{7}$$

This finishes the proof. ■

Define $f : P \to P$ by

$$f(x) := \begin{cases} f_1(x) & \text{if x} \in \text{F} \\ f_2(x) & \text{if x} \in \text{F}' \end{cases}$$

which by construction satisfies the three conditions on $f, F$ and $F'$ necessary to define the corresponding isometry $h_{f,F,F'}$. Note that $f(F) = G$ and $f(F') = G'$.

**Lemma 11** $h = h_{f,F,F'}$.

**Proof:** By Eq. (2), it suffices to prove that if $L$ is a filter of $P$, then

$$h(L) = \{f(x) : x \in L \cap F\} \cup (G' \setminus \{f(x) : x \in L \cap F'\}).$$

Consider $L_1 = L \cap F$ and $L_2 = L \cap F'$. As $L_1 \subseteq L$, by Eq. (3),

$$h(L_1) \cap G \subseteq h(L) \cap G \tag{8}$$

and as $L_2 \subseteq L$ and Eq. (4),

$$h(L) \cap G' \subseteq h(L_2) \cap G' = h(L_2), \tag{9}$$

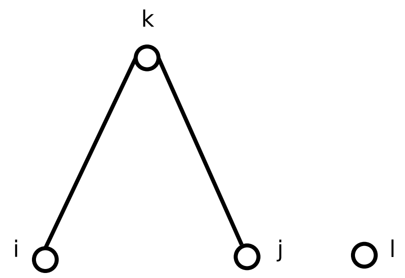


Figure 1: Poset $P$

From Eq. (6), we know that $h(L_1) \cap G = \{f(x) : x \in L \cap F\}$, and from Eq. (7), $h(L_2) = G' \setminus \{f(x) : x \in L \cap F'\}$, so it is enough to show that $h(L_1) \cap G = h(L) \cap G$ and $h(L) \cap G' = h(L_2)$ for which, by Eq. (8) and (9), it suffices to prove that $|h(L_1) \cap G| = |L_1|$ is equal to $|h(L) \cap G|$ and $|h(L) \cap G'| = |h(L_2)|$. Define

$$a := |L_1|,\ a' := |L_2|,\ b := |h(L) \cap G|,\ b' := |h(L) \cap G'|,\ f := |F|,\ f' := |F'|,\ g := |G|,\ g' := |G'|.$$

As $h$ is an isometry, $f' = d^2(\emptyset, F') = d^2(h(\emptyset), h(F')) = d^2(G', \emptyset) = g'$ and $f = d^2(P, F') = d^2(h(P), h(F')) = d^2(G, \emptyset) = g$. Also,

$$f - a + a' = d^2(L, F) = d^2(h(L), h(F)) = d^2(h(L), P) = g - b + g' - b' = f - b + f' - b',$$

and

$$a + a' = d^2(L, \emptyset) = d^2(h(L), h(\emptyset)) = d^2(h(L), G') = b + g' - b' = b + f' - b',$$

from which $a = b$ and $f' - a' = b'$. Then,

$$|h(L_1) \cap G| = |\{f(x) : x \in L \cap F\}| = |L_1| = a = b = |h(L) \cap G|,$$

and

$$|h(L_2)| = |G' \setminus \{f(x) : x \in L \cap F'\}| = f' - a' = b' = |h(L) \cap G'|,$$

whence $h(L) := \{f(x) : x \in L \cap F\} \cup (G' \setminus \{f(x) : x \in L \cap F'\})$, as stated. ■

Then, $h$ and $h_{f,F,F'}$ coincide on the vertices of $O(P)$ and consequently, they must be equal. ■

Let us now see some consequences of this result. Notice that in the proof of Theorem 1, filters $G, G'$ form a disconnection of $P$ (they are disjoint filters) and that $h(\emptyset) = G'$. This provides us with the following corollary:

**Corollary 2** *A necessary condition for a filter to be the image of $\emptyset$ under an isometry $h$ is that its complement is disconnected from it.*

However, the condition is not sufficient:

**Example 2** *Consider $P = \{i, j, k, l\}$ such that $i \prec k, j \prec k$, $i$ and $j$ are not comparable and $l$ is not comparable with $i, j, k$ (see Figure 1).*

*If $h_{f,F,F'}(\{i, j, k\}) = \emptyset$ for a choice of $f, F, F'$, this implies that $F' = \{i, j, k\}$. By Remark 1, this means that $\{i, j, k\}$ is isomorphic to $\overline{f(\{i, j, k\})}$, that is not possible, as no filter of $P$ has this structure.*

Corollary 2 leads to a characterization of the order polytopes whose isometry group is *transitive*, i.e., any vertex can be applied to any other vertex.

**Proposition 1** *The isometry group of $O(P)$ acts transitively on the vertices if and only if $P$ is an antichain.*

**Proof:** Suppose $P$ is an antichain. Consider two vertices of $O(P)$ given by the filters $F_1$ and $F_2$. Define $F := (P \setminus (F_1 \cup F_2)) \cup (F_1 \cap F_2)$ and $F' = P \setminus F$; as $P$ is an antichain, both $F$ and $F'$ are filters. Define $f = Id$. Then, it is easy to check that $h_{f,F,F'}(F_1) = F_2$, whence the isometry group is transitive.

Now suppose that $P$ is not an antichain. Consider $i, j$ in $P$ such that $i \prec j$. Then $j^{\uparrow}$ is not disconnected from its complement and hence, there is no isometry $h$ such that $h(\emptyset) = j^{\uparrow}$ by Corollary 2. Thus, the isometry group of $O(P)$ does not act transitively. ■

Another straight consequence of Theorem 1 is the following:

**Corollary 3** *If $P$ is connected, then either $F = P$ or $F = \emptyset$.*

The proof lays in the fact that $F$ and $F'$ are filters. Now, the following can be proved:

**Proposition 2** *Suppose $P$ is a chain $1 \prec 2 ... \prec n$. Then, the only isometries on $O(P)$ are the identity and the dual application $h_{f,\emptyset,P}$ given by $f(i) = n - i + 1$.*

**Proof:** As $P$ is connected, it is either $F = P$ or $F = \emptyset$. If $F = P$, then $P$ and $f(P)$ are isomorphic by Remark 1, whence $f(P)$ is a chain and thus $f$ is the identity map.

If $F = \emptyset$, then $F' = P$ and necessarily $P$ and $\overline{f(P)}$ are isomorphic by Remark 1. Thus, $f(P)$ is a chain such that $f(n) \prec f(n-1) \prec ... \prec f(1)$, whence the result. ■

# 4 The group $H_0$

As we have proved in the previous section, $\{F, F'\}$ determines a partition of $P$ in two filters. In this section, we are going to study the isometries satisfying $F = P$ or, equivalently, $h(\emptyset) = \emptyset$. We will denote this set by $H_0$. Remark that $H_0$ is given by the mappings $f$ such that $h_{f,P,\emptyset}$ is an isometry. Note also that $H_0$ is never empty, as the identity map determines an isometry in $H_0$. Moreover, it is easy to check that $H_0$ is a subgroup of the group of isometries. The following result completely determines $H_0$.

**Corollary 4** *The group $H_0$ is isomorphic to the group of order automorphisms of $P$ (isomorphisms keeping the order structure of $P$).*

**Proof:** It is a direct consequence of Remark 1. ■

However, $H_0$ is not always a normal subgroup of the group of isometries of $O(P)$, as it is shown in next example.

**Example 3** *Consider an antichain of two elements $i, j$. Then, the corresponding order polytope can be identified with the unit square and thus, the set of isometries corresponds with the set of symmetries of the unit square. The filter $\emptyset$ is associated with a vertex of the square and $H_0$ is the set of symmetries fixing this vertex, and it is a well-known fact that this subgroup is not a normal subgroup of the group of symmetries of the unit square.*

In next proposition, we give a characterization for $H_0$ to be a normal subgroup.

**Proposition 3** *$H_0$ is a normal subgroup of the group of isometries of $O(P)$ if and only if every filter $K$ such that there exists an isometry $h$ satisfying $h(\emptyset) = K$ is a fixed point of every isometry in $H_0$.*

**Proof:** Take $h$ and isometry of $O(P)$ and let us denote $K := h(\emptyset)$. Take $g \in H_0$. Then, $h^{-1}gh \in H_0$ if and only if $h^{-1}(g(h(\emptyset))) = \emptyset$, which is equivalent to $h^{-1}(g(K)) = \emptyset$, which in turn is equivalent to $g(K) = h(\emptyset) = K$, whence the result. ■

For the particular case of connected posets, the following result can be stated.

**Proposition 4** *If $P$ is connected, then:*

*1. $H_0$ is a normal subgroup of $O(P)$.*

*2. If $P$ is autodual, then $H_0$ is a subgroup of index 2.*

*3. If $P$ is not autodual, then $H_0 = O(P)$.*

**Proof:** If $P$ is connected and $h$ is an isometry, then either $h(\emptyset) = \emptyset$ or $h(\emptyset) = P$ by Corollary 2. Thus, $P$ is fixed by all isometries in $H_0$, and we conclude from the previous proposition that $H_0$ is normal.

If $P$ is autodual, then there exists an order reversing bijection $f : P \to P$. Consider the isometry $h_{f,\emptyset,P}$. Clearly, $h_{f,\emptyset,P}(\emptyset) = P$ and hence $h_{f,\emptyset,P} \notin H_0$. Consider another isometry $g \notin H_0$. Since $P$ is connected, from Corollary 3, it is $g(P) = \emptyset$. Then, $g^{-1}(h_{f,\emptyset,P}(\emptyset)) = \emptyset$ and consequently, $g^{-1}h_{f,\emptyset,P} \in H_0$. Thus, $h_{f,\emptyset,P}H_0 = gH_0$, so there are two different cosets of $H_0$ ($H_0$ and $h_{f,\emptyset,P}H_0$).

If $P$ is not autodual, such order reversing bijection does not exist and $h(\emptyset) = \emptyset$ for any isometry, whence $H_0 = O(P)$. ■

For example, if $P$ is a chain, then $P$ is connected and autodual. In Proposition 2 we have shown that there are only two isometries, the identity map and the isometry given by the dual application. In this case $H_0$ has only one element, the identity map, and it is a normal trivial subgroup of the group of isometries on $O(P)$.

# 5 Example of application: Isometries on the set of fuzzy measures

In this section, we are going to apply the previous results to obtain in a very simple way the set of isometries of $\mathcal{FM}(A_1, ..., A_p)$ and $\mathcal{FM}(X)$. The set of isometries of these order polytopes has been already obtained in [18].

As we have stated in Section 2, $\mathcal{FM}(A_1, ..., A_p)$ is the order polytope of the poset

$$P(A_1, ..., A_p) := \{(i_1, ..., i_p) : i_j \in \{0, ..., |A_j|\}, i, j \in \mathbb{Z}\} \setminus \{(0, ..., 0), (|A_1|, ..., |A_p|)\}.$$

As $P(A_1, ..., A_p)$ is connected and autodual [2], we can apply Corollary 4 and Proposition 4 to conclude that it suffices to find the set of order automorphisms of $P(A_1, ..., A_p)$.

On the other hand, note that $P(A_1, ..., A_p)$ is a product of chains except for the top and bottom elements. Thus, the following can be shown:

**Theorem 2** *If a poset $P$ is a product of $p$ chains of sizes $a_1, ..., a_p$ except the top and bottom elements, then the group of automorphisms of $P$ is generated by the functions $f_{j,k}$ given by:*

$$f_{j,k}(c_1, ..., c_j, ..., c_k, ..., c_p) = (c_1, ..., c_k, ..., c_j, ..., c_p),$$

*where $j, k$ are such that $a_j = a_k$. We call this mapping the* **transposition** *between the chains $j$ and $k$.*

**Proof:** Clearly, any function in the conditions of the theorem is an order automorphism of $P$. It suffices to show that any order automorphism can be written as a composition of such functions.

Notice that elements $e_{c,i} := (0, ..., c, ..., 0)$, with $c > 0$ in the $i$-th coordinate are the join-irreducible elements in $P$. Moreover, any element of $P$ can be obtained as a supremum of elements $e_{c,i}$. Thus, any order automorphism of $P$ is determined by the images of the elements $e_{c,i}$. As any order automorphism $f$ takes join-irreducible elements in join-irreducible elements, it follows that $f(e_{c,i}) = e_{d,j}$. On the other hand, $e_{c,i}^{\downarrow}$ has exactly $c$ elements and $e_{c,i}^{\uparrow}$ has $(a_1 + 1)...(a_{i-1} + 1)(a_{i+1} + 1)...(a_p + 1)(a_i - c + 1) - 1$ elements. Hence, straightforward calculus shows that $e_{c,i}$ can only be applied to a $e_{d,j}$ with $c = d$ and $a_i = a_j$, whence the result. ■

Define $g : P(A_1, ..., A_p) \to P(A_1, ..., A_p)$ by $g((c_1, ..., c_p)) = (|A_1| - c_1, ..., |A_p| - c_p)$. Then, $g$ is an isomorphism between $P(A_1, ..., A_p)$ and its dual poset. We will call such mapping $g$ the **dual application**. Thus, by Proposition 4 we conclude:

**Theorem 3** *The group of isometries on $\mathcal{FM}(A_1, ..., A_p)$ is generated by the isometries induced by transpositions between subsets of indifference of the same cardinality, and by the isometry induced by the dual application.*

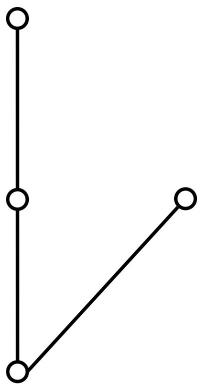

Figure 2: Poset $P$

Suppose that we partitionate $\{A_1, ..., A_p\}$ in $r$ classes such that $A_i$ and $A_j$ are in the same class if they have the same cardinality. Assume that we have $r$ classes $C_1, ..., C_r$ whose cardinalities are $c_1, ..., c_r$, respectively. Then, the number of isometries on $\mathcal{FM}(A_1, ..., A_p)$ is

$$c_1! ... c_r! 2.$$

For the general case $\mathcal{FM}(X)$, all subsets of indifference have the same cardinality. Morevoer, compositions of transpositions are permutations on $X$. Therefore, the following holds:

**Theorem 4** *The group of isometries on $\mathcal{FM}(X)$ is given by the isometries induced by permutations, and compositions of permutations with the dual application. Then, we have $n!2$ isometries on $\mathcal{FM}(X)$.*

For a given permutation $\pi$, the corresponding isometry $h$ in $H_0$ is given by

$$h(\mu)(A) = \mu(\pi(A)),\ A \subseteq X,\ \mu \in \mathcal{FM}(X).$$

Note that it is straightforward to check that these mappings are isometries. What is interesting in this result is that there are no other isometries.

# 6 Upper bound of the number of isometries on $O(P)$

In this section we look for upper and lower bounds for the number of isometries on $O(P)$. We have already shown that at least there is an isometry on $O(P)$ (the identity map). In some cases, this is the only isometry, as next example shows:

**Example 4** *Consider the poset $P$ given in Figure 2. Since $P$ is connected and not autodual, by Proposition 4, it follows that any isometry is in $H_0$. From Corollary 4, $H_0$ is isomorphic to the set of order automorphism of $P$, that in this case is the trivial group.*

In the rest of the section, we are going to look for upper bounds. Related to antichains, the following can be proved:

**Lemma 12** *If $P$ is an antichain, then the number of isometries on $O(P)$ is $|P|!\,2^{|P|}$.*

**Proof:** If $P$ is an antichain, then no pair of elements can be compared; thus, $F$ can be any subset of $P$, whence we have $2^{|P|}$ possible choices. For fixed $F$ and $F'$, $f$ can be any bijection on $P$, i.e. we have $|P|!$ possibilities for $f$. Joining both facts, the result holds. ■

In Example 1 we have found all the eight isometries for $O(P)$ when $P$ is the antichain of two elements. As a corollary, we can conclude the following:

**Corollary 5** $|P|!\,2^{|P|}$ *is an upper bound for the cardinality of the set of isometries on* $O(P)$.

**Proof:** It suffices to remark that for a given partition $\{F, F'\}$, the number of possible choices for $f$ is bounded by $|P|!$. As the number of possible partitions is bounded by $2^{|P|}$, the result holds. ■

Moreover, the following can be shown.

**Lemma 13** *Consider a poset* $P$ *with* $p$ *elements. Then, the number of isometries on* $O(P)$ *is* $p!\,2^p$ *if and only if* $P$ *is an antichain.*

**Proof:** The "if" part is Lemma 12. Let us prove the "only if" part. For this, suppose that $P$ is not an antichain. Then, there exist $i, j \in P$ such that $i \prec j$. As any isometry is given by two filters $F, F'$, we conclude that either $i, j \in F$ or $i, j \in F'$. But then, we are avoiding the possibilities $i \in F, j \in F'$ and $i \in F', j \in F$. Thus, the number of isometries on $O(P)$ is bounded by $p!\,(2^p - 2)$, whence the result. ■

Assume now that $P$ is connected and let us look for an upper bound for the isometries on $O(P)$.

**Proposition 5** *Let* $P$ *be a connected poset such that* $|P| = p$. *Then, the number of isometries on* $O(P)$ *is bounded by* $(p-1)!$ *for* $p > 4$. *If* $p = 1, 2, 3$, *the number of isometries is 2. If* $p = 4$, *the number of isometries is bounded by 8.*

**Proof:** If $p = 1, 2, 3$, then the number of isometries is exactly 2. For $p = 1, 2$, the result is trivial. For $p = 3$, we have two possibilities:

- The chain of length 3, that by Proposition 2 has only two isometries.
- The poset $\{x_1, x_2, x_3\}$ with $x_1 \prec x_2, x_1 \prec x_3$ and its dual. In this case, as $P$ is not autodual, $O(P) = H_0$, that is determined by the group of order isomorphisms on $P$, and this group has clearly two elements.

Therefore, let us assume $p > 3$. By Theorem 1, we know that any isometry on $O(P)$ is given by a mapping $f$ and two filters $F, F'$. As $P$ is connected, we necessarily have by Corollary 3 that either $F = P$ or $F = \emptyset$.

Assume $F = P$. Then, $f$ is isotone on $P$ and $P$ and $f(P)$ are order isomorphic posets. As $P$ and $f(P)$ are order isomorphic, if $f(i) = j$ we necessarily need that $i^{\uparrow}$ is isomorphic to $j^{\uparrow}$ and $i^{\downarrow}$ is isomorphic to $j^{\downarrow}$ by Remark 1.

Let us consider the equivalence relation given by $i\mathcal{R}j$ if and only if $i^{\uparrow}$ and $j^{\uparrow}$ are isomorphic and $i^{\downarrow}$ and $j^{\downarrow}$ are isomorphic. Suppose $i\mathcal{R}j$; then, $i$ and $j$ cannot be compared if $i \neq j$. On the other hand, as $P$ is connected, there is a sequence $i = i_1, i_2, ..., i_s = j$ such that either $i_k \prec i_{k+1}$ or $i_k \succ i_{k+1}, \forall k = 1, ..., s-1$, whence there exists $k$ such that $i \prec k$ or $k \prec i$. Therefore, there are at least two equivalence classes. Consider the different equivalence classes $C_1, ..., C_r$, and let us denote their corresponding cardinalities by $c_1, ..., c_r$, respectively.

Thus, if $F = P$, the number of possible mappings $f$ is bounded by $c_1!...c_r!$, because if $f(i) = j$, we necessarily need $i\mathcal{R}j$, i.e., $i$ and $j$ are in the same equivalence class. Remark that this is just an upper bound, as the image on an equivalence class can determine the image of other elements.

As $c_1 + ... + c_r = p$, and $r \geq 2$ the value $c_1!...c_r!$ attains its maximum when $r = 2$ and $c_1 = p-1, c_2 = 1$. Consequently, an upper bound for the number of isometries when $F = P$ is $(p-1)!$ Note that this bound is attained when there is a top element or a bottom element and the other elements of $P$ cannot be compared.

If $F = \emptyset$ is possible, then $P$ is autodual. Then, we have two possibilities:

- There are only two equivalence classes. As $P$ is autodual, this implies that both have the same number of elements (so $p$ is even), whence the number of isometries when $F = P$ is bounded by $(\frac{p}{2}!)^2$. In this case, by Proposition 4, the number of isometries on $O(P)$ is bounded by $2(\frac{p}{2}!)^2 \leq (p-1)!$ if $p > 4$. If $p = 4$, then $2\frac{4}{2}!^2 > (4-1)!$, so that the number of isometries on $O(P)$ is bounded by 8. This bound is achieved when we consider a poset with four elements $\{i_1, i_2, i_3, i_4\}$ such that $i_1, i_2 \prec i_3, i_4$ and $i_1, i_2$ and $i_3, i_4$ are incomparable.

- Otherwise, there are at least three equivalent classes, whence the number of isometries when $F = P$ is bounded by $(p-2)!$ By Proposition 4, the number of isometries is bounded by $2(p-2)! \leq (p-1)!$ This finishes the proof. ■

Let us now turn to the problem of determining the number of isometries if $P$ has $r$ connected components. Let us denote the connected components by $P_1, ..., P_r$. We will denote the number of isometries on $O(P)$ by $n$. Similarly, the number of isometries on $O(P)$ that are in $H_0$ will be denoted by $s$. For poset $P_i$, we will use the notation $n_i$ and $s_i$, respectively.

**Proposition 6** *Suppose $P$ has $r$ connected components $P_1, ..., P_r$ such that $P_i$ is not isomorphic to $P_j$ nor to the dual of $P_j$ if $j \neq i$. Then, the number of isometries on $O(P)$ is*

$$n = \prod_{i=1}^{r} n_i.$$

**Proof:** By Theorem 1, we know that any isometry on $O(P)$ is given by a partition $\{F, F'\}$ of $P$ and a mapping $f$. As $F, F'$ are filters, this implies that any connected component $P_i$ is contained in either $F$ or $F'$. Then, $f(P_i)$ is order isomorphic to $P_i$ if $P_i \subseteq F$ or $f(P_i)$ is order isomorphic to $\overline{P_i}$ if $P_i \subseteq F'$ by Remark 1. Thus, it follows by hypothesis that $f$ maps $P_i$ in $P_i$ and therefore, the restriction of $h$ to $P_i$ is an isometry on $P_i$. Hence, any isometry on $O(P)$ is a direct product of $r$ isometries, one in each $P_i, i = 1, ..., r$. In other words, $O(P)$ is the direct product of $O(P_i), i = 1, ..., r$. This finishes the proof. ■

**Corollary 6** *Suppose $P$ has $r$ connected components $P_1, ..., P_r$ such that $P_i$ is not isomorphic to $P_j$ nor to $\overline{P_j}, j \neq i$. Suppose also that $P_i$ is not autodual for any $i = 1, ..., r$. Then, the number of isometries on $O(P)$ is*

$$n = \prod_{i=1}^{r} s_i.$$

**Proof:** It suffices to remark that all the isometries on $O(P_i)$ are in $H_0$ by Proposition 4, whence $s_i = n_i, i = 1, ..., r$. Now, it suffices to apply Proposition 6. ■

**Corollary 7** *Suppose $P$ has $r$ connected components $P_1, ..., P_r$ such that $P_i$ is not isomorphic to $P_j$ nor to $\overline{P_j}, j \neq i$. Then, the number of isometries on $O(P)$ is*

$$n = 2^k \prod_{i=1}^{r} s_i,$$

*where $k$ is the number of components that are autodual.*

**Proof:** If $P_i$ is autodual, then $n_i = 2s_i$ by Proposition 4. Now, apply Proposition 6. ■

Let us now allow isomorphisms between the connected components of $P$.

**Corollary 8** *Suppose $P$ has $r$ connected components $P_1, ..., P_r$ such that $P_i$ is not autodual nor isomorphic to $\overline{P_j}, j \neq i$. We can partitionate the set $\{P_1, ..., P_r\}$ in $m$ subsets such that if $P_i$ and $P_j$ are in the same subset, then they are isomorphic. Let us denote by $c_j, j = 1, ..., m$ the number of connected components in the $j$-th class. Then, the number of isometries on $O(P)$ is*

$$n = \prod_{i=1}^{r} s_i \prod_{j=1}^{m} c_j!$$

**Proof:** It suffices to note that by hypothesis $f(P_i)$ is isomorphic to $P_i$. Then, if $P_i$ is in the $j$-th class of equivalence, $f(P_i)$ can be any connected component in the class. As $f$ is one-to-one, the number of possibilities for $f(P_1), ..., f(P_r)$ is $\prod_{j=1}^m c_j!$ Then, for each isometry on $O(P)$ such that $f(P_i) = P_i, \forall i$, we can generate $\prod_{j=1}^m c_j!$ new isometries. As $P_i$ is not autodual by hypothesis, the result follows from Corollary 6. ■

Let us now allow isomorphisms between the connected components of $P$ and their duals.

**Corollary 9** *Suppose $P$ has $r$ connected components $P_1, ..., P_r$ such that $P_i$ is not autodual $i = 1, ..., r$. We can partitionate the set $\{P_1, ..., P_r\}$ in $m$ subsets such that if $P_i$ and $P_j$ are in the same subset, then they are isomorphic or $P_i$ is isomorphic to $\overline{P_j}$. Let us denote by $c_j, j = 1, ..., m$ the number of components $P_i$ in the $j$-th class. Then, the number of isometries on $O(P)$ is*

$$n = \prod_{i=1}^{r} s_i \prod_{j=1}^{m} c_j!$$

Joining all these results, the following can be stated.

**Theorem 5** *Suppose $P$ has $r$ connected components $P_1, ..., P_r$. We can partitionate the set $\{P_1, ..., P_r\}$ in $m$ classes such that if $P_i$ and $P_j$ are in the same class, then they are isomorphic or $P_i$ is isomorphic to $\overline{P_j}$. Let us denote by $c_j, j = 1, ..., m$ the number of components $P_i$ in the $j$-th class. Then, the number of isometries on $O(P)$ is*

$$n = 2^k \prod_{i=1}^{r} s_i \prod_{j=1}^{m} c_j!,$$

*where $k$ is the number of components $P_i$ that are autodual.*

**Proof:** The only case that we need to treat is the case in which there are components that are autodual. Consider a class of autodual components, namely $C_j$. Then, if $P_i$ is in the class, the number of isometries generated by the class is $c_j! 2^{c_j} \prod_{P_i \in C_j} s_i$. Therefore, the result holds. ■

This result allows us to determine upper bounds for the number of isometries on $O(P)$ when $P$ is not connected applying Proposition 5. The following holds:

**Corollary 10** *Suppose $P$ has $r$ connected components $P_1, ..., P_r$. We can partitionate the set $\{P_1, ..., P_r\}$ in $m$ classes such that if $P_i$ and $P_j$ are in the same class, then they are isomorphic or $P_i$ is isomorphic to $\overline{P_j}$. Let us denote by $c_j, j = 1, ..., m$ the number of components $P_i$ in the $j$-th class. Then, the number of isometries on $O(P)$ is bounded by*

$$2^k 8^l \prod_{p_i \geq 5} (p_i - 1)! \prod_{j=1}^{m} c_j!,$$

*where $k$ is the number of components that have one, two or three elements and $l$ is the number of components of 4 elements.*

Note that two classes of isomorphic components are formed by components with one element and with two elements. If $r = p$, then all connected components are singletons and we obtain the result for antichains of Lemma 12. Finally, the following result can be stated.

**Theorem 6** *The number of isometries on $O(P)$ where $P$ is a poset with $r$ connected components and $|P| = p > r - 3$ is bounded by*

$$2^{r-1}(r-1)!(p-r)!$$

*This bound is attained when $r - 1$ connected components are singletons and the other component is such that it has either an upper bound or a lower bound and the other elements of the component cannot be compared.*

*If $p = r + 3$, the component that is not a singleton has 4 elements and the bound is*

$$2^{r-1}(r-1)!8.$$

*If $p = r+2$ or $p = r+1$, the number of isometries*

$$2^r(r-1)!$$

*Finally, if $p = r$, the number of isometries is (see Lemma 12)*

$$2^r r!$$

**Proof:** Suppose there are $u$ connected components with at least 5 elements in $m'$ equivalence classes. Then, $\prod_{p_i \geq 5}(p_i - 1)! \prod_{i=1}^{m'} c_j! \leq (\sum_{i=1}^{u} p_i - 1)!$ On the other hand, if there are $t$ elements summing all the elements in components of 1, 2, 3 or 4 elements, it holds that $2^k 8^l \prod_{i=m'+1}^{r} c_i! \leq 2^t (\sum_{i=m'+1}^{r} c_i)!$ Joining these facts, the result holds. ■

## 7 Conclusions and open problems

In this paper, we have obtained the general form of the group of isometries over an order polytope. Next, we have applied these results for the special order polytopes of the set of fuzzy measures and the set of $p$-symmetric measures when the indifference partition is fixed. We have also studied the subgroup of isometries satisfying $h(\emptyset) = \emptyset$ and we have derived upper bounds for the number of isometries on an order polytope. In all these results, we have shown that the order structure of the poset plays a fundamental role. We think this results could shed light on the properties of these polytopes.

The problem of obtaining the set of isometries is related to the problem of identification of fuzzy measures [6]. On the other hand, it has been shown that the number of vertices of the polytope $\mathcal{FM}(X)$ coincides with the Dedekind numbers [19]. Thus, it is necessary to define a subfamily with a reduced number of vertices. In this sense, if we additionally impose that it comes from a poset, the results in this paper provide us with the set of isometries.

In the theory of fuzzy measures, there is an important subfamily of fuzzy measures called $k$-additive measures [12]. It can be proved that the set of fuzzy measures being at most $k$-additive is a convex polytope, too. However, this polytope is not an order polytope; hence, we cannot apply the results in the paper to determine its group of isometries. On the other hand, as this polytope is a subpolytope of $\mathcal{FM}(X)$, the results that we have obtained could offer a glimpse about the solution of the problem. The same can be said for other subfamilies of fuzzy measures, as belief functions [7, 22] or possibility measures [9].

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